Darcy's law for a compressible thermofluid

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Abstract. A linear system of differential equations describing a joint motion of a thermoelastic porous body with a sufficiently large Lamé's constants (absolutelty rigid body) and a thermofluid, occupying porous space, is considered. The rigorous justification, under various conditions imposed on physical parameters, is fulfilled for homogenization procedures as the dimensionless size of the pores tends to zero, while the porous body is geometrically periodic. As the results we derive Darcy's system of filtration or acoustic equations for thermofluid, depending on ratios between physical parameters. The proofs are based on Nguetseng's two-scale convergence method of homogenization in periodic structures.

Keywords: anisothermic Stokes and Lamé's equations, two-scale convergence, homogenization of periodic structures

0. Introduction

In the present publication we consider a problem of a joint motion of thermoelastic deformable solid (thermoelastic skeleton), perforated by a system of channels (pores) and slightly compressible anisothermic fluid (thermofluid) occupying a porous space. In dimensionless variables (without primes)

$$\mathbf{x}' = L\mathbf{x}, \qquad t' = \tau t, \qquad \mathbf{w}' = L\mathbf{w}, \qquad \theta' = \vartheta_* \frac{L}{\tau v_*}\theta, \qquad \rho_s' = \rho_0 \rho_s, \qquad \rho_f' = \rho_0 \rho_f,$$

the differential equations of the problem in a domain $\Omega \in \mathbf{R}^3$ for the dimensionless displacement vector $\mathbf{w}$ of the continuum medium and the dimensionless temperature θ, have a form:

$$\alpha_\tau \bar{\rho} \frac{\partial^2 \mathbf{w}}{\partial t^2} = \operatorname{div}_x \mathbb{P}, \tag{0.1}$$

$$\alpha_\tau \bar{c}_p \frac{\partial \theta}{\partial t} = \operatorname{div}_x(\bar{\alpha}_\varkappa \nabla_x \theta) - \bar{\alpha}_\theta \frac{\partial}{\partial t} \operatorname{div}_x \mathbf{w}, \tag{0.2}$$

$$\mathbb{P} = \bar{\chi}\mathbb{P}^f + (1-\bar{\chi})\mathbb{P}^s, \tag{0.3}$$

$$\mathbb{P}^f = \alpha_\mu \mathbb{D}\Big(\mathbf{x}, \frac{\partial \mathbf{w}}{\partial t}\Big) - \Big(p - \alpha_\nu \operatorname{div}_x \frac{\partial \mathbf{w}}{\partial t} + \alpha_{\theta f}\theta\Big)\mathbb{I}, \tag{0.4}$$

$$\mathbb{P}^s = \alpha_\lambda \mathbb{D}(x, \mathbf{w}) - \alpha_{\theta s}\theta \mathbb{I}, \tag{0.5}$$

$$p + \bar{\chi}\alpha_p \operatorname{div}_x \mathbf{w} = 0. \tag{0.6}$$

The problem is endowed with initial and boundary conditions

$$\mathbf{w}|_{t=0} = 0, \qquad \frac{\partial \mathbf{w}}{\partial t}\Big|_{t=0} = 0, \qquad \theta|_{t=0} = 0, \qquad \mathbf{x} \in \Omega, \tag{0.7}$$

$$\mathbf{w} = \mathbf{w}_0, \qquad \theta = \theta_0, \qquad \mathbf{x} \in S = \partial\Omega, \qquad t \geqslant 0. \tag{0.8}$$

Here and further we use notations

$$\mathbb{D}(x, \mathbf{u}) = \frac{1}{2}(\nabla_x \mathbf{u} + (\nabla_x \mathbf{u})^{\mathrm{T}}),$$
$$\bar{\rho} = \bar{\chi}\rho_f + (1 - \bar{\chi})\rho_s, \qquad \bar{c}_p = \bar{\chi}c_{pf} + (1 - \bar{\chi})c_{ps},$$
$$\bar{\alpha_{\varkappa}} = \bar{\chi}\alpha_{\varkappa f} + (1 - \bar{\chi})\alpha_{\varkappa s}, \qquad \bar{\alpha}_\theta = \bar{\chi}\alpha_{\theta f} + (1 - \bar{\chi})\alpha_{\theta s},$$

$\mathbb{P}^f$ is a liquid stress tensor, $\mathbb{P}^s$ is a stress tensor in a solid skeleton and p is a liquid pressure.

In this model the characteristic function of the porous space $\bar{\chi}(\mathbf{x})$ is a known function.

For more details about Eqs (0.1)–(0.6) and description of dimensionless constants (all these constants are positive) see [1].

We accept the following constraints

Assumption 1. Domain $\Omega = (0,1)^3$ is a periodic repetition of an elementary cell $Y^\varepsilon = \varepsilon Y$, where $Y = (0,1)^3$ and quantity $1/\varepsilon$ is integer, so that Ω always contains an integer number of elementary cells Y_i^ε. Let Y_s be a "solid part" of Y, and the "liquid part" Y_f – is its open complement. We denote as $\gamma = \partial Y_f \cap \partial Y_s$ and γ is C^1-surface. A pore space Ω_f^ε is the periodic repetition of the elementary cell εY_f, and solid skeleton Ω_s^ε is the periodic repetition of the elementary cell εY_s. A C^1-boundary $\Gamma^\varepsilon = \partial\Omega_s^\varepsilon \cap \partial\Omega_f^\varepsilon$ is the periodic repetition in Ω of the boundary $\varepsilon\gamma$. The "solid skeleton" Ω_s^ε is a connected domain.

In these assumptions

$$\bar{\chi}(\mathbf{x}) = \chi^\varepsilon(\mathbf{x}) = \chi(\mathbf{x}/\varepsilon),$$
$$\bar{c}_p = c_p^\varepsilon(\mathbf{x}) = \chi^\varepsilon(\mathbf{x})c_{pf} + \big(1 - \chi^\varepsilon(\mathbf{x})\big)c_{ps},$$
$$\bar{\rho} = \rho^\varepsilon(\mathbf{x}) = \chi^\varepsilon(\mathbf{x})\rho_f + \big(1 - \chi^\varepsilon(\mathbf{x})\big)\rho_s,$$
$$\bar{\alpha}_\varkappa = \alpha_\varkappa^\varepsilon(\mathbf{x}) = \chi^\varepsilon(\mathbf{x})\alpha_{\varkappa f} + \big(1 - \chi^\varepsilon(\mathbf{x})\big)\alpha_{\varkappa s},$$
$$\bar{\alpha}_\theta = \alpha_\theta^\varepsilon(\mathbf{x}) = \chi^\varepsilon(\mathbf{x})\alpha_{\theta f} + \big(1 - \chi^\varepsilon(\mathbf{x})\big)\alpha_{\theta s}.$$

Let ε be a characteristic size of pores l divided by the characteristic size L of the entire pore body:

$$\varepsilon = \frac{l}{L}.$$

Suppose that all dimensionless parameters depend on the small parameter ε and there exist limits (finite or infinite)

$$\lim_{\varepsilon\searrow 0} \alpha_\mu(\varepsilon) = \mu_0, \qquad \lim_{\varepsilon\searrow 0} \alpha_\lambda(\varepsilon) = \lambda_0, \qquad \lim_{\varepsilon\searrow 0} \alpha_\tau(\varepsilon) = \tau_0,$$
$$\lim_{\varepsilon\searrow 0} \alpha_p(\varepsilon) = p_*, \qquad \lim_{\varepsilon\searrow 0} \alpha_\nu(\varepsilon) = \nu_0, \qquad \lim_{\varepsilon\searrow 0} \alpha_{\varkappa f} = \varkappa_{0f},$$
$$\lim_{\varepsilon\searrow 0} \alpha_{\varkappa s}(\varepsilon) = \varkappa_{0s}, \qquad \lim_{\varepsilon\searrow 0} \alpha_{\theta f}(\varepsilon) = \beta_{0f}, \qquad \lim_{\varepsilon\searrow 0} \alpha_{\theta s}(\varepsilon) = \beta_{0s}, \qquad \lim_{\varepsilon\searrow 0} \frac{\alpha_\mu}{\varepsilon^2} = \mu_1.$$

Different isothermal models have been considered in [4–8]. The first research with the aim of finding limiting regimes in the case when the skeleton was assumed to be an absolutely rigid isothermal body was carried out by Sanchez-Palencia and Tartar. Sanchez-Palencia [5, Section 7.2] formally obtained Darcy's law of filtration using the method of two-scale asymptotic expansions, and Tartar [5, Appendix] mathematically rigorously justified the homogenization procedure. Using the same method of two-scale expansions Keller and Burridge [4] derived formally the system of Biot's equations in the case when the parameter α_μ was of order ε^2, and the rest of the coefficients were fixed independent of ε. Under the same assumptions as in the article [4], the rigorous justification of Biot's model has been given by Nguetseng [6] and later by Clopeau et al. in [7]. The most general case has been studied in [8].

In the present work by means of the Nguetseng's method we investigate all possible homogenized systems for the anisothermic problem (0.1)–(0.8), when

$$\mu_0 = 0; \qquad \lambda_0 = \infty; \qquad \beta_{0s}, \beta_{0f}, \nu_0, \tau_0 < \infty,$$
$$0 < \mu_1 + \tau_0, p_*, \varkappa_{0f}, \varkappa_{0s} < \infty$$

(the condition $\lambda_0 = \infty$ means that the solid skeleton is an absolutely rigid body).

We show that limiting regimes as ε tends to zero are described by Darcy's system of equations of filtration or acoustic equations for the velocity of the liquid component, coupled with corresponding heat equation. Here we do not discuss properties of these homogenized systems, including the uniqueness of the solutions, because it is a special and very serious topic. To do that we must know all properties of the coefficients of homogenized systems. These systems have no a certain type (only for the case $\tau_0 = 0$ the system might be a parabolic one). Of course, there is a "degenerate" case, when $\beta_{0f} = 0$ and each system becomes decoupled and obviously has a unique solution. It is clear that for sufficiently small β_{0f} the systems conserve the uniqueness of solutions, which is in a some sense useful, because shows that they are complete, i.e. the number of boundary and initial conditions correspond to the number of equations.

1. Formulation of the main results

As usual, Eqs (0.1)–(0.2) are understood in the sense of distributions. They involve Eqs (0.1)–(0.5) in a usual sense in the domains Ω_f^ε and Ω_s^ε and the boundary conditions

$$[\vartheta] = 0, \qquad [\mathbf{w}] = 0, \qquad \mathbf{x}_0 \in \Gamma^\varepsilon, \quad t \geqslant 0, \tag{1.1}$$

$$[\mathbb{P} \cdot \mathbf{n}] = 0, \qquad [\alpha_\varkappa^\varepsilon \nabla_x \theta \cdot \mathbf{n}] = 0, \qquad \mathbf{x}_0 \in \Gamma^\varepsilon, \quad t \geqslant 0 \tag{1.2}$$

on the boundary Γ^ε, where $\mathbf{n}$ is a unit normal to the boundary and

$$[\varphi](\mathbf{x}_0) = \varphi_{(s)}(\mathbf{x}_0) - \varphi_{(f)}(\mathbf{x}_0),$$
$$\varphi_{(s)}(\mathbf{x}_0) = \lim_{\substack{\mathbf{x}\to\mathbf{x}_0\\ \mathbf{x}\in\Omega_s^\varepsilon}} \varphi(\mathbf{x}), \qquad \varphi_{(f)}(\mathbf{x}_0) = \lim_{\substack{\mathbf{x}\to\mathbf{x}_0\\ \mathbf{x}\in\Omega_f^\varepsilon}} \varphi(\mathbf{x}).$$

There are various equivalent in the sense of distributions forms of representation of Eqs (0.1)–(0.2) and boundary conditions (1.1)–(1.2). In what follows, it is convenient to write them in the form of the integral equalities.

Definition 1. Functions $(\mathbf{w}^\varepsilon, \theta^\varepsilon, p^\varepsilon, q^\varepsilon)$ are called a generalized solution of the problem (0.1)–(0.8), if they satisfy the regularity conditions

$$\mathbf{w}^\varepsilon, \mathbb{D}(x, \mathbf{w}^\varepsilon), \operatorname{div}_x \mathbf{w}^\varepsilon, q^\varepsilon, p^\varepsilon, \frac{\partial p^\varepsilon}{\partial t}, \theta^\varepsilon, \nabla_x \theta^\varepsilon \in L^2(\Omega_T) \tag{1.3}$$

in the domain $\Omega_T = \Omega \times (0, T)$, boundary conditions (0.8) with functions

$$\mathbf{w}_0^\varepsilon, \theta_0^\varepsilon \in L^2((0,T); W_2^1(\Omega)),$$

equations

$$q^\varepsilon = p^\varepsilon + \frac{\alpha_\nu}{\alpha_p}\frac{\partial p^\varepsilon}{\partial t} + \alpha_{\theta f}\chi^\varepsilon \theta^\varepsilon, \tag{1.4}$$

$$p^\varepsilon + \chi^\varepsilon \alpha_p \operatorname{div}_x \mathbf{w}^\varepsilon = 0 \tag{1.5}$$

a.e. in Ω_T, integral identity

$$\int_{\Omega_T} \Big(\alpha_\tau \rho^\varepsilon \mathbf{w}^\varepsilon \cdot \frac{\partial^2 \boldsymbol{\varphi}}{\partial t^2} - \chi^\varepsilon \alpha_\mu \mathbb{D}(\mathbf{x}, \mathbf{w}^\varepsilon) : \mathbb{D}\Big(x, \frac{\partial \boldsymbol{\varphi}}{\partial t}\Big) \tag{1.6}$$
$$+ \big\{(1-\chi^\varepsilon)\alpha_\lambda \mathbb{D}(x, \mathbf{w}^\varepsilon) - \big(q^\varepsilon + \alpha_{\theta s}(1-\chi^\varepsilon)\theta^\varepsilon\big)\mathbb{I}\big\} : \mathbb{D}(x, \boldsymbol{\varphi})\Big)\, \mathrm{d}x\, \mathrm{d}t = 0$$

for all smooth vector-functions $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\mathbf{x}, t)$ such that $\boldsymbol{\varphi}|_{\partial\Omega} = \boldsymbol{\varphi}|_{t=T} = \partial\boldsymbol{\varphi}/\partial t|_{t=T} = 0$ and integral identity

$$\int_{\Omega_T} \Big((\alpha_\tau c_p^\varepsilon \theta^\varepsilon + \alpha_\theta^\varepsilon \operatorname{div}_x \mathbf{w}^\varepsilon) \frac{\partial \xi}{\partial t} - \alpha_\varkappa^\varepsilon \nabla_x \theta^\varepsilon \cdot \nabla_x \xi \Big)\, \mathrm{d}x\, \mathrm{d}t = 0 \tag{1.7}$$

for all smooth functions $\xi = \xi(\mathbf{x}, t)$ such that $\xi|_{\partial\Omega} = \xi|_{t=T} = 0$.

In this definition we changed the form of representation of the stress tensor $\mathbb{P}$ in the integral identity (1.6) by introducing a new unknown function q^ε, which in a certain way have a sense of pressure. We have done it because it simplifies the use of homogenization procedure. In what follows we call equation (1.4) as a state equation and equation (1.5) as a continuity equation.

In (1.6) by $\mathbb{A} : \mathbb{B}$ we denote the convolution (or, equivalently, the inner tensor product) of two second-rank tensors along the both indexes, i.e., $\mathbb{A} : \mathbb{B} = \operatorname{tr}(\mathbb{B}^* \circ \mathbb{A}) = \sum_{i,j=1}^3 A_{ij} B_{ji}$.

We suppose the next assumption to be held:

Assumption 2. Sequences $\{\sqrt{\alpha_\mu}\chi^\varepsilon |\nabla \partial \mathbf{w}_0^\varepsilon/\partial t|\}$, $\{\sqrt{\alpha_\lambda}(1-\chi^\varepsilon)\nabla\partial\mathbf{w}_0^\varepsilon/\partial t\}$, $\{\sqrt{\alpha_p}\chi^\varepsilon \operatorname{div}_x \partial\mathbf{w}_0^\varepsilon/\partial t\}$, $\{\sqrt{\alpha_\nu}\chi^\varepsilon \operatorname{div}_x \partial\mathbf{w}_0^\varepsilon/\partial t\}$, $\{\sqrt{\alpha_\theta^\varepsilon} \operatorname{div} \partial\mathbf{w}_0^\varepsilon/\partial t\}$, $\{\partial^2\mathbf{w}_0^\varepsilon/\partial t^2\}$, $\{\partial\theta_0^\varepsilon/\partial t\}$, $\{\nabla\theta_0^\varepsilon\}$ are uniformly in ε bounded in $L^2(\Omega)$ and $\mathbf{w}_0^\varepsilon = 0$ on $S_s^\varepsilon = \partial\Omega_s^\varepsilon \cap \partial\Omega$.

It is clear that we need all these restrictions just to bound solutions of the problem (0.1)–(0.8) uniformly with respect to parameter ε. They are sufficiently cumbersome and at the first glance it is very hard to taste them. Nevertheless the set of functions $\{\mathbf{w}_0^\varepsilon, \theta_0^\varepsilon\}$ is wide enough. For example, we may consider $\theta_0^\varepsilon = \theta_0(\mathbf{x}, t) \in L^2((0,T); W_2^1(\Omega))$ and for the case $\alpha_\mu = \mu_1\varepsilon^2$ we put

$$\frac{\partial \mathbf{w}_0^\varepsilon}{\partial t}(\mathbf{x}, t) = \varphi(\mathbf{x}, t)\mathbf{v}_0\left(\frac{\mathbf{x}}{\varepsilon}\right),$$

where $\varphi \in C^\infty$ and periodic in variable $\mathbf{y}$ function $\mathbf{v}_0(\mathbf{y})$ satisfies the following conditions:

$$\mathbf{v}_0 \in W_2^1(Y_f), \qquad \operatorname{div}\mathbf{v}_0 = 0, \quad \mathbf{y} \in Y_f, \qquad \mathbf{v}_0 = 0, \quad \mathbf{y} \in Y_s, \qquad \mathbf{v}_0 \neq 0, \quad \mathbf{y} \in \partial Y_f.$$

Such a function $\mathbf{v}_0$ always exists (see [9]) and we may choose it from the condition

$$\langle \mathbf{v}_0 \rangle_Y = \int_Y \mathbf{v}_0 \,\mathrm{d}y \neq 0.$$

Obviously functions $\mathbf{w}_0^\varepsilon$ satisfy all conditions of the Assumption 2 because derivatives

$$\sqrt{\alpha_\mu}\chi^\varepsilon \nabla \frac{\partial \mathbf{w}_0^\varepsilon}{\partial t} = \sqrt{\mu_1}\left(\varepsilon \nabla\varphi(\mathbf{x}, t) \otimes \mathbf{v}_0\left(\frac{\mathbf{x}}{\varepsilon}\right) + \varphi(\mathbf{x}, t)\nabla_y \mathbf{v}_0\left(\frac{\mathbf{x}}{\varepsilon}\right)\right),$$
$$\operatorname{div}\frac{\partial \mathbf{w}_0^\varepsilon}{\partial t} = \nabla\varphi(\mathbf{x}, t) \cdot \mathbf{v}_0\left(\frac{\mathbf{x}}{\varepsilon}\right)$$

are uniformly bounded in ε and

$$\frac{\partial \mathbf{w}_0^\varepsilon}{\partial t} \to \varphi(\mathbf{x}, t)\langle \mathbf{v}_0 \rangle_Y$$

weakly in $L^2(\Omega_T)$ as $\varepsilon \searrow 0$.

In what follows all parameters may take all permitted values. If, for example, $\tau_0 = 0$, then all terms in final equations containing this parameter disappear.

The following Theorems 1–2 are the main results of the paper.

Theorem 1. *For all $\varepsilon > 0$ on the arbitrary time interval $[0,T]$ there exists a unique generalized solution of the problem* (0.1)–(0.7) *and*

$$\max_{0\leqslant t\leqslant T} \| |\mathbf{w}^\varepsilon(t)| + \sqrt{\alpha_\mu}\chi^\varepsilon |\nabla_x \mathbf{w}^\varepsilon(t)| + (1-\chi^\varepsilon)\sqrt{\alpha_\lambda}|\nabla_x \mathbf{w}^\varepsilon(t)| \|_{2,\Omega} \leqslant C_0, \tag{1.8}$$

$$\|\theta^\varepsilon\|_{2,\Omega_T} + \|\nabla_x \theta^\varepsilon\|_{2,\Omega_T} \leqslant C_0, \tag{1.9}$$

$$\|q^\varepsilon\|_{2,\Omega_T} + \|p^\varepsilon\|_{2,\Omega_T} + \frac{\alpha_\nu}{\alpha_p}\left\|\frac{\partial p^\varepsilon}{\partial t}\right\|_{2,\Omega_T} \leqslant C_0 \tag{1.10}$$

where C_0 does not depend on the small parameter ε.

Theorem 2. *Let*

$$\mathbf{w}_0 = \lim_{\varepsilon\searrow 0} \mathbf{w}_0^\varepsilon, \qquad \mathbf{v}_0 = \lim_{\varepsilon\searrow 0} \frac{\partial \mathbf{w}_0^\varepsilon}{\partial t}, \qquad \theta_0 = \lim_{\varepsilon\searrow 0} \theta_0^\varepsilon.$$

Sequences $\{\chi^\varepsilon \mathbf{w}^\varepsilon\}$, $\{p^\varepsilon\}$ and $\{q^\varepsilon\}$ converge weakly in $L^2(\Omega_T)$ to $\mathbf{w}^f$, p and q, respectively. The sequence $\{\theta^\varepsilon\}$ converge weakly in $L^2((0,T);W_2^1(\Omega))$ to function θ. Functions $\mathbf{w}^\varepsilon$ admit an extension $\mathbf{u}^\varepsilon$ from $\Omega_s^\varepsilon \times (0,T)$ into Ω_T such that the sequence $\{\nabla \mathbf{u}^\varepsilon\}$ converge strongly in $L^2((0,T);W_2^1(\Omega))$ to zero. Weak and strong limits $\mathbf{v} = \partial \mathbf{w}^f/\partial t$, q, p and θ satisfy in the domain Ω_T the state equation

$$q = p + \frac{\nu_0}{p_*}\frac{\partial p}{\partial t} + m\beta_{0f}\theta, \tag{1.11}$$

the continuity equation

$$\frac{1}{p_*}\frac{\partial p}{\partial t} + \operatorname{div} \mathbf{v} = 0, \tag{1.12}$$

the heat equation

$$\tau_0 \hat{c}_p \frac{\partial \theta}{\partial t} - \frac{\beta_{0f}}{p_*}\frac{\partial p}{\partial t} = \operatorname{div}\big(B^\theta \cdot \nabla \vartheta\big) \tag{1.13}$$

and the relation

$$\mathbf{v} = -\frac{1}{m}\int_0^t B_1(\mu_1, t-\tau)\cdot \nabla q(\mathbf{x},\tau)\,\mathrm{d}\tau \tag{1.14}$$

in the case of $\tau_0 > 0$ and $\mu_1 > 0$, or Darcy's law in the form

$$\mathbf{v} = -\frac{1}{m} B_2(\mu_1)\cdot \nabla q \tag{1.15}$$

in the case of $\tau_0 = 0$ or, finally, the balance of momentum equation in the form

$$\tau_0 \rho_f \frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{m}(m\mathbb{I} - B_3)\cdot \nabla q, \tag{1.16}$$

in the case of $\mu_1 = 0$. The problem is supplemented by the homogeneous initial condition (for the case $\tau_0 > 0$) and the boundary condition

$$(\mathbf{v} - \mathbf{v}_0)\cdot \mathbf{n}(\mathbf{x}) = 0, \quad \mathbf{x} \in S,\ t > 0 \tag{1.17}$$

for the velocity of the liquid component and initial and boundary conditions

$$\tau_0\theta(\mathbf{x},0) = 0; \qquad \theta(\mathbf{x},t) = \theta_0(\mathbf{x},t), \quad \mathbf{x}\in S,\ t>0 \tag{1.18}$$

for the temperature.

In Eq. (1.13) *symmetric strictly positively defined matrix* B^{θ} *is given below by formula* (4.22) *and in Eqs* (1.14)–(1.16) $\mathbf{n}(\mathbf{x})$ *is the unit normal vector to* S *at a point* $\mathbf{x} \in S$, *and matrix* $B_1(\mu_1, t)$ *and symmetric strictly positively defined matrices* $B_2(\mu_1)$ *and* B_3 *are given below by formulas* (4.28), (4.30) *and* (4.32).

2. Preliminaries

2.1. Two-scale convergence

Justification of Theorems 1–2 relies on systematic use of the method of two-scale convergence, which had been proposed by Nguetseng [2] and has been applied recently to a wide range of homogenization problems (see, for example, the survey [3]).

Definition 2. A sequence $\{\varphi^{\varepsilon}\} \subset L^2(\Omega_T)$ is said to be *two-scale convergent* to a limit $\varphi \in L^2(\Omega_T \times Y)$ if and only if for any 1-periodic in $\mathbf{y}$ function $\sigma = \sigma(\mathbf{x}, t, \mathbf{y})$ the limiting relation

$$\lim_{\varepsilon \searrow 0} \int_{\Omega_T} \varphi^{\varepsilon}(\mathbf{x}, t)\sigma(\mathbf{x}, t, \mathbf{x}/\varepsilon)\, \mathrm{d}x\, \mathrm{d}t = \int_{\Omega_T}\int_{Y} \varphi(\mathbf{x}, t, \mathbf{y})\sigma(\mathbf{x}, t, \mathbf{y})\, \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}t \tag{2.1}$$

holds.

Existence and main properties of weakly convergent sequences are established by the following fundamental theorem [2,3]:

Theorem 3 (Nguetseng's theorem).

1. *Any bounded in* $L^2(Q)$ *sequence contains a subsequence, two-scale convergent to some limit* $\varphi \in L^2(\Omega_T \times Y)$.
2. *Let sequences* $\{\varphi^{\varepsilon}\}$ *and* $\{\varepsilon\nabla_x\varphi^{\varepsilon}\}$ *be uniformly bounded in* $L^2(\Omega_T)$. *Then there exist a* 1*-periodic in* $\mathbf{y}$ *function* $\varphi = \varphi(\mathbf{x}, t, \mathbf{y})$ *and a subsequence* $\{\varphi^{\varepsilon}\}$ *such that* $\varphi, \nabla_y\varphi \in L^2(\Omega_T \times Y)$, *and* φ^{ε} *and* $\varepsilon\nabla_x\varphi^{\varepsilon}$ *two-scale converge to* φ *and* $\nabla_y\varphi$, *respectively.*
3. *Let sequences* $\{\varphi^{\varepsilon}\}$ *and* $\{\nabla_x\varphi^{\varepsilon}\}$ *be bounded in* $L^2(Q)$. *Then there exist functions* $\varphi \in L^2(\Omega_T)$ *and* $\psi \in L^2(\Omega_T \times Y)$ *and a subsequence from* $\{\varphi^{\varepsilon}\}$ *such that* ψ *is* 1*-periodic in* $\mathbf{y}$, $\nabla_y\psi \in L^2(\Omega_T \times Y)$, *and* φ^{ε} *and* $\nabla_x\varphi^{\varepsilon}$ *two-scale converge to* φ *and* $\nabla_x\varphi(\mathbf{x}, t) + \nabla_y\psi(\mathbf{x}, t, \mathbf{y})$, *respectively.*

Corollary 1. *Let* $\sigma \in L^2(Y)$ *and* $\sigma^{\varepsilon}(\mathbf{x}) := \sigma(\mathbf{x}/\varepsilon)$. *Assume that a sequence* $\{\varphi^{\varepsilon}\} \subset L^2(\Omega_T)$ *two-scale converges to* $\varphi \in L^2(\Omega_T \times Y)$. *Then the sequence* $\sigma^{\varepsilon}\varphi^{\varepsilon}$ *two-scale converges to* $\sigma\varphi$.

2.2. An extension lemma

The typical difficulty in homogenization problems while passing to a limit as $\varepsilon \searrow 0$ arises because of the fact that the bounds on the gradient of displacement $\nabla_x\mathbf{w}^{\varepsilon}$ may be distinct in liquid and rigid phases. The classical approach in overcoming this difficulty consists of constructing of extension to the whole Ω of the displacement field defined merely on Ω_s. The following lemma is valid due to the well-known results from [10,11]. We formulate it in appropriate for us form:

Lemma 2.1. *Suppose that Assumption* 1 *on geometry of periodic structure holds,* $\psi^\varepsilon \in W_2^1(\Omega_s^\varepsilon)$ *and* $\psi^\varepsilon = 0$ *on* $S_s^\varepsilon = \partial\Omega_s^\varepsilon \cap \partial\Omega$ *in the trace sense. Then there exists a function* $\sigma^\varepsilon \in W_2^1(\Omega)$ *such that its restriction on the sub-domain* Ω_s^ε *coincide with* ψ^ε*, i.e.,*

$$(1 - \chi^\varepsilon(\mathbf{x}))(\sigma^\varepsilon(\mathbf{x}) - \psi^\varepsilon(\mathbf{x})) = 0, \quad \mathbf{x} \in \Omega, \tag{2.2}$$

and, moreover, the estimate

$$\|\sigma^\varepsilon\|_{2,\Omega} \leqslant C\|\psi^\varepsilon\|_{2,\Omega_s^\varepsilon}, \qquad \|\nabla_x \sigma^\varepsilon\|_{2,\Omega} \leqslant C\|\nabla_x \psi^\varepsilon\|_{2,\Omega_s^\varepsilon} \tag{2.3}$$

hold true, where the constant C *depends only on geometry* Y *and does not depend on* ε.

2.3. Friedrichs–Poincaré's inequality in periodic structure

The following lemma was proved by Tartar in [5, Appendix]. It specifies Friedrichs–Poincaré's inequality for ε-periodic structure.

Lemma 2.2. *Suppose that assumptions on the geometry of* Ω_f^ε *hold true. Then for any function* $\varphi \in \overset{\circ}{W}{}_2^1(\Omega_f^\varepsilon)$ *the inequality*

$$\int_{\Omega_f^\varepsilon} |\varphi|^2 \, \mathrm{d}\mathbf{x} \leqslant C\varepsilon^2 \int_{\Omega_f^\varepsilon} |\nabla_x \varphi|^2 \, \mathrm{d}\mathbf{x} \tag{2.4}$$

holds true with some constant C*, independent of* ε.

We also need this Friedrichs–Poincaré's inequality for our particular case just to estimate functions in the ε-layer Q^ε of the boundary S. This domain Q^ε consists of all elementary cells εY touching the boundary $\partial\Omega$. We consider special class of functions u^ε, which are extensions of functions $w^\varepsilon \in W_2^1(\Omega_s^\varepsilon)$, vanishing on the part $S_s^\varepsilon = \partial\Omega_s^\varepsilon \cap \partial\Omega$ of the boundary $S = \partial\Omega$, from subdomain Ω_s^ε onto whole domain Ω (see Lemma 2.1). Due to supposition on the structure of the pore space, the intersection of the boundary of the "solid part" Y_s with each sides of the boundary ∂Y is a set with nonempty interior and strictly positive measure. Therefore on the each side of the boundary S the function u^ε is equal to zero on some set with nonempty interior, periodic structure and strictly positive measure, independent of ε.

Lemma 2.3. *Suppose that assumptions on the geometry of* Ω_s^ε *hold true. Then for any function* $u^\varepsilon \in W_2^1(\Omega)$ *such that* $u^\varepsilon = 0$ *on the part* $S_s^\varepsilon = \partial\Omega_s^\varepsilon \cap \partial\Omega$ *of the boundary* S*, the inequality*

$$\int_{Q^\varepsilon} |u^\varepsilon|^2 \, \mathrm{d}x \leqslant C\varepsilon^2 \int_{Q^\varepsilon} |\nabla u^\varepsilon|^2 \, \mathrm{d}x \tag{2.5}$$

holds true with some constant C *independent of the small parameter* ε.

2.4. *Some notation*

Further we denote

(1) $\langle \Phi \rangle_Y = \int_Y \Phi \, \mathrm{d}y$, $\langle \Phi \rangle_{Y_f} = \int_{Y_f} \Phi \, \mathrm{d}y$, $\langle \Phi \rangle_{Y_s} = \int_{Y_s} \Phi \, \mathrm{d}y$.

(2) If **a** and **b** are two vectors then the matrix $\mathbf{a} \otimes \mathbf{b}$ is defined by the formula

$$(\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c})$$

for any vector **c**.

3. Proof of Theorem 1

Under restriction $\tau_0 > 0$ estimates (1.8)–(1.9) follow from estimates

$$\begin{aligned} &\max_{0<t<T} \Big(\sqrt{\alpha_\lambda} \big\| (1-\chi^\varepsilon) \nabla_x \mathbf{w}^\varepsilon(t) \big\|_{2,\Omega} + \sqrt{\alpha_\tau} \Big\| \frac{\partial \mathbf{w}^\varepsilon}{\partial t}(t) \Big\|_{2,\Omega} \\ &\qquad + \sqrt{\alpha_p} \| \chi^\varepsilon \operatorname{div} \mathbf{w}^\varepsilon(t) \|_{2,\Omega} + \sqrt{\alpha_\tau} \| \theta^\varepsilon(t) \|_{2,\Omega} \Big) + \| \nabla_x \theta^\varepsilon \|_{2,\Omega_T} \\ &\quad + \sqrt{\alpha_\mu} \Big\| \chi^\varepsilon \nabla_x \frac{\partial \mathbf{w}^\varepsilon}{\partial t} \Big\|_{2,\Omega_T} + \sqrt{\alpha_\nu} \Big\| \chi^\varepsilon \operatorname{div} \frac{\partial \mathbf{w}^\varepsilon}{\partial t} \Big\|_{2,\Omega_T} \leqslant C_0, \end{aligned} \tag{3.1}$$

where C_0 is independent of ε due to Assumption 2. These estimates we obtain if we multiply equation for $\mathbf{w}^\varepsilon$ by $(\partial \mathbf{w}^\varepsilon/\partial t - \partial \mathbf{w}_0^\varepsilon/\partial t)$, equation for θ^ε multiply by $(\theta^\varepsilon - \theta_0^\varepsilon)$, integrate by parts sum the result and use Hölder's and Gronwall inequalities.

The same estimates (3.1) guaranties the existence and uniqueness of the generalized solution for the problem (0.1)–(0.8).

Estimates (1.10) for pressures follows from Eqs (1.4)–(1.5) and estimates (3.1).

Estimation of $\mathbf{w}^\varepsilon$ and θ^ε in the case $\tau_0 = 0$ is not simple, and we outline it in more detail.

First of all we use estimates (3.1) in the form

$$\sqrt{\alpha_\mu} \| \chi^\varepsilon \nabla_x \mathbf{w}^\varepsilon(t) \|_{2,\Omega} + \sqrt{\alpha_\lambda} \big\| (1-\chi^\varepsilon) \nabla_x \mathbf{w}^\varepsilon(t) \big\|_{2,\Omega} \leqslant C_0. \tag{3.2}$$

Next, on the strength of Lemma 2.1, we construct extension $\mathbf{u}^\varepsilon$ of the function $\mathbf{w}^\varepsilon$ from Ω_s^ε into Ω_f^ε, such that $\mathbf{u}^\varepsilon = \mathbf{w}^\varepsilon$ in Ω_s^ε and

$$\| \nabla_x \mathbf{u}^\varepsilon \|_{2,\Omega} \leqslant C \big\| (1-\chi^\varepsilon) \nabla_x \mathbf{w}^\varepsilon \big\|_{2,\Omega}.$$

Next we use the property of functions $\mathbf{u}^\varepsilon$, that $\mathbf{u}^\varepsilon = 0$ on the part $S_s^\varepsilon = \partial \Omega_s^\varepsilon \cap \partial \Omega$ of the boundary $S = \partial \Omega$. Due to this property

$$\| \mathbf{u}^\varepsilon \|_{2,\Omega} \leqslant C \| \nabla_x \mathbf{u}^\varepsilon \|_{2,\Omega}, \tag{3.3}$$

where C does not depend on ε.

We prove this estimate (3.3) in the Appendix (Lemma A.1)

After that we estimate $\|\mathbf{w}^\varepsilon\|_{2,\Omega}$ with the help of Friedrichs–Poincaré's inequality in periodic structure (Lemma 2.2) for the difference $(\mathbf{u}^\varepsilon - \mathbf{w}^\varepsilon)$ and estimates (3.2):

$$\begin{aligned}
\|\mathbf{w}^\varepsilon\|_{2,\Omega} &\leqslant \|\mathbf{u}^\varepsilon\|_{2,\Omega} + \|\mathbf{u}^\varepsilon - \mathbf{w}^\varepsilon\|_{2,\Omega} \leqslant \|\mathbf{u}^\varepsilon\|_{2,\Omega} + C\varepsilon\|\chi^\varepsilon \nabla_x(\mathbf{u}^\varepsilon - \mathbf{w}^\varepsilon)\|_{2,\Omega} \\
&\leqslant \|\mathbf{u}^\varepsilon\|_{2,\Omega} + C\varepsilon\|\nabla_x \mathbf{u}^\varepsilon\|_{2,\Omega} + C(\varepsilon\alpha_\mu^{-1/2})\|\chi^\varepsilon\sqrt{\alpha_\mu}\nabla_x \mathbf{w}^\varepsilon\|_{2,\Omega} \\
&\leqslant \|\mathbf{u}^\varepsilon\|_{2,\Omega} + \frac{C\varepsilon}{\sqrt{\alpha_\lambda}}\|\sqrt{\alpha_\lambda}\nabla_x \mathbf{w}^\varepsilon\|_{2,\Omega_s^\varepsilon} + C(\varepsilon\alpha_\mu^{-1/2})\|\sqrt{\alpha_\mu}\nabla_x \mathbf{w}^\varepsilon\|_{2,\Omega_f^\varepsilon} \leqslant C_0.
\end{aligned}$$

The norm $\|\theta^\varepsilon\|_{2,\Omega}$ we estimate with the help of the usual Poincaré's inequality for the difference $(\theta^\varepsilon - \theta_0^\varepsilon)$ and estimate (3.1).

The rest of the proof is the same as for the case $\tau_0 > 0$.

4. Proof of Theorem 2

4.1. Weak and two-scale limits of sequences of displacement, temperatures and pressures

On the strength of Theorem 1, the sequences $\{\theta^\varepsilon\}$, $\{\nabla\theta^\varepsilon\}$, $\{p^\varepsilon\}$, $\{q^\varepsilon\}$ and $\{\mathbf{w}^\varepsilon\}$ are uniformly in ε bounded in $L^2(\Omega_T)$. Hence there exist a subsequence of small parameters $\{\varepsilon > 0\}$ and functions θ, p, q, π and $\mathbf{w}$ such that

$$\theta^\varepsilon \to \theta, \qquad \nabla\theta^\varepsilon \to \nabla\theta, \qquad p^\varepsilon \to p, \qquad q^\varepsilon \to q, \qquad \mathbf{w}^\varepsilon \to \mathbf{w}$$

weakly in $L^2(\Omega_T)$ as $\varepsilon \searrow 0$.

Due to Lemma 2.1 there is a function $\mathbf{u}^\varepsilon \in L^\infty((0,T); W_2^1(\Omega))$ such that $\mathbf{u}^\varepsilon = \mathbf{w}^\varepsilon$ in $\Omega_s \times (0,T)$, the family $\{\mathbf{u}^\varepsilon\}$ is uniformly in ε bounded in $L^\infty((0,T); W_2^1(\Omega))$. Moreover,

$$\|\nabla_x \mathbf{u}^\varepsilon\|_{2,\Omega} \leqslant C\|(1-\chi^\varepsilon)\nabla_x \mathbf{w}^\varepsilon\|_{2,\Omega} \leqslant \frac{C}{\sqrt{\alpha_\lambda}}$$

and

$$\nabla\mathbf{u}^\varepsilon \to 0, \qquad \mathbf{u}^\varepsilon \to 0, \quad \text{strongly in } L^2(\Omega_T) \tag{4.1}$$

as $\varepsilon \searrow 0$.

The last assertion follows from the Assumption 2 if we use the Friedrichs–Poincaré's inequality for $\mathbf{u}^\varepsilon$ in the ε-layer of the boundary S (see Lemma 2.3), and convergence of the sequence $\{\mathbf{u}^\varepsilon\}$ to $\mathbf{u}$ strongly in $L^2(\Omega_T)$ and weakly in $L^2((0,T); W_2^1(\Omega))$. For more details see Appendix (Lemma A.2).

Note also, that

$$\chi^\varepsilon\alpha_\mu\mathbb{D}(\mathbf{x}, \mathbf{w}^\varepsilon) \to 0,$$

strongly in $L^2(\Omega_T)$ as $\varepsilon \searrow 0$.

Relabeling if necessary, we assume that the sequences converge themselves.

On the strength of Nguetseng's theorem, there exist 1-periodic in $\mathbf{y}$ functions $\Theta(\mathbf{x},t,\mathbf{y})$, $P(\mathbf{x},t,\mathbf{y})$, $Q(\mathbf{x},t,\mathbf{y})$ and $\mathbf{W}(\mathbf{x},t,\mathbf{y})$, such that the sequences $\{\nabla\theta^{\varepsilon}\}$, $\{p^{\varepsilon}\}$, $\{q^{\varepsilon}\}$ and $\{\mathbf{w}^{\varepsilon}\}$ two-scale converge to $\nabla_x\theta+\nabla_y\Theta(\mathbf{x},t,\mathbf{y})$, $P(\mathbf{x},t,\mathbf{y})$, $Q(\mathbf{x},t,\mathbf{y})$ and $\mathbf{W}(\mathbf{x},t,\mathbf{y})$ respectively.

Note that the sequence $\{\operatorname{div}\mathbf{w}^{\varepsilon}\}$ weakly converges to $\operatorname{div}\mathbf{w}$.

The same arguments we apply for the functions $\mathbf{w}_0^{\varepsilon}$ and θ_0^{ε}:

$$\mathbf{w}_0^{\varepsilon}\to\mathbf{w}_0$$

weakly in $L^2(\Omega_T)$ as $\varepsilon\searrow 0$,

$$\theta_0^{\varepsilon}\to\theta_0$$

weakly in $L^2((0,T);W_2^1(\Omega))$ as $\varepsilon\searrow 0$ and the sequence $\{\operatorname{div}\mathbf{w}_0^{\varepsilon}\}$ weakly converges to $\operatorname{div}\mathbf{w}_0$.

4.2. Micro- and macroscopic equations

Lemma 4.1. *For all $\mathbf{x}\in\Omega$ and $\mathbf{y}\in Y$ weak and two-scale limits of the sequences $\{\theta^{\varepsilon}\}$, $\{p^{\varepsilon}\}$, $\{q^{\varepsilon}\}$, $\{\mathbf{w}^{\varepsilon}\}$, $\{\nabla_x\vartheta^{\varepsilon}\}$ and $\{\nabla_x\mathbf{u}^{\varepsilon}\}$ satisfy the relations*

$$P=\frac{1}{m}\chi p,\qquad Q=\frac{1}{m}\chi q; \tag{4.2}$$

$$(1-\chi)\mathbf{W}=0; \tag{4.3}$$

$$q=p+\frac{\nu_0}{p_*}\frac{\partial p}{\partial t}+m\beta_{0f}\theta; \tag{4.4}$$

$$\frac{1}{p_*}p+\operatorname{div}\mathbf{w}=0; \tag{4.5}$$

$$(\mathbf{w}-\mathbf{w}_0)\cdot\mathbf{n}(\mathbf{x})=0,\quad \mathbf{x}\in S; \tag{4.6}$$

$$\operatorname{div}_y\mathbf{W}=0. \tag{4.7}$$

Proof. The weak and two-scale limiting passage in Eq. (1.4) yield that Eq. (4.4) and

$$Q(\mathbf{x},t,\mathbf{y})=P(\mathbf{x},t,\mathbf{y})+\frac{\nu_0}{p_*}\frac{\partial P}{\partial t}(\mathbf{x},t,\mathbf{y})+\chi(\mathbf{y})\beta_{0f}\theta(\mathbf{x},t),\quad \mathbf{y}\in Y_f. \tag{4.8}$$

In order to prove Eq. (4.2), into Eq. (1.6) insert a test function $\psi^{\varepsilon}=\varepsilon\psi(\mathbf{x},t,\mathbf{x}/\varepsilon)$, where $\psi(\mathbf{x},t,\mathbf{y})$ is an arbitrary 1-periodic and finite on Y_f function in $\mathbf{y}$. Passing to the limit as $\varepsilon\searrow 0$, we get

$$\nabla_y Q=0,\quad \mathbf{y}\in Y_f. \tag{4.9}$$

Next, fulfilling the two-scale limiting passage in the equality

$$(1-\chi^{\varepsilon})p^{\varepsilon}=0$$

we arrive at

$$(1-\chi)P=0$$

which along with Eqs (4.8)–(4.9) justifies Eq. (4.2).

Equation (4.5) and boundary condition (4.6) are derived quite similarly if we represent Eq. (1.5) in the form

$$\frac{1}{\alpha_p}p^\varepsilon+\operatorname{div}\big(\mathbf{w}^\varepsilon-\mathbf{w}_0^\varepsilon\big)=\big(1-\chi^\varepsilon\big)\operatorname{div}\mathbf{u}^\varepsilon-\operatorname{div}\mathbf{w}_0^\varepsilon, \tag{4.10}$$

multiply by an arbitrary function $\psi^\varepsilon=\psi(\mathbf{x},t)$, integrate, and then pass to the limit as $\varepsilon\searrow 0$. Using now in (4.10) test functions in the form $\psi^\varepsilon=\varepsilon\psi(\mathbf{x},t,\mathbf{x}/\varepsilon)$ we obtain (4.7).

In order to prove Eq. (4.3) it is sufficient to consider the two-scale limiting relation in

$$\big(1-\chi^\varepsilon\big)\big(\mathbf{w}^\varepsilon-\mathbf{u}^\varepsilon\big)=0. \qquad \square$$

Lemma 4.2. *For all* $(\mathbf{x},t)\in\Omega_T$ *the microscopic equation*

$$\operatorname{div}_y\big\{K(\mathbf{y})(\nabla_x\theta+\nabla_y\Theta)\big\}=0,\quad \mathbf{y}\in Y, \tag{4.11}$$

holds true. Here $K=\varkappa_{0f}\chi+\varkappa_{0s}(1-\chi)$.

Proof. First of all, using continuity equation (1.5) we rewrite the heat equation in the form

$$\alpha_\tau c_p^\varepsilon\frac{\partial\theta^\varepsilon}{\partial t}=\operatorname{div}\big(\alpha_\varkappa^\varepsilon\nabla_x\theta^\varepsilon\big)-\alpha_{\theta s}\big(1-\chi^\varepsilon\big)\frac{\partial}{\partial t}\big(\operatorname{div}\mathbf{u}^\varepsilon\big)+\frac{\alpha_{\theta s}}{\alpha_p}\frac{\partial p^\varepsilon}{\partial t}. \tag{4.12}$$

Substituting now a test function of the form $\psi^\varepsilon=\varepsilon\psi(\mathbf{x},t,\mathbf{x}/\varepsilon)$, where $\psi(\mathbf{x},t,\mathbf{y})$ is an arbitrary 1-periodic in $\mathbf{y}$ function vanishing on the boundary $\partial\Omega$, into corresponding integral identity, and passing to the limit as $\varepsilon\searrow 0$, we arrive at the desired microscopic relation on the cell Y. $\square$

In the same way, using a test function independent of the fast variable $\mathbf{y}/\varepsilon$, we get from Eq. (4.12)

Lemma 4.3. *For all* $(\mathbf{x},t)\in\Omega_T$ *the macroscopic equation*

$$\tau_0\hat{c}_p\frac{\partial\theta}{\partial t}-\frac{\beta_{0f}}{p_*}\frac{\partial p}{\partial t}=\operatorname{div}\big\{\hat{\varkappa}_0\nabla_x\vartheta+\langle K\nabla\Theta\rangle_Y\big\} \tag{4.13}$$

hold true. Here $\hat{c}_p=mc_{pf}+(1-m)c_{ps}$, $\hat{\varkappa}_0=\langle K\rangle_Y$.

Now we pass to the microscopic equations for the velocities in the liquid.

Lemma 4.4. *Let* $\mathbf{V}=\chi\,\partial\mathbf{W}/\partial t$. *Then*

$$\tau_0\rho_f\frac{\partial\mathbf{V}}{\partial t}=\mu_1\Delta_y\mathbf{V}-\nabla_yR-\frac{1}{m}\nabla_xq,\quad \mathbf{y}\in Y_f,\quad t>0; \tag{4.14}$$

$$\mathbf{V}=0,\quad \mathbf{y}\in\gamma;\quad \mathbf{V}(\mathbf{y},0)=0,\quad \mathbf{y}\in Y_f \tag{4.15}$$

in the case $\mu_1 > 0$, and

$$\tau_0\rho_f\frac{\partial \mathbf{V}}{\partial t} = -\nabla_y R - \frac{1}{m}\nabla_x q, \quad \mathbf{y}\in Y_f, \quad t>0; \tag{4.16}$$

$$\mathbf{V}\cdot\mathbf{n} = 0, \quad \mathbf{y}\in\gamma, \quad t>0; \quad \mathbf{V}(\mathbf{y},0)=0, \quad \mathbf{y}\in Y_f \tag{4.17}$$

in the case $\mu_1 = 0$.

In Eq. (4.17) $\mathbf{n}$ *is the unit normal to* γ.

Proof. Differential equations (4.14) and (4.16) follow as $\varepsilon \searrow 0$ from integral equality (1.6) with the test function $\psi = \varphi(x\varepsilon^{-1})\cdot h(\mathbf{x},t)$, where φ is solenoidal and finite in Y_f.

Boundary condition in (4.15) is the consequence of the two-scale convergence of $\{\alpha_\mu^{1/2}\nabla_x \mathbf{w}^\varepsilon\}$ to the function $\mu_1^{1/2}\nabla_y \mathbf{W}(\mathbf{x},t,\mathbf{y})$. On the strength of this convergence, the function $\nabla_y\mathbf{W}(\mathbf{x},t,\mathbf{y})$ is L^2-integrable in Y. The boundary condition (4.17) follows from Eq. (4.7). □

4.3. Homogenized equations

Lemma 4.5. *Weak and strong limits q and θ satisfy the initial-boundary value problem*

$$\tau_0\hat{c}_p\frac{\partial\theta}{\partial t} - \frac{\beta_{0f}}{p_*}\frac{\partial p}{\partial t} = \operatorname{div}(B^\theta\cdot\nabla\vartheta), \quad \mathbf{x}\in\Omega, t>0; \tag{4.18}$$

$$\tau_0\theta(\mathbf{x},0)=0, \quad \mathbf{x}\in\Omega; \quad \theta(\mathbf{x},t)=\theta_0(\mathbf{x},t), \quad \mathbf{x}\in S, t>0, \tag{4.19}$$

where a symmetric strictly positively defined matrix B^θ is defined by formula (4.22).

Proof. For $i=1,2,3$ we consider the periodic in $\mathbf{y}$ model problems

$$\operatorname{div}_y\{K(\mathbf{y})(\nabla_y\Theta_i + e_i)\} = 0, \quad \mathbf{y}\in Y, \tag{4.20}$$

where $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ are the standard Cartesian basis vectors, and put

$$\Theta = \sum_{i=1}^{3}(\Theta_i\otimes\mathbf{e}_i)\cdot\nabla_x\theta. \tag{4.21}$$

Then Θ solves the problem (4.11) and

$$B^\theta = \hat{\varkappa}_0\mathbb{I} + \sum_{i=1}^{3}\langle K\nabla_y\Theta_i^s\rangle_Y\otimes\mathbf{e}_i. \tag{4.22}$$

All properties of the matrix B^θ are well known (see [5,11]). □

Lemma 4.6. *The strong and weak limits θ, $\mathbf{v} = \langle\mathbf{V}\rangle_{Y_f}$, p and q satisfy in Ω_T Eq.* (4.5), *the relation*

$$\mathbf{v} = -\frac{1}{m}\int_0^t B_1(\mu_1, t-\tau)\cdot\nabla q(\mathbf{x},\tau)\,\mathrm{d}\tau \tag{4.23}$$

in the case of $\tau_0 > 0$ *and* $\mu_1 > 0$*, or Darcy's law in the form*

$$\mathbf{v} = -\frac{1}{m} B_2(\mu_1) \cdot \nabla q \tag{4.24}$$

in the case of $\tau_0 = 0$ *or, finally, the balance of momentum equation in the form*

$$\tau_0 \rho_f \frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{m}(m\mathbb{I} - B_3) \cdot \nabla q, \tag{4.25}$$

in the case of $\mu_1 = 0$*. The problem is supplemented by the boundary condition*

$$(\mathbf{v} - \mathbf{v}_0) \cdot \mathbf{n}(\mathbf{x}) = 0, \quad \mathbf{x} \in S, \quad t > 0, \tag{4.26}$$

in Eqs (4.23)–(4.26) $\mathbf{n}(\mathbf{x})$ *is the unit normal vector to* S *at a point* $\mathbf{x} \in S$*, and matrices* $B_1(\mu_1, t)$*,* $B_2(\mu_1)$*, and* B_3 *are given below by formulas* (4.28), (4.30) *and* (4.32).

Proof. (a) If $\mu_1 > 0$ and $\tau_0 > 0$, then the solution of the system of microscopic equations (4.7), (4.14), and (4.15), provided with the homogeneous initial data, is given by formula

$$m\mathbf{V} = -\int_0^t \mathbf{B}_1^f(\mathbf{y}, t - \tau) \cdot \nabla_x q(\mathbf{x}, \tau)\, \mathrm{d}\tau, \qquad mR = -\int_0^t R_f(\mathbf{y}, t - \tau) \cdot \nabla_x q(\mathbf{x}, \tau)\, \mathrm{d}\tau,$$

in which

$$\mathbf{B}_1^f(\mathbf{y}, t) = \sum_{i=1}^{3} \mathbf{V}^i(\mathbf{y}, t) \otimes \mathbf{e}_i, \qquad R_f(\mathbf{y}, t) = \sum_{i=1}^{3} R^i(\mathbf{y}, t)\mathbf{e}_i,$$

and the functions $\mathbf{V}^i(\mathbf{y}, t)$ and $R^i(\mathbf{y}, t)$ are defined by virtue of the periodic initial-boundary value problem

$$\begin{cases} \tau_0 \rho_f \dfrac{\partial \mathbf{V}^i}{\partial t} - \mu_1 \Delta \mathbf{V}^i + \nabla R^i = 0, \qquad \mathrm{div}_y\, \mathbf{V}^i = 0, \quad \mathbf{y} \in Y_f, t > 0, \\ \mathbf{V}^i = 0, \quad \mathbf{y} \in \gamma, t > 0; \qquad \tau_0 \rho_f \mathbf{V}^i(y, 0) = \mathbf{e}_i, \quad \mathbf{y} \in Y_f. \end{cases} \tag{4.27}$$

In Eq. (4.27) $\mathbf{e}_i$ is the standard Cartesian basis vector.

Therefore

$$B_1(\mu_1, t) = \langle \mathbf{B}_1^f \rangle_{Y_f}(t). \tag{4.28}$$

(b) If $\tau_0 = 0$ and $\mu_1 > 0$ then the solution of the stationary microscopic equations (4.7), (4.14) and (4.15) is given by formula

$$\mathbf{V} = -\mathbf{B}_2^f(\mathbf{y}) \cdot \nabla q,$$

in which

$$\mathbf{B}_2^f(\mathbf{y}) = \sum_{i=1}^{3} \mathbf{U}^i(\mathbf{y}) \otimes \mathbf{e}_i,$$

and the functions $\mathbf{U}^i(\mathbf{y})$ are defined from the periodic boundary value problem

$$\begin{cases} -\mu_1 \Delta \mathbf{U}^i + \nabla R^i = \mathbf{e}_i, & \operatorname{div}_y \mathbf{U}^i = 0, \quad \mathbf{y} \in Y_f, \\ \mathbf{U}^i = 0, \quad \mathbf{y} \in \gamma. \end{cases} \tag{4.29}$$

Thus

$$B_2(\mu_1) = \langle \mathbf{B}_2^f(\mathbf{y}) \rangle_{Y_f}. \tag{4.30}$$

Matrix $B_2(\mu_1)$ is symmetric and positively defined ([5, Chapter 8], [8]).

(c) If $\tau_0 > 0$ and $\mu_1 = 0$ then in the process of solving the system (4.7), (4.16) and (4.17) we firstly find the pressure $R(\mathbf{x}, t, \mathbf{y})$ by virtue of solving the Neumann problem for Laplace's equation in Y_f. If

$$mR(\mathbf{x}, t, \mathbf{y}) = -\sum_{i=1}^{3} R_i(\mathbf{y})\mathbf{e}_i \cdot \nabla_x q(\mathbf{x}, t),$$

where $R^i(\mathbf{y})$ is the solution of the problem

$$\Delta_y R_i = 0, \qquad \mathbf{y} \in Y_f; \qquad \nabla_y R_i \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{e}_i, \quad \mathbf{y} \in \gamma. \tag{4.31}$$

Formula (4.25) appears as the result of homogenization of Eq. (4.16) and

$$B_3 = \sum_{i=1}^{3} \langle \nabla R_i(\mathbf{y}) \rangle_{Y_f} \otimes \mathbf{e}_i, \tag{4.32}$$

where the matrix $(m\mathbb{I} - B_3)$ is symmetric and positively definite ([5, Chap. 8], [8]). □

Appendix

Lemma A.1. *Under assumptions on the geometry of the domains Ω_f^ε and Ω_s^ε let*

$$u \in W_2^1(\Omega)$$

and $u = 0$ on the part $S_s^\varepsilon = \partial\Omega_s^\varepsilon \cap \partial\Omega$ of the boundary $S = \partial\Omega$. Then

$$\|u\|_{2,\Omega} \leqslant C \|\nabla_x u\|_{2,\Omega}, \tag{A.1}$$

where C does not depend on ε.

Proof. To prove this statement we use Friedrichs–Poincaré's inequality in the form (2.5). In fact, standard representation

$$u^2(x_1,x') = u^2(x_1^0,x') + 2\int_{x_1^0}^{x_1} u(y_1,x')\frac{\partial u}{\partial y_1}(y_1,x')\,\mathrm{d}y_1,$$

where $x' = (x_2, x_3)$, and $(x_1^0, x') \in Q^\varepsilon$, and integration with respect to the variable $x_1 \in (0,1)$ give us

$$\int_0^1 u^2(x_1,x')\,\mathrm{d}x_1 \leqslant u^2(x_1^0,x') + 2\Big(\int_0^1 u^2(x_1,x')\,\mathrm{d}x_1\Big)^{1/2}\Big(\int_0^1\Big|\frac{\partial u}{\partial x_1}(x_1,x')\Big|^2\,\mathrm{d}x_1\Big)^{1/2}.$$

Now we repeat the integration with respect to variable x_1 such that $(x_1^0, x') \in Q^\varepsilon$ and with respect to the variable $x' \in (0,1)\times(0,1)$:

$$2\varepsilon\int_\Omega u^2\,\mathrm{d}x \leqslant \int_{Q^\varepsilon} u^2\,\mathrm{d}x + 4\varepsilon\Big(\int_\Omega u^2\,\mathrm{d}x\Big)^{1/2}\Big(\int_\Omega |\nabla u|^2\,\mathrm{d}x\Big)^{1/2}.$$

Next we apply the inequality (2.5) and get

$$2\varepsilon\int_\Omega u^2\,\mathrm{d}x \leqslant C\varepsilon^2\int_\Omega |\nabla u|^2\,\mathrm{d}x + 4\varepsilon\Big(\int_\Omega u^2\,\mathrm{d}x\Big)^{1/2}\Big(\int_\Omega |\nabla u|^2\,\mathrm{d}x\Big)^{1/2},$$

which is obviously equivalent to (A.1). □

Lemma A.2. *Under assumptions on the geometry of the domains Ω_f^ε and Ω_s^ε let*

$$\Big\|\frac{\partial \mathbf{v}^\varepsilon}{\partial t}\Big\|_{2,\Omega_T} + \Big\|\nabla\frac{\partial \mathbf{v}^\varepsilon}{\partial t}\Big\|_{2,\Omega_T} \leqslant C_0, \tag{A.2}$$

$$\max_{0\leqslant t\leqslant T}\big(\|\mathbf{v}^\varepsilon(t)\|_{2,\Omega} + \|\nabla\mathbf{v}^\varepsilon(t)\|_{2,\Omega}\big) \leqslant C_0, \tag{A.3}$$

where C_0 does not depend on the small parameter ε, and $\mathbf{v}^\varepsilon = 0$ on the part $S_s^\varepsilon = \partial\Omega_s^\varepsilon \cap \partial\Omega$ of the boundary $S = \partial\Omega$.

Then there exist a subsequence of $\{\varepsilon > 0\}$ and function

$$\mathbf{v} \in L^\infty(0,T; W_2^1(\Omega)),$$

such that

(1) $\mathbf{v}^\varepsilon(\cdot,t) \to \mathbf{v}(\cdot,t)$ weakly in $W_2^1(\Omega)$ as $\varepsilon \searrow 0$ for all $t \in [0,T]$,

and

(2) $\mathbf{v}(\cdot,t) \in \overset{\circ}{W}{}_2^1(\Omega)$ for all $t \in [0,T]$.

Proof. First of all note, that there are a subsequence of small parameters $\{\varepsilon > 0\}$ and function $\mathbf{v}$, such that

$$\mathbf{v},\ \frac{\partial \mathbf{v}}{\partial t} \in L^2(0,T; W_2^1(\Omega))$$

and

$$\mathbf{v}^\varepsilon(\cdot,t) \to \mathbf{v}(\cdot,t) \quad \text{weakly in } L^2(0,T;W_2^1(\Omega)) \text{ as } \varepsilon \searrow 0.$$

Let now $\varphi(\mathbf{x})$ be an arbitrary smooth function and

$$J_\varphi^\varepsilon(t) = \int_\Omega \big((\mathbf{v}^\varepsilon(\mathbf{x},t) - \mathbf{v}(\mathbf{x},t)) \cdot \varphi(\mathbf{x}) + \nabla\big(\mathbf{v}^\varepsilon(\mathbf{x},t) - \mathbf{v}(\mathbf{x},t)\big) \cdot \nabla\varphi(\mathbf{x})\big)\,\mathrm{d}x.$$

By the construction

$$\int_0^T J_\varphi^\varepsilon(t)\psi(t)\,\mathrm{d}t \to 0 \quad \text{as } \varepsilon \searrow 0 \text{ for any } \psi \in L^2(0,T).$$

The first statement of the lemma means that

$$J_\varphi^\varepsilon(t) \to 0 \quad \text{as } \varepsilon \searrow 0 \text{ for all } t \in [0,T].$$

Estimates (A.2) and (A.3) imply

$$\int_0^T \left| \frac{\mathrm{d}J_\varphi^\varepsilon}{\mathrm{d}t}(t) \right|^2 \mathrm{d}t \leqslant C_0^2.$$

Using this last estimate, initial condition $J_\varphi^\varepsilon(0) = 0$ and weak convergence in $L^2(0,T)$ of the sequence $\{J_\varphi^\varepsilon\}$ to zero, one may easily prove that

$$J_\varphi^\varepsilon(t) \to 0 \quad \text{in } C[0,T],$$

which proves the first part of the lemma.

To prove the second part of the lemma note, that

$$\mathbf{v}^\varepsilon(\cdot,t) \to \mathbf{v}(\cdot,t) \quad \text{strongly in } L^2(S) \text{ as } \varepsilon \searrow 0 \text{ for all } t \in [0,T].$$

This fact follows from the well-known imbedding theorem, which states that any weakly convergent sequence in $W_2^1(\Omega)$ converges strongly in $L^2(S)$.

Now we use Lemma 2.3 and estimate (2.5) to conclude that

$$\max_{0\leqslant t\leqslant T} \|\mathbf{v}^\varepsilon(t)\|_{2,S}^2 \leqslant \varepsilon C_0. \tag{5.4}$$

In fact, we may prove it for each facet separately. Considering, for example, the facet $S_{3,0} = \{x_3 = 0, x' = (x_1,x_2) \in (0,1)\times(0,1)\}$ one has

$$\begin{aligned} |\mathbf{v}^\varepsilon(x',0,t)|^2 &= |\mathbf{v}^\varepsilon(x',x_3,t)|^2 + 2\int_0^{x_3} \mathbf{v}^\varepsilon(x',y_3,t)\frac{\partial \mathbf{v}^\varepsilon}{\partial y_3}(x',y_3,t)\,\mathrm{d}y_3 \\ &\leqslant |\mathbf{v}^\varepsilon(x',x_3,t)|^2 + 2\left(\int_0^\varepsilon |\mathbf{v}^\varepsilon(x',y_3,t)|^2\,\mathrm{d}y_3\right)^{1/2}\left(\int_0^\varepsilon \left|\frac{\partial \mathbf{v}^\varepsilon}{\partial y_3}(x',y_3,t)\right|^2 \mathrm{d}y_3\right)^{1/2} \end{aligned}$$

and consequently, after integration over $S_{3,0}$ and interval $x_3 \in (0,\varepsilon)$,

$$\varepsilon \int_{S_{3,0}} |\mathbf{v}^\varepsilon|^2 \,\mathrm{d}x' \leqslant \int_{Q^\varepsilon} |\mathbf{v}^\varepsilon|^2 \,\mathrm{d}x + 2\varepsilon \left(\int_{Q^\varepsilon} |\mathbf{v}^\varepsilon|^2 \,\mathrm{d}x \right)^{1/2} \left(\int_\Omega |\nabla \mathbf{v}^\varepsilon|^2 \,\mathrm{d}x \right)^{1/2}.$$

Using estimates (2.5) and (A.3) we finally get estimate (5.4), which means that

$$\mathbf{v}^\varepsilon(\cdot,t) \to 0 \quad \text{strongly in } L^2(S)$$

as $\varepsilon \searrow 0$ for all $t \in [0,T]$ and that $\mathbf{v} = 0$ on the boundary S. □

Coming back to the relation (4.1) we put

$$\mathbf{v}^\varepsilon(\mathbf{x},t) = \int_0^t \mathbf{u}^\varepsilon(\mathbf{x},\tau)\,\mathrm{d}\tau.$$

Then $\mathbf{v}^\varepsilon$ satisfies all conditions of the Lemma A.2,

$$\mathbf{v}^\varepsilon \to \mathbf{v}, \qquad \mathbf{u}^\varepsilon \to \mathbf{u} \quad \text{strongly in } L^2(\Omega_T)$$

as $\varepsilon \searrow 0$,

$$\mathbf{v}(\mathbf{x},t) = \int_0^t \mathbf{u}(\mathbf{x},\tau)\,\mathrm{d}\tau$$

and

$$\mathbf{v}(\cdot,t) \in \overset{\circ}{W}{}_2^1(\Omega) \quad \text{for all } t \in [0,T].$$

On the other hand,

$$\nabla \mathbf{u}^\varepsilon \to 0 \quad \text{strongly in } L^2(\Omega_T) \text{ as } \varepsilon \searrow 0,$$

which implies

$$\nabla \mathbf{v}^\varepsilon \to 0 \quad \text{strongly in } L^2(\Omega_T) \text{ as } \varepsilon \searrow 0.$$

Therefore, $\mathbf{v} = 0$ and, consequently, $\mathbf{u} = 0$.

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