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DIFFUSION MODELS OF SLOPE DEVELOPMENT

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ABSTRACT

This paper represents a systematic investigation of slope evolution diffusion models and has the following sections: (1) The model of slope development with linear coefficient $k = k_0 x$; (2) The model of slope development with quadratic coefficient in x; (3) Slope development model with vertical lowering of base level (downcutting); (4) Slope development model with the base level a horizontal variable (undercutting); (5) Steady-state regime of undercut slopes; (6) Model of a pediment and scree slope formation.

The comparison is made between mathematical and classical methods of slope evolution analysis.

KEY WORDS Diffusion models Slope development Steady-state solution Undercut slopes

INTRODUCTION

Diffusion models have been used to describe various geomorphological processes concerned with denudation and material accumulation. Such models were used to describe slope development more than 20 years ago by Culling (1960, 1963, 1965). A balance equation of the material (continuity equation) forms the basis of these models

$$\frac{\partial y}{\partial t} = -\frac{\partial q}{\partial x} \tag{1}$$

where q is the material discharge, defined as the flow past a unit plane perpendicular to the direction of flow; $y(x, t)$ is the elevation, x the horizontal distance, and t is the time. Taking the condition that the material discharge is proportional to the surface gradient

$$q = -k(x, t)\frac{\partial y}{\partial x} \tag{2}$$

substitution into equation (1) gives

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial x}\left(k(x, t)\frac{\partial y}{\partial x}\right) \tag{3}$$

the diffusion model of slope development.

By its structure equation (3) is analogous to the models applied to problems of heat conduction and diffusion. In which case expression (2) is analogous to Fourier's and Fick's first laws of heat conduction and

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diffusion, respectively. The diffusion model was investigated by Culling for the case of a constant coefficient corresponding to stable conditions both in climate and lithology. The model proved to describe well the long term evolution of slopes, their flattening, denudation on convex parts, and accumulation on concave ones.

Culling (1960, 1963) worked out a great number of boundary-value problems for equation (3) with constant coefficients corresponding to different geomorphological situations basing his treatment on the standard work of Carslaw and Jaeger (1959) on the conduction of heat in solids. For the given case (k = constant) a solution in classical terms gives the slope development as a Fourier series (of trigonometrical functions). On the other hand where the coefficient is a variable and depends upon the space coordinate $k = k(x)$ then different kinds of Fourier series are obtained. It is useful to consider the problem for the variable coefficient as it allows the taking into account of the spatial variability of slope forming factors.

When solving boundary value problems over a finite interval and for $k = k(x)$ two kinds of Fourier series are involved (Trofimov and Moskovkin, 1976, 1983):

1. A Fourier–Bessel series appears when $k(x) = ax + b$ and when $k(x) = ax^n$, $n \neq 2$.
2. A Fourier–Legendre series appears when $k(x) = ax^2 + bx + c$, $a < 0$.

We proceed to consider these two cases in detail.

THE MODEL OF SLOPE DEVELOPMENT WITH LINEAR COEFFICIENT $k = k_0x$.

The possibility of using a linear coefficient in the diffusion model has been discussed in a number of works (Mizutani, 1970; Kirkby, 1971; Carson and Kirkby, 1972; Hirano, 1975, 1976) but non-stationary cases were not covered. In this connection we have developed the solution for the case corresponding to a slope with fixed base level developing under the action of sheetwash with constant precipitation (Trofimov and Moskovkin, 1983). In an established (steady) phase of run-off the unit water discharge (Q) increases linearly down the slope and so

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial x}\left(k_0 x \frac{\partial y}{\partial x}\right),\ y(l, t) = 0,\ y(x, 0) = f(x),\ 0 \leqslant x \leqslant l \tag{4}$$

where x is the horizontal distance from the divide, y is the elevation, l is the horizontal equivalent (slope length), k_0 is a constant depending upon the run-off coefficient, the intensity of rainfall, the physical quality of the surface and on soil particle transport by overland flow, while $f(x)$ is the initial slope profile.

One can assume that when a slope has homogenous physical properties then the unit sediment discharge (q) is proportional to the unit water discharge (Q) for a given gradient. The constant of proportionality (ρ) gives the contribution to the turbidity (sediment concentration) of the flow which itself is proportional to the gradient. Thus, $q = \rho Q$, $\rho = -c\partial y/\partial x$, where c is an erosion coefficient while $Q = bIx$, where b is the run-off coefficient, and I is the intensity of rainfall. It follows that K_0 can be determined from observations of unit water (V_m, m^2) and sediment yield (W_m, m^2) according to the formula

$$k_0 = cbI = W_m bI / V_m i \tag{5}$$

where rainfall intensity I is measured in m/min and $i = -\partial y/\partial x$, regarded as a constant. Note that V_m and W_m can be taken as the total water and sediment yield in the same measurement units (m^3, tonnes). The general solution of equation (4) is as follows

$$\begin{cases} y(x, t) = \sum_{n=1}^{\infty} C_n \exp\left(-\frac{k_0 \mu_n^2 t}{4l}\right) J_0(\mu_n \sqrt{(x/l)}) \\ C_n = \frac{2}{J_1^2(\mu_n)} \int_0^1 z f(lz^2) J_0(\mu_n z) \mathrm{d}z \end{cases} \tag{6}$$

where μ_n are the positive roots of the equation $J_0(z) = 0$ (see Appendix 1).

For large t when all but the first term of the Fourier–Bessel series are reduced to negligible proportions because of the rapid convergence of the exponential factor (as the sequence of roots squared), one obtains the regular solution

$$y_{\text{reg}}(x, t) = C_1 \exp\left(-\frac{k_0 \mu_1^2 t}{4l}\right) J_0(\mu_1 \sqrt{(x/l)}) \tag{7}$$

which implies that a slope profile in the course of time tends to the convex form of $J_0(u)$ between $u = 0$ and $u = \mu_1$, and then flattens and slowly tends to a horizontal surface.

The notion of a regular regime in geomorphology was introduced by Devdariani (1963, 1967) with the analogy of regular regime theory in heat conduction. According to this theory the higher terms of a series solution disappear with time leaving only the first harmonic (Fourier, 1822). Such a regime in heat conduction theory at which the influence of the initial conditions disappears was termed a regular regime (Boussinesq, 1901).

The independence upon initial conditions here means that a common slope configuration does not depend upon initial profile configuration any longer and is determined by the first harmonic form only (convex form). Here the initial condition (initial slope profile) with the help of which C_0 is defined (in integral (6)) describes the first harmonic amplitude only but not the character of its form (convexity, concavity and so on).

We may now consider the possibility of using equation (6) to determine the soil loss on slopes. The soil loss volume (W) during the interval τ from a section of the slope of unit width and of horizontal length (l_0) from the point of initial run-off ($x = 0$) will, in accordance with slope profile lowering, be given by

$$W(\tau) = \int_0^{l_0} [f(x) - y(x, \tau)] \, dx \tag{8}$$

where $f(x)$ and $y(x, \tau)$ are the slope profiles at $t = 0$ and $t = \tau$ respectively.

Substituting for $t = \tau$ from equation (6) and performing the integration

$$W(\tau) = \int_0^{l_0} f(x) dx - 4\sqrt{(l_0 l)} \sum_{n=1}^{\infty} \frac{J_1(\mu_n \sqrt{(l_0/l)})}{\mu_n J_1^2(\mu_n)} \exp\left(-k_0 \frac{\mu_n^2 \tau}{4l}\right) \int_0^1 z f(lz^2) J_0(\mu_n z) dz \tag{9}$$

where $l_0 \leqslant l$ (see Appendix 2).

This expression may be used to calculate the soil loss from slopes of arbitrary configuration and demands the minimum of initial parameter values. However, the complexity will necessitate numerical methods in the general case. A simpler expression is obtained for the computation when the initial slope is rectilinear with length $l_0 = l$: $f(x) = h(1 - (x/l))$. Substituting into equation (9) and integrating we obtain (see Appendix 2)

$$W(\tau) = \frac{hl}{2} - 16hl \sum_{n=1}^{\infty} \exp\left(-\frac{k_0 \mu_n^2}{4l} \tau\right) \Big/ \mu_n^4 \tag{10}$$

We now make a verification of formula (10) in accordance with some artificial rainfall experiments conducted at the Kirov Pedagogical Institute by Sheklein (Chitishvily, 1974); with the values, $I = 1$ mm/min $= 10^{-3}$ m/min, $b = 0{\cdot}04$, $\tau = 58{\cdot}1$ min, $i = \tan 7° = 0{\cdot}1228$, $W_m = 0{\cdot}01373\,\text{m}^2$ (from the conversion: sediment yield —3·57 tonnes/hectare, bulk density —1·3 tonnes/m^3), $V_m = bIl\tau = 0{\cdot}1162\,\text{m}^2$.

From formula (5) we have for k_0 the value $3{\cdot}8446 \times 10^{-5}$ m/min. The height difference for a length $l = 50$ m is given by $h = \tan 7°\, 50\,\text{m} = 6{\cdot}14$ m. The positive roots of $J_0(z) = 0$ are available in tables. Computation shows that taking the first six terms in equation (10) leads to a fourfold excess of computed soil loss (W) over measured volume (W_m). The extension of the calculation to 15–20 terms is enough for a 5 per cent accuracy.

Consider now the mathematical model for the longitudinal profile of a river valley in steady state conditions under the action continuous oblique uplift. Water discharge is assumed to increase linearly downstream. An oblique uplift results from the tilting of the plane of the landscape and is characterized in this case by a linear increase in height or velocity from lower to upper reaches of the river. Such uplift appears to have first been described in detail by Makkaveev (1955) and corresponds to valley development on the limb of a growing structure or within the limits of a tilting block.

Taking into account the additional volume above base level introduced by the tectonic factor

$$\frac{\partial}{\partial x}\left[\alpha(ax + Q_0)\frac{\partial y}{\partial x}\right] + V_0(1 - x/l) = 0 \tag{11}$$

where the expression in square brackets is the sediment discharge, taken as directly proportional to the product of the water discharge and the gradient, water discharge being taken as increasing linearly along the length of the river beginning at the source; Q_0 represents water discharge at source (inflow); a = constant, the increment in the intensity of water discharge with distance downstream; α = constant, a coefficient characteristic of the physical properties of the bedrock and sediment load; V_0 is the velocity of uplift at the source ($x = 0$); l is the horizontal distance of stream length; while y is the river profile elevation. The second term represents an oblique uplift with a linear decrease in velocity from source to mouth, where it is taken as zero.

Two boundary conditions are considered; (i) constant base-level at river mouth; and (ii) constant sediment discharge at source:

$$y(l) = 0$$

and

$$\alpha Q_0 \left.\frac{\partial y}{\partial x}\right|_{x=0} = -q_0 = \text{const.} \tag{12}$$

By integrating twice and using equation (12) we obtain as a solution

$$y(x) = \left(\frac{V_0 Q_0}{\alpha a^2} + \frac{V_0 Q_0^2}{2l\alpha a^3} - \frac{q_0}{\alpha a}\right) \ln\left(\frac{ax + Q_0}{al + Q_0}\right) + \frac{V_0}{4l\alpha a}(x^2 - l^2) + \frac{V_0}{\alpha a}\left(1 + \frac{Q_0}{2la}\right)(l - x) \tag{13}$$

Solution (13) is a monotonic decreasing function and allows for the disclosure of a number of non-trivial regularities. We may determine the existence of a point of inflection separating an upper convex segment from a lower concave one. Thus setting the second derivative to zero we derive the point of inflection (x_0)

$$x_0 = \left[\frac{2lQ_0}{a}\left(1 + \frac{Q_0}{2la}\right) - \frac{2q_0 l}{V_0}\right]^{1/2} - \frac{Q_0}{a} \tag{14}$$

The existence of an inflection point within the segment $(0, l)$ depends upon

$$\frac{Q_0}{a} > \frac{q_0}{V_0} \tag{15}$$

Note this inequality was obtained under the assumption that $x_0 > 0$ in which case the result $x_0 < l$ is automatically satisfied and there exists a real root in expression (14). Inequality (15) can be rewritten as

$$V_0/a > q_0/Q_0 = \rho_0$$

where ρ_0 is a non-dimensional parameter of sediment concentration at the river source. As $al = Q_l - Q_0$, where Q_l is the water discharge at the mouth (outflow), then the inequality can be reformulated as

$$[V_0 l/(Q_l - Q_0)] > \rho_0,$$

thus giving a simple criterion of the existence of a point of inflection in longitudinal river profiles.

Theoretically with increase in the value of V_0 (other things being equal) the inflection point migrates downstream to a definite limit within $[0, l]$. This limit is found from the expression (14)

$$\lim_{V_0 \to \infty} x_0 = \left[\frac{2lQ_0}{a}\left(1 + \frac{Q_0}{2la}\right)\right]^{1/2} - \frac{Q_0}{a} \tag{16}$$

Thus in conditions of oblique uplift the convexity in a river's upper course is due to the uplift while the concavity in the lower course is caused by the increase in water discharge. These results do not change qualitatively with a more complex dependence of sediment to water discharge, e.g. $q = -\alpha Q^n \frac{\partial y}{\partial x}$, $n > 1$ and with a non-linear increase in water discharge downstream.

THE MODEL OF SLOFE DEVELOPMENT WITH QUADRATIC COEFFICIENT IN x

For the case when the coefficient k increases, at first as in the previous model (4) and thereafter diminishes due to an increase in the infiltration rate in the lower part of the slope, the behaviour of the coefficient is approximated by the quadratic function $k(x) = ax^2 + bx + c$, $a < 0$, and we have the model (Trofimov and Moskovkin, 1976, 1983),

$$\begin{cases} \dfrac{\partial y}{\partial t} = \dfrac{\partial}{\partial x}\left[(ax^2 + bx + c)\dfrac{\partial y}{\partial x}\right], \ y(x, 0) = f(x) \\ x_1 \leqslant x \leqslant x_2 \end{cases} \tag{17}$$

where x_1 and x_2 are the roots of the quadratic equation $ax^2 + bx + c = 0$, $a < 0$.

The points x_1 and x_2 are the singular points of an ordinary hypergeometric equation derived from (17) by separation of the variables. A solution to equation (17) has been obtained in Fourier–Legendre form (Trofimov and Moskovkin, 1976, 1983) (see Appendix 3):

$$\begin{cases} y(z, t) = \sum\limits_{n=0}^{\infty} C_n P_n(z) \exp[a(n+1)nt] \\ C_n = \left(n + \dfrac{1}{2}\right)\displaystyle\int_{-1}^{+1} f(z) P_n(z) \mathrm{d}z \end{cases} \tag{18}$$

where

$$x = x_1 + (x_2 - x_1)\left(\frac{1-z}{2}\right)$$

$p_n(z)$ is the Legendre polynomial of order n. The regular and limiting solutions are available from (18) (Trofimov and Moskovkin, 1983).

$$\begin{cases} y_{\text{reg}}(x, t) = C_0 + C_1\left[1 - \dfrac{2(x - x_1)}{(x_2 - x_1)}\right]\exp(2at) \\ y_{\text{lim}}(x, t) = C_0 = \text{const.} \end{cases} \tag{19}$$

Hence the slope profile in the given model in the regular regime with t large tends to a straight form and a horizontal surface. To determine the relative relief after limited flattening, with the help of the expression for C_n

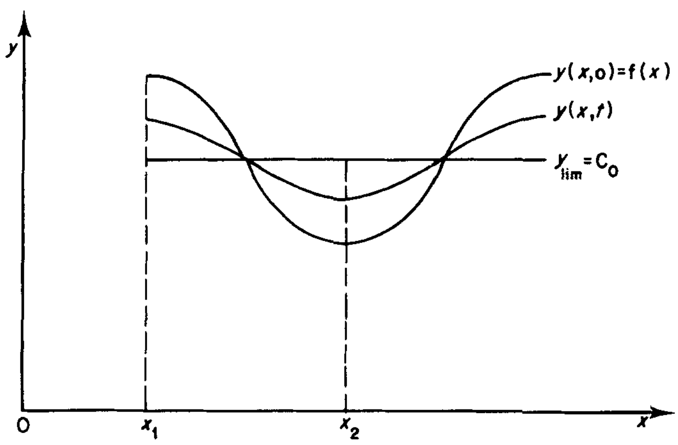

Figure 1. To the problem of the evolution both symmetrical and conjugate slopes

(18) we have for $n = 0$

$$y_{\lim} = C_0 = \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} f(x)\mathrm{d}x \tag{20}$$

This is the average height of the slope form and consequently in the evolutionary process the square of the curvilinear trapezium enclosed by the profile $y(x, t)$ on the segment $[x_1, x_2]$ remains constant. For a natural slope this means that the amount of material removed from the upper parts of the slope is equalled by the amount accumulated at the base.

Unless the lower point is fixed spatially by some external factor, e.g. a stream, it will migrate as a diluvial shelf is built up. However, if we take an image about the vertical plane through x_2, as in Figure 1, the model corresponds to the filling of a symmetrical through, the evolution being given by a series of symmetrically-conjugate convex–concave curves, as in Figure 1.

The case considered, that of an increase in water discharge in the upper parts of the slope and a decrease in the lower parts, is typical of arid mountain regions particularly if a loose permeable material lies on the base slopes (Makkaveev, 1955).

SLOPE DEVELOPMENT MODEL WITH VERTICAL LOWERING OF BASE LEVEL (DOWNCUTTING)

We now proceed to consider problems where there are variable conditions on the boundaries. These problems appear when account is taken of undercutting (both in deepening and of lateral erosion/abrasion) in general slope development processes. In the diffusion model this dependence can be simulated by placing conditions on the boundary conditions. Two principal cases arise: (1) slope development with a vertical change in base-level (downcutting); and (2) slope development with a horizontal change in the lower boundary (undercutting). Let us analyse the first case.

When modelling this type of slope development Culling (1963) gave a solution to the problem of describing the evolution when the controls were on one side the divide and on the other a downcutting stream:

$$\frac{\partial y}{\partial t} = k \frac{\partial^2 y}{\partial x^2}; \left.\frac{\partial y}{\partial x}\right|_{x=0} = 0; \; y(l, t) = \phi(t); \; y(x, 0) = f(x) \tag{21}$$

Carson and Kirkby (1972) considered the particular case of the establisbed stationary solution of this problem when the rate of lowering of base level is a constant, $\phi(t) = \beta t$, with an initial surface $y(x, 0) = 0$ finding that

$$y(x, t) = -\beta t + \frac{\beta(x^2 - l^2)}{2k} \tag{22}$$

Ahnert (1973) has applied numerical metbods in the simulation of this process using his comprehensive model for viscoplastic flow modelling a steady-state convex profile. Armstrong (1976, 1980) has obtained a solution analogous to equation (22) using his three-dimensional process–response model.

The more general solution of the problem corresponding to the two cuttings that occur according to arbitrary laws $\phi_1(t)$ and $\phi_2(t)$ was developed by Culling (1965, p. 252). It covers all possible cases of rejuvenation by vertical fall of local base levels. The solution when $\phi_1 = \phi_2 = Kt$, K being tbe constant rate of lowering of base level, is given in bls paper. In addition to that solution consider the problem of two base levels falling at constant but differing velocities (β, γ = const) and cutting an initial horizontal surface

$$\frac{\partial y}{\partial t} = k \frac{\partial^2 y}{\partial x^2}; \; y(0, t) = -\beta t; \; y(l, t) = -\gamma t; \; y(x, 0) = 0 \tag{23}$$

Consider the case $\beta > \gamma$ (the case for $\beta < \gamma$ is analogous). The solution of equation (23) when t is large enough for the exponential terms to be negligible is (Esin and Moskovkin, 1980)

$$y(x, t) = \frac{(\beta - \gamma)tx}{l} + \frac{(\beta - \gamma)x(x^2 - l^2)}{6kl} + \frac{\beta x}{2k}(l - x) - \beta t \tag{24}$$

We analyse two variants:

1. When $\gamma = \beta$, then by Culling (1965, p. 252)

$$y(x, t) = \frac{\beta x}{2k}(l-x) - \beta t \tag{25}$$

i.e. a convex parabolic interfluve profile is formed with a relative relief of $h = \beta l^2/8k$ (the greater the rate of lowering of base level (β) or of the landform dimension (l), or the smaller the coefficient k, the greater the relative relief).

2. When $\beta > \gamma$, the intervalley divide migrates towards the gentler slope on the right until it reaches a limit at the point $x = l$ (when $\beta < \gamma$ the divide moves to the left, i.e. always towards the gentler rate of downcutting). A divide displacement law can be found from the condition $\partial y/\partial x = 0$

$$x_d = \frac{l\beta}{\beta-\gamma} - \left(\frac{\gamma\beta l^2}{(\beta-\gamma)^2} + \frac{l^2}{3} - 2kt\right)^{1/2} \tag{26}$$

The disappearance of a local maximum of equation (24), i.e. when the divide reaches the point $x = l$, will occur in time

$$t^* = (l^2/6k)[(2\gamma+\beta)/(\beta-\gamma)]. \tag{27}$$

The time t^* is measured from the moment at which the solution (24) is established and this time is estimated from the general solution of equation (23). Both Ahnert (1973) and Armstrong (1976) have obtained from their models the displacement of the divide towards the gentler slopes. This migration of the divide is a demonstration of the unequal slopes law (Gilbert, 1877). According to this law, if opposite slopes have differing steepness then the steeper slope recesses faster than the gentler one and so the divide moves towards the gentler slope.

SLOPE DEVELOPMENT MODEL WITH THE BASE-LEVEL A HORIZONTAL VARIABLE (UNDERCUTTING)

Such a model arises when one considers lateral erosion or abrasion factors. Culling (1963), while simulating thls process, considered a slope form of finite dimensions moving with a constant velocity. This is equivalent to the material discharge set as

$$q = -k\frac{\partial y}{\partial x} + cy$$

In this case the material balance equation (1) was reduced to a diffusion equation plus an additional convective (Culling's mass transport) term $-c\partial y/\partial x$. Later Hirano (1975, 1976) considered similar equations.

We wish to consider the process of undercutting according to equation (3) with a zero condition on the elevation y at the lower base level control which moves horizontally at constant height. If the lateral displacement has a constant velocity b (velocity of slope undercutting) we have the variable boundary value problem (Trofimov and Moskovkin, 1976)

$$\frac{\partial y}{\partial t} = k\frac{\partial^2 y}{\partial x^2};\ bt \leqslant x < +\infty;\ y(bt, t) = 0;\ y(x, 0) = f(x) \tag{28}$$

the solution of which is as follows (see Appendix 4)

$$y(x, t) = \frac{\exp\left(\frac{b^2 t}{4k} - \frac{bx}{2k}\right)}{2\sqrt{(\pi k t)}} \int_0^{+\infty} f(z)\exp\left(\frac{bz}{2k}\right)\left\{\exp\left[-\frac{(z-x+bt)^2}{4kt}\right] - \exp\left[-\frac{(z+x-bt)^2}{4kt}\right]\right\} dz \tag{29}$$

The solution of equation (29) for an initial straight profile $f(x) = ax$ (Esin and Moskovkin, 1980) has shown a convexity at the base of the slope. The gradient increases at the base and in time would lose stability. Note there exists no particular solution to equation (28) of the type $y = a(x-bt)$, corresponding to the parallel retreat of a straight slope as this does not satisfy the problem for any combination of values of a, b, and k. It follows that any slope satisfying a model with constant coefficient cannot retreat strictly parallel i.e. preserving their straightness. Note, the solution $y = a(x - bt)$ is one of the kinematic models of Scheidegger (1970), describing the development of steep slopes. Below we show what kind of steady-state solutions may exist corresponding to parallel retreat of slopes undercut with constant velocity.

STEADY-STATE REGIMES OF UNDERCUT SLOPES

We have obtained (Moskovkin and Trofimov, 1980) the steady-state solution of the boundary-value problem (28) corresponding to a steady-state stage of parallel retreat of an undercut slope (see Appendix 5).

$$y(x, t) = h\left[1 - \exp\left(-\frac{b}{k}(x - bt)\right)\right] \tag{30}$$

Culling (1963) when solving the diffusion equation with a convection term over a semi-infinite region came to the similar steady-state solution (30) for an initial vertical (scarp) slope. It should be noted that Culling's model for the semi-infinite region moving with a constant velocity is mathematically equivalent to our problem (28) on the interval $bt \leqslant x < +\infty$ if in the latter case one changes the variables, $z = x - bt$, $t = t$.

Solution (30) has been used in the analysis of the stability of an initial slope with profile similar to (30) (see Appendix 5)

$$y(x, 0) = h[1 - \exp(-cx)] \tag{31}$$

where h is the limiting horizontal surface of the slope. The value $b = ck$ is critical and defines two different regimes of slope development: when $b > ck$ the gradient at the base of the slope increases, the brow moving gradually up the slope; when $b < ck$ the slope is subject to downwearing. The parameter can be expressed in terms of the gradient (i_0) of equation (31); when $x = 0, c = i_0/h$. The studies of Quigley and Gelinas (1976) on the shores of Lake Erie confirm the existence of these regimes. They distinguish three morphodynamic types of shore slope: (1) The eroded cliff slowly retreats preserving its profile form due to natural abrasive processes at the base and slump and talus formation of the upper parts. This type is developed when the lake has average levels and when a critical equilibrium is obtained between slope development and abrasion processes; (2) The steepness of the eroded cliff increases rapidly due to active undercutting. This type is developed under high water levels and active abrasion; (3) The gradual downwasting of the cliff face occurring in periods of low water level.

Further analysis shows that equation (29) with an initial exponential profile (31) tends in the limit as $t \to \infty$ to the steady-state solution (30), i.e. it is the asymptotic solution to problems with initial conditions (31). Thus as previously obtained the increasing and decreasing regimes of the slope base are transient and in the course of time a steady-state regime will be established. Moreover the establishment of equation (30) will take place from an arbitrary initial profile satisfying the conditions of equation (30), namely

$$y(bt, t) = 0$$

and

$$\lim_{x \to \infty} y(x, t) = h = \text{const.}$$

This can be proved with the help of the asymptotic behaviour of the integral in equation (29) for large t. For this it is sufficient to replace $f(z)$ by its asymptotic value, i.e. $f(z) = h$ in the integral in equation (29). Then integrating after letting $t \to \infty$ we arrive at the steady-state solution (30) (see Appendix 5).

The obtained results are of great importance for geomorphology. It is almost impossible, due to our lack of knowledge, to determine the precise form of initial slope profiles and so trace their subsequent development to compare with present-day states. The steady-state solution does not require such knowledge.

Introducing subsidence into the diffusion model (28)

$$\frac{\partial y}{\partial t} = k\frac{\partial^2 y}{\partial x^2} - V \tag{32}$$

where $V = \text{const.}$, $t > 0$ gives the velocity of subsidence. There exists a steady-state solution for this case (Trofimov and Moskovkin, 1983) (see Appendix 6)

$$y(x, t) = \frac{Ak}{b}\left(1 - \frac{V}{Ab}\right)\left[1 - \exp\left(-\frac{b}{k}(x - bt)\right)\right] + \frac{V}{b}(x - bt) \tag{33}$$

where $y(bt, t) = 0$, $dy/dx|_{x=bt} = A > 0$, is the steady-state gradient at the base of the slope (the point of slope profile which is at the constant zero level which is taken as the foundation of the slope, according to $y(bt, t) = 0$, is being considered).

The slope profile according to this expression once there is sufficient distance from its basement ($y(bt, t) = 0$) tends to a straight line with gradient $i = V/b$. Expression (33) shows: (1) When the ratio of subsidence velocity to velocity of undercutting (V/b) is less numerically than the gradient (A) at the base of the steady-state form then the profile will be convex; (2) When $V/b = A$ the slope becomes rectilinear; (3) When $V/b > A$ the slope becomes concave. Thus by an examination of the valley cross profile one can judge on the ratio of subsidence to lateral undercutting. In the absence of subsidence ($V = 0$), the solution (33) when the condition

$$\lim_{x \to \infty} y(x, t) = h = Ak/b$$

is used instead of

$$\left.\frac{\partial y}{\partial x}\right|_{x=bt} = A$$

will transform into the steady-state solution (30).

The steady-state solutions (30, 33) conform with other theoretical and empirical studies. The first version of a river valley slope development in mathematical terms was given by Gerber (Scheidegger, 1970). His conclusions about slope profiles were based upon an estimate of the elementary volume of waste material with the help of the increment height (y) and the horizontal coordinate (x). It enabled him to write down an ordinary differential equation to determine the slope profile $y(x)$. One of the solutions of this approach leads to (Scheidegger, 1970)

$$y(x) = a_1\left(1 - \frac{\operatorname{tg}\alpha_2}{\operatorname{tg}\alpha_1}\right)\left[1 - \exp\left(-\frac{\operatorname{tg}\alpha_1}{a_1}x\right)\right] + x\operatorname{tg}\alpha_2 \tag{34}$$

which points to the asymptotic tendency of a profile to a straight line

$$y(x) = x\operatorname{tg}\alpha_2 + a_1(1 - [\operatorname{tg}\alpha_2/\operatorname{tg}\alpha_1])$$

It can be seen that Gerber's solution obtained by a geometrical method coincides completely with our steady-state solution with a tectonic factor (33). It should be noted that the Gerber model is similar to our model mathematically but not geomorphologically (for a tectonic factor has not been introduced into it directly). This last equation allows us to put a physical meaning to the formal parameters in Gerber's equation; thus, $\operatorname{tg}\alpha_1 = A$, $\operatorname{tg}\alpha_2 = V/b$ (when a slope has developed sufficiently far from the initial slope it tends to a gradient dependent upon the ratio of subsistence to undercutting), $a_1 = Ak/b$ is the dimension coefficient characterizing the process ratio of material removed from the slope (Ak is the material discharge) and undercutting. When $\operatorname{tg}\alpha_2 = 0$ then the curve of the slope profile tends asymptotically to a constant height $a_1 = h$, and we arrive at the expression

$$y(x) = h[1 - \exp([-\operatorname{tg}\alpha_1/h]x)] \tag{35}$$

which conforms to the steady-state solution in the absence of a tectonic factor (30) for a fixed time moment.

The empirical profiles of the undercut slopes of the Volga, Kama, and Vyatka river valleys and their tributaries lead to an equation of form (35) (Trofimov, 1974) which is geomorphological confirmation of the diffusion model applied to an undercut slope and to the existence in nature of steady-state slope regimes according to equation (30).

Note that the steady-state solution (30) may be used in river valley asymmetry analysis, especially in the case where the main factors are lateral erosion (due to the coriolis force) and slope processes. Characterized in (30) by the parameters b and k, respectively. One can easily characterize slope asymmetry by differences in the base gradients. The base gradient may be found from equation (30) as follows

$$i = \left.\frac{\partial y}{\partial x}\right|_{x=bt} = hb/k$$

Analysis shows (for the northern hemisphere) that when the height at the left and right shores slopes is equal (in comparable hydrologic and climatolithological conditions) the maximum asymmetry of river valleys is shown in the East–West current direction. In this case the maximum microclimatic difference in solar radiation occurs. So both factors (insolation and coriolis force) operate in harmony to produce the consequent increase in steepness on the right hand slope.

MODEL OF A PEDIMENT AND SCREE SLOPE FORMATION

An analysis of pediment and scree slope development has led to a new type of boundary value problem, one that has not been discussed in theoretical geomorphology before. Previously we have considered: (1) conditions on the function y; and (2) with conditions on $-k\,\partial y/\partial x$, the material discharge, in boundary value problems for the diffusion equation of slope development. However, while analysing pediment and scree slope development a third type of boundary condition arose, namely a linear combination of the previous two.

Suppose at $t = 0$ a vertical scarp of height H. The coordinate system has origin in the scarp base. Over the subsequent interval $t > 0$ the scarp retreats parallel at a constant velocity v and in such a way that material accumulates at the base in the positive direction ($y(x, 0) = 0$). The exposed height of scarp face gradually reduces. This leads to a decrease in the supply of material to the accumulating slope (a negative feedback system; a slope of accumulation-scarp). The mathematical problem thus set is (Trofimov and Moskovkin, 1983)

$$\begin{cases} \dfrac{\partial y}{\partial t} = k\dfrac{\partial^2 y}{\partial x^2}, \quad -vt \leqslant x < +\infty \\ y(x,0) = 0,\ 0 < x < +\infty,\ y(0,0) = H \\ -k\left.\dfrac{\partial y}{\partial x}\right|_{x=-vt} = k_p.v\,(H - y|_{x=-vt}) \end{cases} \tag{36}$$

where k_p. is the coefficient characterizing the relationship between the porosity of rock *in situ* and loose rock in terms of the volume change.

The boundary condition in the contiguity point between the slope of accumuiation and the eroded core of ground rock with a scarp retreating with constant velocity v is of the third type. It describes the interaction mechanism between the scarp and the slope via the process of supply of material from the scarp. This condition is analogous to the case of heat exchange with the environment according to Newton's law of cooling in problems of heat conduction (Carslaw and Jaeger, 1959).

The analysis of model (36) has shown that there is no steady-state solution of a 'running wave' type $y(x, t) = f(x + vt)$, satisfying the boundary condition of the third kind and the additional condition that

$$\lim_{x\to\infty} y(x,t) = 0.$$

The important conclusion can be drawn that the scarp will vanish in the limit ($t \to \infty$) independent of the value of the coefficient K. The steady-state solution that would represent a scarp of constant height can only exist in the presence of undercutting of the slope base.

In this connection we cannot fully agree with Kartashov's (1975) suggestion that the accumulation at the base of a scarp continues up to a definite limit even in the case where undercutting is absent. It is well known that intensive removal of material from the base of a slope is needed for pediment formation. In spite of this, even with a large material mobility coefficient k, model (30) will predict that the scarp will be overtopped in the limit. This is a crucial observation. In nature a pediment overlapping a scarp will take a very long time in relation to the formation of scree slopes. Very often the scarp corresponding to the pediment disappears but due to the lowering produced by a similar process operating on the opposite side. Surface lowering processes due to the convergence from opposing directions are well described in classical geomorphological literature. In this connection we consider the above mentioned equilibrium of Kartashov to be relative the scarp overlapping process still takes place but at a small and declining velocity.

It follows from the boundary condition that the gradient at the contiguity point ($x = -vt$) decreases in the course of time tending to zero. With the help of a change of variables $\xi = x + vt$, $t = t$ problem (36) is reduced to a stable boundary value problem solvable by use of the Laplace transformation (Carslaw and Jaeger, 1959) with common initial conditions $y(\xi, 0) = V_0 =$ const. When $V_0 = 0$ the solution in our choice of variables may be written (Trofimov and Moskovkin, 1983) (see Appendix 7)

$$y(x,t) = \frac{H}{2}\left\{\Phi^*\left(\frac{x}{2\sqrt{(kt)}}\right) + \left(\frac{k_{p^\cdot}}{k_{p^\cdot}-1}\right)\exp\left[\frac{v}{k}(x+vt)\right]\Phi^*\left(\frac{2vt+x}{2\sqrt{(kt)}}\right) - \right.$$
$$\left. -\left(\frac{2k_{p^\cdot}-1}{k_{p^\cdot}-1}\right)\exp\left[\frac{k_{p^\cdot}v}{k}(x+k_{p^\cdot}vt)\right]\Phi^*\left(\frac{x+2k_{p^\cdot}vt}{2\sqrt{(kt)}}\right)\right\} \tag{37}$$

where

$$\Phi^*(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty \exp(-\eta^2)\,d\eta,\ \Phi^*(z) = \text{erfc}(z) = 1 - \text{erf}(z)$$

where erf (z) is the error function for argument z and is well tabulated. Solution (37) is true when $k_{p^\cdot} > 1$, which is always the case if the removed material is not removed from the system by a river or by wave action. The scarp overlap is theoretically achieved in the limit

$$\lim_{t\to\infty} y(-vt, t) = H$$

Solution (37) in addition to giving the evolution of the scree slope also gives the development of the rock core. It should be noted that the model is true for vertical scarps only; otherwise it is not possible to write down the boundary conditions as of the third type.

To determine values of the coefficient k we need to find a maximum slope gradient at the initial instant from the boundary conditions of solution (36) when

$$y = 0:\ -\left.\frac{\partial y}{\partial x}\right|_{x=0} = -k_p.vH/k.$$

It follows that for $k_p = 1$–3; $v = 10^{-3}$ m/y; $H = 100$ m, for natural gentle slopes ($0 < i < 1{\cdot}0$) the inequality $k > 0{\cdot}1$ m^2/y should be satisfied; and when $v = 10^{-2}$ m/y and with the same values of k_p and $H - k > 1{\cdot}0$ m^2/y.

The model analysis shows that the character of slope development substantially depends upon two non-dimensional parameters (criteria similarity) $k_1 = vH/k$, $k_2 = k_p$. The first is the more important and ranges over $0 < k_1 < M$, (M a real number not equal to ∞). The second has a smaller range, $1 < k_2 < 3$ (where the upper limit is approximate). If several slopes possess the same criteria values then they should develop in the same way. Numerical computations of (37) have shown that the accumulation slope development with a weak core takes place mainly with small values of criterion k_1, ($0 < k_1 < 10$). Due to this criteria dependence the model shows a complex space–time change of denudation, transition, and accumulation zones. Several numerical computation variants are given in Figures 2 and 3. In Figure 2 ($k = 0{\cdot}01$ m^2/y, $v = 10^{-3}$ m/y, $k_{p^\cdot} = 2$, $H = 100$ m, $k_1 = 10$), we show the denudation of the rock core, and in Figure 3 ($k = 0{\cdot}1$ m^2/y, $v = 10^{-3}$ m/y, $k_{p^\cdot} = 2$, $H = 100$ m, $k_1 = 1$), the process of forming the base rock convex core. For the first variant (Figure 2) the exposed height of the scarp face at a finite moment (96×10^3 y) has reached 0·2 m within a horizontal distance of about 200 m with an accumulation slope of about 100 m. For the second variant (Figure 3) the exposed height of the scarp at (736×10^3 y) has reached 0·7 m with a horizontal distance of about 1600 m and an accumulation slope of about 900 m.

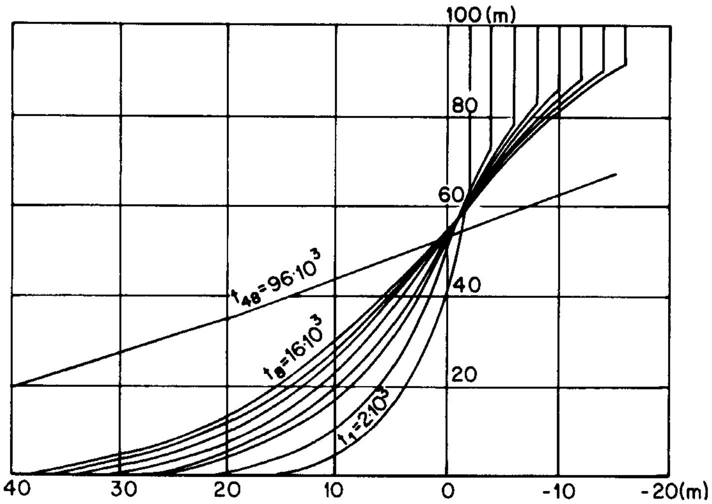

Figure 2. Series of curves for solution (37) (for selected values of time) (parameters in the text)

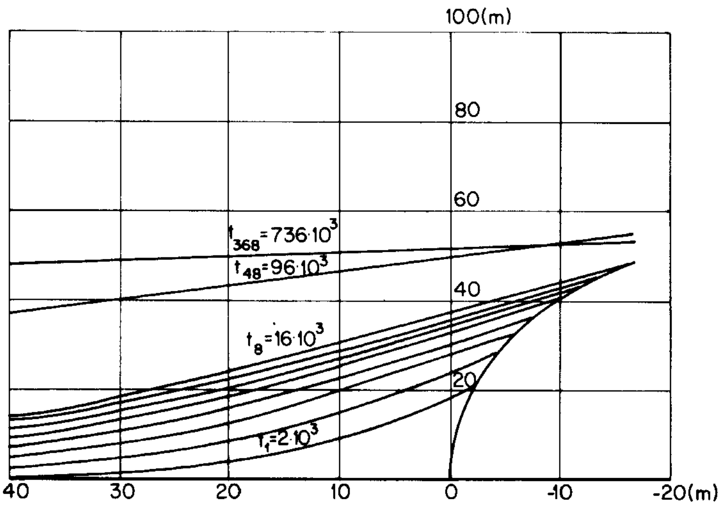

Figure 3. Series of curves for solution (37) (for selected values of time) (parameters in the text)

Note, numerical simulation by thls model may be used when mountain territory is brought under cultivation and to forecast when an advancing accumulation slope will reach roadways or other installations.

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APPENDIX 1

TO DEDUCE SOLUTION (6)

Putting $z = \sqrt{(lx)}$ in equation (4) transforms it into

$$\left(\frac{4}{k_0 l}\right)\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial z^2} + \frac{1}{z}\frac{\partial y}{\partial z} \tag{38}$$

Further reasoning is analogous to Culling's work (1963) in which landform evolution with radial symmetry was analysed. We give only initial stage of this reasoning.

The change $y = \exp(-k_0 \mu^2 t/4l)\, u$ converts equation (38) to Bessel's equation of order zero

$$\frac{d^2 u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \frac{\mu^2}{l^2} u = 0 \tag{39}$$

which has the solution

$$u = AJ_0(\mu\sqrt{(x/l)}) + BY_0(\mu\sqrt{(x/l)}) \tag{40}$$

In accordance with the condition of the problem it is necessary to bound the function in the interval $0 < x < l$, i.e. $B = 0$ and equation (40) may be simplified as follows $u = AJ_0(\mu\sqrt{(x/l)})$.

To satisfy the boundary condition $y(l, t) = 0$, μ must be one of the roots of the equation $J_0(z) = 0$ and so on (see Culling, 1963, p. 156–157).

APPENDIX 2

TO DEDUCE EXPRESSIONS (9) AND (10)

When we obtained expression (9) with the help of equation (6), the integral

$$\int_0^l J_0(\mu_n\sqrt{(x/l)})\,dx$$

was obtained by substituting $v = \mu_n\sqrt{(x/l)}$ and using tables of integrals (Dwight, 1961)

$$\int_0^{\mu_n\sqrt{(l_0/l)}} J_0(v)\,v\,dv = \mu_n\sqrt{(l_0/l)}\cdot J_1(\mu_n\sqrt{(l_0/l)})$$

The integral of expression (9) when $f(x) = h(1-[x/l])$ is transformed as follows:

$$A = \int_0^1 z f(lz^2) J_0(\mu_n z)\,dz = h\int_0^1 zJ_0(\mu_n z)\,dz - h\int_0^1 z^3 J_0(\mu_n z)\,dz \tag{41}$$

where the first integral has been analysed above and equals $(h/\mu_n) J_1(\mu_n)$ and the second one has been found as an indefinite integral with the help of integration by parts and relations for Bessel's function

$$\int v^3 J_0(v)\,dv = 2v^2 J_0(v) + (v^3 - 4v) J_1(v) \tag{42}$$

Using equation (42) with the help of $J_0(\mu_n) = 0$ the expression (41) is transformed in the following way

$$A = (h/\mu_n) J_1(\mu_n) - (h/\mu_n^4)\int_0^{\mu_n} v^3 J_0(v)\,dv = (h/\mu_n) J_1(\mu_n) - (h/\mu_n^4)[2\mu_n^2 J_0(\mu_n) + (\mu_n^3 - 4\mu_n) J_1(\mu_n)]$$
$$= (4h/\mu_n^3) J_1(\mu_n) \tag{43}$$

Putting it into equation (9), we obtain

$$W(\tau) = hl_0\left(1-\frac{l_0}{2l}\right) - 16\sqrt{(l_0 l)}\,h \sum_{n=1}^{\infty} \frac{J_1(\mu_n\sqrt{l_0/l})\exp(-k_0\mu_n^2\tau/4l)}{J_1(\mu_n)\mu_n^4} \tag{44}$$

When $l = l_0$ the expression (43) transforms into (10).

APPENDIX 3

SOLUTION OF THE EQUATION (17) BY FOURIER'S METHOD

Using Fourier's method for the function $y(x, t)$ in equation (17) we obtain: $y(x, t) = X(x)T(t)$. Putting this in equation (17) and differentiating we have

$$X\frac{dT}{dt} = (2ax+b)\frac{dX}{dx}T + (ax^2+bx+c)\frac{d^2X}{dx^2}T \tag{45}$$

For separation of the variables divide both sides of (45) by XT and equate them with the indefinite separation parameter m

$$\frac{dT}{dt}\frac{1}{T} = \frac{(2ax+b)}{X}\frac{dX}{dx} + \frac{(ax^2+bx+c)}{X}\frac{d^2X}{dx^2} = -m \tag{46}$$

Thus we have obtained two independent ordinary differential equations

$$\begin{aligned} &\frac{dT}{dt} + mT = 0 \\ &(ax^2+bx+c)\frac{d^2X}{dx^2} + (2ax+b)\frac{dX}{dx} + mX = 0 \end{aligned} \tag{47}$$

Solution of the second equation of the system (47).

By the change of variables $X = \eta(\xi)$, $x = x_1(x_2 - x_1)\xi$ reduce it to the hypergeometric equation

$$\xi(\xi-1)\frac{d^2\eta}{d\xi^2} + \left[2\xi + \frac{2ax_1+b}{a(x_2-x_1)}\right]\frac{d\eta}{d\xi} + \frac{m}{a}\xi = 0 \tag{48}$$

where x_1 and x_2 are the roots of the quadratic equation $ax^2+bx+c = 0$, $a < 0$.

Using the formula for the roots of a quadratic equation obtain the second term (in square brackets)

$$\frac{2ax_1+b}{a(x_2-x_1)} = \left[2a\left(\frac{-b-\sqrt{(b^2-4ac)}}{2a}\right)+b\right]\Big/ a\left(\frac{2\sqrt{(b^2-4ac)}}{2a}\right) = -1$$

Then equation (48) can be rewritten as follows:

$$\xi(\xi-1)\frac{d^2\eta}{d\xi^2} + (2\xi-1)\frac{d\eta}{d\xi} + \frac{m}{a}\eta = 0 \tag{49}$$

Using $\xi = (1-z)/2$ reduce equation (49) to Legendre's equation, noting that

$$\frac{d\eta}{d\xi} = -2\frac{d\eta}{dz},\ \frac{d^2\eta}{d\xi^2} = 4\frac{d^2\eta}{dz^2}$$

$$\frac{d}{dz}\left[(1-z)^2\frac{d\eta}{dz}\right] + \lambda\eta = 0 \tag{50}$$

where $\lambda = -m/a$, $-1 < z < 1$.

The Legendre's polynomials $P_n(z)$ are fundamental functions that correspond to the eigenvalues $\lambda_n = (n+1)n$ and they are the solutions of the following problem (Tikhonov and Samarsky, 1966, p. 665): 'Find λ for which there are the non-trivial solutions on the segment $[-1, 1]$ of Legendre's equation (50), bounded at $z = \pm 1$ ($\eta(\pm 1) < \infty$) and satisfying the condition of normalization $\eta(1) = 1$'. It is the solution of the eigenvalue problem or the Sturm–Liouville problem. No concrete boundary conditions are needed except the boundedness of the function at the limits.

We give the first few Legendre's polynomials: $P_0(z) = 1, P_1(z) = z, P_2(z) = (1/2)(3z^2 - 1), P_3(z) = (1/2)(5z^2 - 3z), \ldots$ Of these $P_0(z)$ and $P_1(z)$ have been used in obtaining the regular and limited solutions (19).

Solution of the first equation of the system (47)

Using the separation parameter $\lambda_n = (n+1)n = -m/a$ we obtain

$$T_n(t) = C_n \exp(a(n+1)nt) \tag{51}$$

where C_n are integration constants corresponding to the eigenvalue $\lambda_n = (n+1)n$.

General solution

The general solution of equation (17) by Fourier's method is given as the series

$$y(z, t) = \sum_{n=0}^{\infty} C_n P_n(z) \exp(a(n+1)nt) \tag{52}$$

where z is connected with the previous variable x in this way: $x = x_1 + (x_2 - x_1)[(1-z)/2]$.

The constants C_n may be found from the condition of orthogonality of Legendre's polynomials

$$\int_{-1}^{+1} P_m(z) P_n(z) \, dz = \begin{cases} 0; & m \neq n \\ \dfrac{2}{2n+1}; & m = n \end{cases} \tag{53}$$

The general solution (52) when $t = 0$ is

$$y(z, 0) = f(z) = \sum_{n=0}^{\infty} C_n P_n(z) \tag{54}$$

When both parts of equality (54) are multiplied by $P_m(z)$ and the expressions obtained are integrated over the interval from -1 to $+1$ we find

$$\int_{-1}^{+1} f(z) P_m(z) \, dz = \sum_{n=0}^{\infty} C_n \int_{-1}^{+1} P_n(z) P_m(z) \, dz \tag{55}$$

from which using the orthogonality of Legendre's polynomials, we obtain

$$C_n = \left(n + \frac{1}{2}\right) \int_{-1}^{+1} f(z) P_n(z) \, dz \tag{56}$$

The expressions (52 and 56) give the final solution of the problem.

APPENDIX 4

THE SOLUTION OF BOUNDARY VALUE PROBLEM (28)

With the help of change of variables $\xi = x - bt, t = t$, the variable boundary value problem (28) is transformed to a stable boundary value problem

$$\frac{\partial y}{\partial t} = k \frac{\partial^2 y}{\partial \xi^2} + b \frac{\partial y}{\partial \xi}; \; y(0, t) = 0; \; y(\xi, 0) = f(\xi); \; 0 < \xi < +\infty \tag{57}$$

Let us illustrate Fourier's method for problem solution on a semi-infinite interval considering problem (57) as an example.

Separating variables $y(\xi, t) = X(\xi)T(t)$ in (57) we obtain the system of ordinary differential equations

$$\begin{aligned} &\frac{dT(t)}{dt} + \mu^2 kT(t) = 0 \\ &\frac{d^2 X(\xi)}{d\xi^2} + \frac{b}{k}\frac{dX(\xi)}{d\xi} + \mu^2 X(\xi) = 0 \end{aligned} \tag{58}$$

where μ is separation parameter.

The solution of second equation of this system is

$$X(\xi) = \exp\left(-\frac{b}{2k}\xi\right)[A(\lambda)\cos\lambda\xi + B(\lambda)\sin\lambda\xi] \tag{59}$$

The solution of first equation of the system is

$$T(t) = \exp[-k(\lambda^2 + b^2/4k^2)] \tag{60}$$

where $\mu^2 = \lambda^2 + b^2/4k^2$; $A(\lambda)$, $B(\lambda)$ are some functions of the parameter λ.

By multiplying (59) and (60) we obtain a partial solution. The general solution is as follows

$$y(\xi, t) = \int_0^{+\infty} \exp\left(-\frac{b}{2k}\xi\right)\exp\left[-kt\left(\lambda^2 + \frac{b^2}{4k^2}\right)\right][A(\lambda)\cos\lambda\xi + B(\lambda)\sin\lambda\xi]\,d\lambda \tag{61}$$

When $t = 0$ the solution (61) is transformed in the following way

$$y(\xi, 0) = f(\xi) = \int_0^{+\infty} \exp\left(-\frac{b}{2k}\xi\right)[A(\lambda)\cos\lambda\xi + B(\lambda)\sin\lambda\xi]\,d\lambda \tag{62}$$

To make it possible to apply Fourier's formula we consider the function $f(\xi)$ to be defined on an infinite interval. Then from (62) we obtain

$$\begin{aligned} A(\lambda) &= \frac{1}{\pi}\int_{-\infty}^{+\infty} f(z)\exp\left(\frac{b}{2k}z\right)\cos\lambda z\,dz \\ B(\lambda) &= \frac{1}{\pi}\int_{-\infty}^{+\infty} f(z)\exp\left(\frac{b}{2k}z\right)\sin\lambda z\,dz \end{aligned} \tag{63}$$

Putting $A(\lambda)$ and $B(\lambda)$ from (63) in the general solution (61) and assuming that $f(\xi)$ is an absolutely integrable function in the interval $(-\infty, +\infty)$, interchanging integrations on λ and on z, and calculating the internal integral (Fihtengoltz, 1960)

$$\int_0^{\infty} \exp(-k\lambda^2 t)\cos\lambda(z-\xi)\,d\lambda = \frac{1}{2}\sqrt{(\pi/kt)}\exp\left[-\frac{(z-\xi)^2}{4kt}\right]$$

we obtain

$$y(\xi, t) = \frac{\exp(-b^2 t/4k)}{2\sqrt{(\pi kt)}}\int_{-\infty}^{+\infty} f(z)\exp\left[\frac{b}{2k}(z-\xi)\right]\exp\left[-\frac{(z-\xi)^2}{4kt}\right]dz \tag{64}$$

In order to come to problem (57) for the semi-infinite interval with zero boundary condition prolongate the function $f(\xi)\exp([b/2k]\xi)$ by odd form and then come to solution (29), respectively.

APPENDIX 5

STEADY-STATE SOLUTION (30) AS ASYMPTOTIC SOLUTION OF PROBLEM (28)

In connection with the condition of the problem the slope's base retreats with constant velocity b. In this case a steady-state solution of the problem (28) may be found as follows

$$y = \phi(z),\ z = x - bt \tag{65}$$

Putting (65) into equation (28) and performing the differentiations we obtain a second-order ordinary differential equation

$$\frac{d^2\phi}{dz^2} + \frac{b}{k}\frac{d\phi}{dz} = 0 \tag{66}$$

With the help of a change of variables $d\phi/dz = w(z)$ and taking into account both the main boundary condition $\phi(0) = 0$ and supplementary ones

$$\lim_{x\to\infty} y(x, t) = \lim_{z\to\infty} \phi(z) = h$$

(h is the limiting horizontal surface of the slope) the solution of (66) is obtained as (30). For initial slope profile stahility analysis (31) put (30) into (29) and calculate the integrals to obtain the solution (Moskovkin and Trofimov, 1980)

$$y(x,t) = \frac{h}{2}\left\{1 + \Phi\left(\frac{x}{2\sqrt{(kt)}}\right) - \exp\left(-\frac{b}{k}(x-bt)\right)\left[1 - \Phi\left(\frac{x-2bt}{2\sqrt{(kt)}}\right)\right] + \exp\left[\frac{(ck-b)}{k}((ck-b)t+x)\right]\right.$$
$$\left.\times\left[1 - \Phi\left(\frac{2t(ck-b)+x}{2\sqrt{(kt)}}\right)\right] - \exp(c(ckt-x))\left[1 - \Phi\left(\frac{2ckt-x}{2\sqrt{(kt)}}\right)\right]\right\}, \tag{67}$$

where

$$\Phi(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(\eta^2)\,d\eta$$

is the error function.

When $c = b/k$ solution (67) transforms into the steady-state solution (30). The analysis (67) shows that the value $b = ck$ is critical and defines two different regimes of slope development (see the section *Steady-state regime of undercut slopes*).

When $c \neq b\,k$ solution (67) asymptotically (when t is large) transforms into the steady-state sohition (30). It follows from the state that the first and last two error functions in solution (67) when $x > bt$ tend to zero and

$$\lim_{t\to\infty} \Phi\left(\frac{x-2bt}{2\sqrt{kt}}\right) = \begin{cases} \Phi(\infty) = 1,\ x > 2bt \\ \Phi(-\infty) = -1,\ bt < x < 2bt \end{cases}$$

From this when $bt < x < 2bt$ we have solution (30) and when $x > 2bt$ we have

$$\lim_{t\to\infty} y(x, t) = h$$

Note, that Culling (1963, p. 153) has shown that the steady-state solution (30) was an asymptotic solution to the problem for an initial vertical slope (scarp).

As we see from the section *Steady-state regime of undercut slopes*, the asymptotic solution of problem (28) may be found from the general solution (29) when $f(z) = h$. Putting it into equation (29) we obtain after integration

$$\frac{h}{2}\left\{\Phi\left(\frac{x}{2\sqrt{(kt)}}\right) - \exp\left(-\frac{b}{k}(x-bt)\right)\left[1 - \Phi\left(\frac{x-2bt}{2\sqrt{(kt)}}\right)\right]\right\}$$

From this we obtain the same results as when analysing asymptotic behaviour of the solution (67).

APPENDIX 6

THE SOLUTION OF STEADY-STATE PROBLEM (33)

Find a solution similar to (65) (see Appendix 5). Putting (65) into equation (32) and differentiating, we obtain a second-order ordinary differential equation:

$$\frac{d^2\phi}{dz^2}+\frac{b}{k}\frac{d\phi}{dz}-\frac{V}{k}=0 \tag{68}$$

With the help of a change of variables $d\phi/dz = w(z)$, $d^2\phi/dz^2 = dw/dz$ reduce (68) to the first-order ordinary differential equation:

$$\frac{dw}{dz}+\frac{b}{k}w-\frac{V}{k}=0 \tag{69}$$

A solution may then be found with the help of a constant variation method

$$w(z) = C_1 \exp\left(-\frac{b}{k}z\right)+\frac{V}{b} \tag{70}$$

where C_1 is a constant of integration.

Using the boundary condition

$$\left.\frac{dy}{dx}\right|_{x=bt} = \left.\frac{d\phi}{dz}\right|_{z=0} = w(0) = A$$

from equation (70) find the constant of integration: $C_1 = A - V/b$. Now find $\phi(z)$ by integration of (70) on z

$$\phi(z) = -\frac{k}{b}\left(A-\frac{V}{b}\right)\exp\left(-\frac{b}{k}z\right)+\frac{V}{b}z+C_2 \tag{71}$$

The second constant of integration is found with the help of the second boundary condition $y(bt, t) = \phi(0) = 0$ from the solution (71): $C_2 = (k/b)\,(A - V/b)$. Putting this constant in solution (71) and returning to the previous variables we have the desired solution (30).

APPENDIX 7

TO DEDUCE SOLUTION (37)

With the help of a change of variables $\xi = x + vt$, $t = t$ we reduce (30) to the stable boundary value problem

$$\begin{gathered}\frac{\partial y}{\partial t} = k\frac{\partial^2 y}{\partial \xi^2}-v\frac{\partial y}{\partial \xi}; \quad 0<\xi<+\infty \\ y(\xi, 0) = 0, \quad -k\left.\frac{\partial y}{\partial \xi}\right|_{\xi=0} = k_p v(H - y|_{\xi=0})\end{gathered} \tag{72}$$

Using the Laplace transform

$$\bar{y} = \int_0^{+\infty} e^{-pt}\, y\, dt$$

we obtain the second-order ordinary differential equation

$$\frac{d^2\bar{y}}{d\xi^2}-\frac{v}{k}\frac{d\bar{y}}{d\xi}-\frac{p}{k}\bar{y}=0 \tag{73}$$

the solution of which is

$$\bar{y} = C_1 \exp\left[\frac{v}{2k}\xi - \left(\frac{v^2}{4k^2} + \frac{p}{k}\right)^{1/2}\xi\right] \tag{74}$$

where C_1 is an integration constant.

Note, that the second linearly independent solution of equation (73) was rejected because of its unboundedness when $\xi \to \infty$, i.e. the second integration constant is taken as equal to zero ($c_2 = 0$).

The boundary condition, with the help of the Laplace transform, reduces to

$$\left(\frac{d\bar{y}}{d\xi} - \frac{k_p.v}{k}\bar{y}\right)\bigg|_{\xi=0} = -\frac{k_p.vH}{kp}, \tag{75}$$

Putting (74) when $\xi = 0$ into equation (75) we obtain the integration constant C_1 as follows

$$C_1 = \left(-\frac{k_p.vH}{kp}\right)\bigg/\left[\frac{v}{2k} - \left(\frac{v^2}{4k^2} + \frac{p}{k}\right)^{1/2} - \frac{k_p.v}{k}\right] \tag{76}$$

Putting expression C_1 into equation (74) we obtain the final expression for $\bar{y}(\xi, p)$. Using the inverse Laplace transform obtain solution (37) for function $y(\xi, t)$ or $y(x, t)$ (see also Carslaw and Jaeger, 1959).

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