Mathematical Modeling of Short-Time Filtration and Acoustic Processes in Porous Media[1]

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This paper considers the joint motion problem of a deformable solid perforated by a system of channels (pores), which are filled with a fluid or a gas. Such media are called elastic porous media; they approximate fairly well real-life consolidated grounds. The solid component of such a medium is called the skeleton of the ground, and the domain occupied by the fluid is called the porous space. In dimensionless (not marked by asterisks) variables, we have

$$\mathbf{x}_* = \mathbf{x}L, \quad t_* = t\tau, \quad \mathbf{w}_* = \mathbf{w}L, \quad \mathbf{F}_* = \mathbf{F}g,$$
$$\rho_f^* = \rho_f\rho_0, \quad \rho_s^* = \rho_s\rho_0;$$

the differential equations for the dimensionless displacement vector $\mathbf{w}$ in the domain Ω at $t > 0$ have the form

$$\alpha_\tau\bar{\rho}\frac{\partial^2\mathbf{w}}{\partial t^2} = \mathrm{div}_x\mathbb{P} + \bar{\rho}\mathbf{F}, \qquad (1)$$

where the stress tensor $\mathbb{P} = \bar{\chi}\,\mathbb{P}^f + (1-\bar{\chi})\mathbb{P}^s$ of the continuous medium coincides with the tensor $\mathbb{P}^s = \alpha_\lambda\mathbb{D}(x, \mathbf{w}) + \alpha_\eta\mathrm{div}_x\mathbf{w}\mathbb{I}$ of elastic potentials in the solid skeleton (here, $\mathbb{I}$ is the spherical tensor) and with the tensor $\mathbb{P}^f = \alpha_\mu\mathbb{D}\left(x, \frac{\partial\mathbf{w}}{\partial t}\right) + \left(-p + \alpha_\nu\mathrm{div}_x\frac{\partial\mathbf{w}}{\partial t}\right)\mathbb{I}$ of viscous stresses in the porous space, and

$$D(x, \mathbf{u}) = \frac{1}{2}(\nabla_x\mathbf{u} + (\nabla_x\mathbf{u})^T), \quad \bar{\rho} = \bar{\chi}\rho_f + (1-\bar{\chi})\rho_s.$$

The characteristic function $\bar{\chi}(\mathbf{x})$ of the porous space and the dimensionless vector $\mathbf{F} = \mathbf{F}(\mathbf{x}, t)$ of distributed mass forces are assumed to be known; the pressure p is determined from the continuity equation

$$p = -\alpha_p\mathrm{div}_x\mathbf{w}. \qquad (2)$$

The dimensionless constants α_i ($i = \tau, \nu, \mu, \ldots$) are defined by

$$\alpha_\tau = \frac{L}{g\tau^2}, \quad \alpha_\nu = \frac{\nu}{\tau Lg\rho_0}, \quad \alpha_\mu = \frac{2\mu}{\tau Lg\rho_0},$$
$$\alpha_p = \frac{c^2\rho_f}{Lg}, \quad \alpha_\eta = \frac{\eta}{Lg\rho_0}, \quad \alpha_\lambda = \frac{2\lambda}{Lg\rho_0},$$

where L is the characteristic macroscopic size (the diameter of the domain under consideration), τ is the characteristic time of physical processes, ρ_0 is the mean density of water at atmospheric pressure, ρ_f and ρ_s are the mean dimensionless densities of the liquid and solid phases (related to the mean water density ρ_0), respectively, g is the acceleration due to gravity, p_0 is the atmospheric pressure, μ is the viscosity of the fluid (gas), ν is the volume viscosity of the fluid (gas), λ and η are the Lame elastic constants, and c is the speed of sound in the fluid under consideration. The problem is supplemented by the homogeneous initial and boundary conditions

$$\mathbf{w}|_{t=0} = 0, \quad \left.\frac{\partial\mathbf{w}}{\partial t}\right|_{t=0} = 0, \quad \mathbf{w}|_S = 0, \quad S = \partial\Omega. \qquad (3)$$

The mathematical model (1)–(3) is commonly accepted (see [1–6]) and contains the natural small parameter ε, which equals the ratio of the mean size l of the pores to the characteristic size L of the domain: $\varepsilon = \frac{l}{L}$. Thus, it is reasonable to seeklimiting regimes in the exact model as the small parameter tends to zero. Such an approximation substantially simplifies the initial problem, while retaining all of its basic properties. But even in the presence of a small parameter, the problem is too difficult, and some additional simplifying assumptions are necessary. One of the possible geomet-

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ric assumptions is that the porous space is periodic. Analytically, this is equivalent to the equality

$$\bar{\chi}(\mathbf{x}) = \chi^{\varepsilon}(\mathbf{x}) = \chi\left(\frac{\mathbf{x}}{\varepsilon}\right),$$

where $\chi(\mathbf{y})$ is a given 1-periodic function determining the elementary cell of the porous space.

The limiting regimes in the initial-boundary value problem (1)–(3) for

$$\tau_0, \mu_0, \lambda_0^{-1}, \nu_0, p_*^{-1}, \eta_0^{-1} < \infty,$$

where

$$\lim_{\varepsilon \searrow 0}\alpha_\mu(\varepsilon) = \mu_0, \quad \lim_{\varepsilon \searrow 0}\alpha_\lambda(\varepsilon) = \lambda_0,$$

$$\lim_{\varepsilon \searrow 0}\alpha_\tau(\varepsilon) = \tau_0, \quad \lim_{\varepsilon \searrow 0}\alpha_\nu(\varepsilon) = \nu_0,$$

$$\lim_{\varepsilon \searrow 0}\alpha_p(\varepsilon) = p_*, \quad \lim_{\varepsilon \searrow 0}\alpha_\eta(\varepsilon) = \eta_0,$$

$$\lim_{\varepsilon \searrow 0}\frac{\alpha_\mu}{\varepsilon^2} = \mu_1, \quad \lim_{\varepsilon \searrow 0}\frac{\alpha_\lambda}{\varepsilon^2} = \lambda_1$$

were studied by many authors (see [1–6]). Thus, in [1–5], the special cases where $\tau_0 = 1$, $\mu_0 < \infty$, and $0 < \lambda_0 < \infty$ were studied. The most complete results were obtained in [6]. In particular, it was shown in [6] that, depending on the relation between the dimensionless parameters, the limiting regimes are described either by the Biot system of equations (if $\mu_0 = 0$; $\tau_0 = 0$; and $0 < \mu_1, \lambda_0 < \infty$), by the Lame system of anisotropic equations (if the porous space is disconnected or $\mu_0 = 0$, $\mu_1 = \infty$, and $\lambda_0 < \infty$), by the Lame anisotropic system for the solid component coupled with the acoustic equations for the liquid component (if $\mu_0 = 0$; $\tau_0 > 0$; and $\lambda_0, \mu_1 < \infty$), by the filtration equations (the Darcy law) or the acoustic equations for the liquid component in the first approximation (in this case, the solid skeleton is assumed to be fixed) and the Lame anisotropic system for the solid skeleton in the second approximation (if $\lambda_0 = \infty$), or by the viscoelasticity system (if $0 < \mu_0$ and $\lambda_0 < \infty$).

In this paper, we complete the study of [6] in the cases where

$$p_*^{-1}, \eta_0^{-1}, \mu_0, \nu_0 < \infty, \quad \tau_0 = 1, \quad \lambda_0 = 0. \tag{4}$$

These cases include, in particular, the situation in which the dimensionless time of the process is small ($\tau_0 = \infty$); it suffices to renormalize the displacement as $\mathbf{w} \to \alpha_\tau\mathbf{w}$. They model short-time processes, such as, e.g., hydraulic fracturing.

We show that the limiting regimes are described by one of the following averaged systems of equations: the anisotropic system of Stokes equations for the liquid component coupled with the acoustics equations for the solid component (if $\mu_0 > 0$ and $\lambda_1 < \infty$); the anisotropic Stokes system for a one-velocity continuum (if $\mu_0 > 0$ and $\lambda_1 = \infty$); and various versions of the acoustics equations for a two- or one-velocity continuum (if $\mu_0 = 0$). The proofs systematically use Nguetseng's two-scale convergence method [7], and the derivation of the averaged equations relies on the information contained in the coefficients, which depend on the fast variable $\mathbf{y} = \frac{\mathbf{x}}{\varepsilon}$. This gives a solution to the corresponding microscopic equations on the elementary cell $Y = (0, 1) \times (0, 1) \times (0, 1)$. The structure of this cell determines the geometry of the solid skeleton and the porous space. If $Y = Y_s \cup Y_f \cup \gamma$, then the solid component of the medium is a periodic repetition of the elementary cell εY_s, the porous space is a periodic repetition of the cell εY_f, and the boundary between the solid skeleton and the porous space is a periodic repetition of the boundary $\varepsilon\gamma$. By analogy with the pressure p in the liquid phase, we introduce the new unknown functions $q = p + \frac{\alpha_\nu}{\alpha_p}\frac{\partial p}{\partial t}$ and $\pi = -\alpha_\eta \operatorname{div}_x \mathbf{w}$, which we also call pressures.

We make the following assumption.

Assumption 1. The boundary γ is a Lipschitz surface, the porous space and the solid skeleton are connected sets, and the functions $\mathbf{F}$, $\frac{\partial \mathbf{F}}{\partial t}$, and $\frac{\partial^2 \mathbf{F}}{\partial t^2}$ are bounded in $L^2(\Omega)$.

Under this assumption, the following theorem is valid.

Theorem 1. *For any* $\varepsilon > 0$, *problem* (1)–(3) *has a unique solution* $\{\mathbf{w}^\varepsilon, p^\varepsilon, q^\varepsilon, \pi^\varepsilon\}$ *on an arbitrary time interval* $[0, T]$, *and the sequences* $\{\mathbf{w}^\varepsilon\}$, $\{\chi\mathbf{w}^\varepsilon\}$, $\{(1 - \chi)\mathbf{w}^\varepsilon\}$, $\{p^\varepsilon\}$, $\{q^\varepsilon\}$, and $\{\pi^\varepsilon\}$ converge as $\varepsilon \searrow 0$ weakly in $L^2(\Omega_T)$ to functions $\mathbf{w}$, $\mathbf{w}^f$, $\mathbf{w}^s$, p, q, and π, respectively.

(I) Let $\mu_0 > 0$. Then, each function $\frac{\partial \mathbf{w}^\varepsilon}{\partial t}$ admits an extension $\mathbf{v}^\varepsilon$ from the porous space Ω_f^ε to the domain Ω for all $t \in (0, T)$, so that the sequence $\{\mathbf{v}^\varepsilon\}$ converges as $\varepsilon \searrow 0$ strongly in $L^2(\Omega_T)$ and weakly in $L^2((0, T); W_2^1(\Omega))$ to a function $\mathbf{v}$.

(i) If $\lambda_1 = \infty$, then $\frac{\partial \mathbf{w}^s}{\partial t} = (1 - m)\mathbf{v} = (1 - m)\frac{\partial \mathbf{w}}{\partial t}$, and the weak and strong limits q, p, π, and $\mathbf{v}$ satisfy the following initial-boundary value problem on the domain Ω_T:

$$\hat{\rho}\frac{\partial \mathbf{v}}{\partial t} = \operatorname{div}_x\left\{\mu_0 A_0^f : D(x, \mathbf{v}) + B_0^f\pi + B_1^f \operatorname{div}_c \mathbf{v} + \int_0^t B_2^f(t-\tau)\operatorname{div}_x \mathbf{v}(\mathbf{x}, \tau)d\tau\right\} - \nabla(q+\pi) + \hat{\rho}\mathbf{F}, \quad (5)$$

$$p_*^{-1}\frac{\partial p}{\partial t} + C_0^f : D(x, \mathbf{v}) + a_0^f\pi + (a_1^f + m)\operatorname{div}_x \mathbf{v} + \int_0^t a_2^f(t-\tau)\operatorname{div}_x \mathbf{v}(\mathbf{x}, \tau)d\tau = 0, \quad (6)$$

$$q = p + \frac{\nu_0}{p_*}\frac{\partial p}{\partial t}, \quad \frac{1}{p_*}\frac{\partial p}{\partial t} + \frac{1}{\eta_0}\frac{\partial \pi}{\partial t} + \operatorname{div}_x \mathbf{v} = 0, \quad (7)$$

where $\hat{\rho} = m\rho_f + (1-m)\rho_s$, $m = \int_Y \chi \, dy$, and the symmetric strictly positive definite constant fourth-rank tensor A_0^f, the matrices C_0^f, B_0^f, B_1^f, and $B_2^f(t)$, and the scalars a_0^f, a_1^f, and $a_2^f(t)$ are determined by solving periodic boundary value problems on the elementary cell Y_f.

The differential equations (5) are supplemented by the homogeneous initial and boundary conditions (3).

(ii) If $\lambda_1 < \infty$, then the weak and strong limits $\mathbf{w}^s$, q, p, π, and $\mathbf{v}$ satisfy the initial-boundary value problem on Ω_T consisting of the anisotropic Stokes equations

$$\rho_f m\frac{\partial \mathbf{v}}{\partial t} + \rho_s\frac{\partial^2 \mathbf{w}^s}{\partial t^2} + \nabla(q+\pi) - \hat{\rho}\mathbf{F} = \operatorname{div}_x\left\{B_0^f\pi + \mu_0 A_0^f : D(x, \mathbf{v}) + B_1^f\operatorname{div}_x \mathbf{v} + \int_0^t B_2^f(t-\tau)\operatorname{div}_x \mathbf{v}(\mathbf{x}, \tau)d\tau\right\}, \quad (8)$$

$$p_*^{-1}\frac{\partial p}{\partial t} + C_0^f : D(x, \mathbf{v}) + a_0^f\pi + (a_1^f + m)\operatorname{div}_x \mathbf{v} + \int_0^t a_2^f(t-\tau)\operatorname{div}_x \mathbf{v}(\mathbf{x}, \tau)d\tau = 0, \quad (9)$$

$$q = p + \frac{\nu_0}{p_*}\frac{\partial p}{\partial t} \quad (10)$$

for the liquid component coupled with the continuity equation

$$\frac{1}{p_*}\frac{\partial p}{\partial t} + \frac{1}{\eta_0}\frac{\partial \pi}{\partial t} + \operatorname{div}_x\frac{\partial \mathbf{w}^s}{\partial t} + m\operatorname{div}_x \mathbf{v} = 0 \quad (11)$$

by the relation

$$\frac{\partial \mathbf{w}^s}{\partial t} = \mathbf{v}(\mathbf{x}, t) + \int_0^t B_1^s(t-\tau)\cdot \mathbf{z}(\mathbf{x}, \tau)d\tau, \quad (12)$$

$$\mathbf{z}(\mathbf{x}, t) = -\frac{1}{1-m}\nabla_x\pi(\mathbf{x}, t) + \rho_s\mathbf{F}(\mathbf{x}, t) - \rho_s\frac{\partial \mathbf{v}}{\partial t}(\mathbf{x}, t)$$

(if $\lambda_1 > 0$) or by the momentum conservation law in the form

$$\rho_s\frac{\partial^2 \mathbf{w}^s}{\partial t^2} = \rho_s B_2^s\frac{\partial \mathbf{v}}{\partial t} + ((1-m)I - B_2^s)\left(-\frac{1}{1-m}\nabla_x\pi + \rho_s\mathbf{F}\right) \quad (13)$$

for the solid component (if $\lambda_1 = 0$). The problem is supplemented by the initial and boundary conditions (3) on the velocity $\mathbf{v}$ of the liquid component, the homogeneous initial conditions (3), and the boundary condition

$$\mathbf{w}^s(\mathbf{x}, t)\cdot \mathbf{n}(\mathbf{x}) = 0, \quad (\mathbf{x}, t) \in S = \partial\Omega, \quad t > 0 \quad (14)$$

on the displacement $\mathbf{w}^s$ of the solid component. In Eqs. (11)–(13), $\mathbf{n}(\mathbf{x})$ is the unit normal vector to S at $\mathbf{x} \in S$ and the matrices $B_1^s(t)$ and B_2^s are determined by solving periodic problems on the elementary cell Y_s. Moreover, the matrix $((1-m)I - B_2^s)$ is symmetric and positive definite.

(II) Suppose that $\mu_0 = 0$ and $p_*, \eta_0 < \infty$. Then, there exist functions $\mathbf{w}_f^\varepsilon$, $\mathbf{w}_s^\varepsilon \in L^\infty(0, T; W_2^1(\Omega))$ such that $\mathbf{w}_f^\varepsilon = \mathbf{w}^\varepsilon$ on $\Omega_f^\varepsilon \times (0, T)$, $\mathbf{w}_s^\varepsilon = \mathbf{w}^\varepsilon$ on $\Omega_s^\varepsilon(0, T)$ ($\Omega_s^\varepsilon = \Omega\backslash\overline{\Omega}_f^\varepsilon$), and the sequences $\{\mathbf{w}_f^\varepsilon\}$ and $\{\mathbf{w}_s^\varepsilon\}$ converge as $\varepsilon \searrow 0$ weakly in $L^2(\Omega_T)$ to functions $\mathbf{w}_f$ and $\mathbf{w}_s$, respectively. Moreover,

(iii) if $\mu_1 = \lambda_1 = \infty$, then $\mathbf{w}_f = \mathbf{w}_s = \mathbf{w}$ and, on the domain Ω_T, the functions $\mathbf{w}$, p, q, and π satisfy the acoustic system

$$\hat{\rho}\frac{\partial^2 \mathbf{w}}{\partial t^2} = -\frac{1}{(1-m)}\nabla_x\pi + \hat{\rho}\mathbf{F}, \quad (15)$$

$$\frac{1}{p_*}p + \frac{1}{\eta_0}\pi + \operatorname{div}_x \mathbf{w} = 0, \quad (16)$$

$$q = p + \frac{\nu_0}{p_*}\frac{\partial p}{\partial t}, \quad \frac{1}{m}q = \frac{1}{1-m}\pi, \quad (17)$$

the homogeneous initial conditions (3), and the homogeneous boundary condition (14);

(iv) if $\mu_1 = \infty$ and $\lambda_1 < \infty$, then, on Ω_T, the functions $\mathbf{w} = \mathbf{w}_f$, $\mathbf{w}^s$, p, q, and π satisfy the acoustic system consisting of the state equation (17), the momentum preservation law

$$\rho_f m\frac{\partial^2 \mathbf{w}_f}{\partial t^2} + \rho_s\frac{\partial^2 \mathbf{w}^s}{\partial t^2} = -\frac{1}{(1-m)}\nabla_x \pi + \hat{\rho}\mathbf{F} \quad (18)$$

for the liquid component, the continuity equation

$$\frac{1}{p_*}p + \frac{1}{\eta_0}\pi + m\,\mathrm{div}_x \mathbf{w}_f + \mathrm{div}_x \mathbf{w}^s = 0, \quad (19)$$

and the relation

$$\frac{\partial \mathbf{w}^s}{\partial t} = (1-m)\frac{\partial \mathbf{w}_f}{\partial t} + \int_0^t B_1^s(t-\tau)\cdot \mathbf{z}^s(\mathbf{x},\tau)d\tau, \quad (20)$$

$$\mathbf{z}^s(\mathbf{x}, t) = -\frac{1}{1-m}\nabla_x \pi(\mathbf{x}, t) + \rho_s \mathbf{F}(\mathbf{x}, t) - \rho_s\frac{\partial^2 \mathbf{w}_f}{\partial t^2}(\mathbf{x}, t)$$

(if $\lambda_1 > 0$) or the momentum conservation law in the form

$$\rho_s\frac{\partial^2 \mathbf{w}^s}{\partial t^2} = \rho_s B_2^s\frac{\partial^2 \mathbf{w}_f}{\partial t^2} + ((1-m)I - B_2^s)\cdot\left(-\frac{1}{1-m}\nabla_x\pi + \rho_s\mathbf{F}\right) \quad (21)$$

for the solid component (if $\lambda_1 = 0$). Problem (17), (18)–(21) is supplemented by the initial conditions (3) on the displacements in the liquid and solid components and the boundary condition (14) for the displacements $\mathbf{w} = m\mathbf{w}_f + \mathbf{w}^s$.

(v) If $\mu_1 < \infty$ and $\lambda_1 = \infty$, then the functions $\mathbf{w}^f$, $\mathbf{w} = \mathbf{w}_s$, p, q, and π satisfy the acoustic system on Ω_T, which consists of the state equation (17), the momentum conservation law for the solid component

$$\rho_f\frac{\partial^2 \mathbf{w}^f}{\partial t^2} + \rho_s(1-m)\frac{\partial^2 \mathbf{w}_s}{\partial t^2} = -\frac{1}{(1-m)}\nabla_x\pi + \hat{\rho}\mathbf{F}, \quad (22)$$

the continuity equation

$$\frac{1}{p_*}p + \frac{1}{\eta_0}\pi + \mathrm{div}_x\mathbf{w}^f + (1-m)\mathrm{div}_x\mathbf{w}_s = 0, \quad (23)$$

and the relation

$$\frac{\partial \mathbf{w}^f}{\partial t} = m\frac{\partial \mathbf{w}_s}{\partial t} + \int_0^t B_1^f(t-\tau)\cdot\mathbf{z}^f(\mathbf{x},\tau)d\tau, \quad (24)$$

$$\mathbf{z}^f(\mathbf{x}, t) = -\frac{1}{m}\nabla_x q(\mathbf{x}, t) + \rho_f\mathbf{F}(\mathbf{x}, t) - \rho_f\frac{\partial^2 \mathbf{w}_s}{\partial t^2}(\mathbf{x}, t)$$

(if $\mu_1 > 0$) or the momentum conservation law in the form

$$\rho_f\frac{\partial^2 \mathbf{w}^f}{\partial t^2} = \rho_f B_2^f\frac{\partial^2 \mathbf{w}_s}{\partial t^2} + \left((mI - B_2^f)\cdot\left(-\frac{1}{m}\nabla_x q + \rho_f\mathbf{F}\right)\right) \quad (25)$$

for the liquid component (if $\mu_1 = 0$). Problem (17), (22)–(25) is supplemented by the initial conditions (3) on the displacements in the liquid and solid components and the boundary condition (14) on the displacements $\mathbf{w} = \mathbf{w}^f + (1-m)\mathbf{w}_s$.

In (24) and (25), the matrices $B_1^f(t)$ and B_2^f are determined by solving periodic problems on the elementary cell Y_f. Moreover, the matrix $(mI - B_2^f)$ is symmetric and positive definite.

(vi) If $\mu_1 < \infty$ and $\lambda_1 < \infty$, then the functions $\mathbf{w}$, p, $q\pi$, and Ω_T satisfy the acoustic system on Ω_T, which consists of the continuity and state equations (16) and (17) and the relation

$$\frac{\partial \mathbf{w}}{\partial t} = \int_0^t B^\pi(t-\tau)\cdot\nabla\pi(\mathbf{x},\tau)d\tau + \mathbf{f}(\mathbf{x}, t), \quad (26)$$

where $B^\pi(t)$ and $\mathbf{f}(\mathbf{x}, t)$ are determined by solving periodic problems on the elementary cell Y.

Problem (16), (17), (26) is supplemented by the homogeneous initial and boundary conditions (3) and (14) on the displacements $\mathbf{w}$.

Note that the parameters of the model can take any values allowed by the conditions of the theorems. For example, if $p_*^{-1} = 0$ (which corresponds to an incompressible liquid component) or $\eta_0^{-1} = 0$ (i.e., the solid component is incompressible), then the terms containing these quantities vanish in all equations.

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