DERIVATION OF THE EQUATIONS OF NONISOTHERMAL ACOUSTICS IN ELASTIC POROUS MEDIA

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Abstract: We consider the problem of the joint motion of a thermoelastic solid skeleton and a viscous thermofluid in pores, when the physical process lasts for a few dozens of seconds. These problems arise in describing the propagation of acoustic waves. We rigorously derive the homogenized equations (i.e., the equations not containing fast oscillatory coefficients) which are different types of nonclassical acoustic equations depending on relations between the physical parameters and the homogenized heat equation. The proofs are based on Nguetseng's two-scale convergence method.

Keywords: nonisothermal Stokes and Lamé's equations, equations of acoustics, two-scale convergence, homogenization of periodic structures

Introduction

In this article we consider the problem of modeling rapid nonisothermal processes in an elastic deformable medium perforated by a system of channels and pores filled with a fluid (elastic porous media). The solid component of a medium of this kind is a *skeleton*; and the domain, filled with a fluid, is a *pore space.*

In dimensionless variables (without primes)

$$\boldsymbol{x}' = L\boldsymbol{x}, \quad t' = \tau t, \quad \boldsymbol{w}' = \frac{L^2}{g\tau^2}\boldsymbol{w}, \quad \theta' = \theta_* \frac{L}{g\tau^2}\theta$$

the differential equations of the model in the domain $\Omega \in \boldsymbol{R}^3$ for $t > 0$ for the small perturbations of the dimensionless displacement vector $\boldsymbol{w}$ and the dimensionless temperature θ have the form

$$\bar{\rho}\frac{\partial^2 \boldsymbol{w}}{\partial t^2} = \operatorname{div}\mathbb{P} + \bar{\rho}\boldsymbol{F}, \tag{0.1}$$

$$\bar{c}_p\frac{\partial \theta}{\partial t} = \operatorname{div}(\bar{\alpha}_{\varkappa}\nabla\theta) - \bar{\alpha}_\theta\frac{\partial}{\partial t}(\operatorname{div}\boldsymbol{w}) + \Psi, \tag{0.2}$$

$$\mathbb{P} = \bar{\chi}\mathbb{P}^f + (1-\bar{\chi})\mathbb{P}^s, \tag{0.3}$$

$$\mathbb{P}^f = \alpha_\mu\mathbb{D}\left(x, \frac{\partial \boldsymbol{w}}{\partial t}\right) - (p_f + \alpha_{\theta f}\theta)\mathbb{I}, \tag{0.4}$$

$$\mathbb{P}^s = \alpha_\lambda\mathbb{D}(x, \boldsymbol{w}) + (\alpha_\eta\operatorname{div}\boldsymbol{w} - \alpha_{\theta s}\theta)\mathbb{I}, \tag{0.5}$$

$$p_f + \bar{\chi}\alpha_p\operatorname{div}\boldsymbol{w} = 0. \tag{0.6}$$

Here and subsequently we use the notation:

$$\mathbb{D}(x, \boldsymbol{u}) = (1/2)(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T), \quad \bar{\rho} = \bar{\chi}\rho_f + (1-\bar{\chi})\rho_s,$$

$$\bar{c}_p = \bar{\chi} + (1-\bar{\chi})c_p^0, \quad \bar{\alpha}_\varkappa = \bar{\chi}\alpha_{\varkappa f} + (1-\bar{\chi})\alpha_{\varkappa s}, \quad \bar{\alpha}_\theta = \bar{\chi}\alpha_{\theta f} + (1-\bar{\chi})\alpha_{\theta s};$$

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$\mathbb{I}$ is the unit tensor; the given function $\bar{\chi}(\boldsymbol{x})$ is a characteristic function of the pore space $\Omega_f \subset \Omega$; the given function $\boldsymbol{F}(\boldsymbol{x},t)$ is a dimensionless vector of distributed mass forces; the function $\Psi(\boldsymbol{x},t)$ is a given density of heat sources; $\mathbb{P}^f$ is the stress tensor of the fluid; $\mathbb{P}^s$ is the stress tensor in the solid skeleton; and p_f is the pressure of the fluid. The differential equations (0.1)–(0.6) mean that the displacement vector $\boldsymbol{w}$ and the temperature θ satisfy the nonisothermal Stokes equations in Ω_f ($\bar{\chi}=1$) and the nonisothermal Lamé equations in $\Omega_s = \Omega \backslash \overline{\Omega}_f$ ($\bar{\chi}=0$).

On the common "solid skeleton–pore space" boundary $\Gamma = \partial\Omega_f \cap \partial\Omega_s$ the displacement vector $\boldsymbol{w}$, the temperature θ, and the stress tensor of the continuum medium satisfy the continuity conditions

$$[\boldsymbol{w}](\boldsymbol{x}_0,t)=0, \quad [\theta](\boldsymbol{x}_0,t)=0, \quad \boldsymbol{x}_0 \in \Gamma,\ t \geq 0, \tag{0.7}$$

and the momentum conservation law and the energy conservation law in the form

$$[\mathbb{P}\cdot \mathbf{n}](\boldsymbol{x}_0,t)=0, \quad [\bar{\alpha}_\varkappa \nabla\theta \cdot \mathbf{n}](\boldsymbol{x}_0,t)=0, \quad \boldsymbol{x}_0 \in \Gamma,\ t \geq 0, \tag{0.8}$$

where $\mathbf{n}(\boldsymbol{x}_0)$ is the unit normal to the boundary at $\boldsymbol{x}_0 \in \Gamma$ and

$$[\varphi](\boldsymbol{x}_0,t)=\varphi_{(s)}(\boldsymbol{x}_0,t)-\varphi_{(f)}(\boldsymbol{x}_0,t),$$
$$\varphi_{(s)}(\boldsymbol{x}_0,t)=\lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}_0\\ \boldsymbol{x}\in\Omega_s}}\varphi(\boldsymbol{x},t), \quad \varphi_{(f)}(\boldsymbol{x}_0,t)=\lim_{\substack{l\boldsymbol{x}\to\boldsymbol{x}_0\\ \boldsymbol{x}\in\Omega_f}}\varphi(\boldsymbol{x},t).$$

The derivation of (0.1)–(0.8) and description of strictly positive dimensionless constants may be found in [1]. In particular,

$$\alpha_\mu = \frac{2\mu\tau}{L^2\rho_0}, \quad \alpha_\lambda = \frac{2\lambda\tau^2}{L^2\rho_0}, \quad \alpha_p = \rho_f c_f^2 \frac{\tau^2}{L^2},$$
$$\alpha_\eta = \rho_s c_s^2 \frac{\tau^2}{L^2}, \quad \alpha_{\varkappa f} = \frac{\tau\varkappa_f}{L^2 c_{pf}}, \quad \alpha_{\varkappa s} = \frac{\tau\varkappa_s}{L^2 c_{pf}}, \quad c_p^0 = \frac{c_{ps}}{c_{pf}},$$

where μ is the viscosity of the fluid, λ is the Lamé constant, c_f is the speed of sound in the fluid, c_s is the speed of sound in the solid skeleton, L is the characteristic size of the domain under consideration, τ is a characteristic time of the given physical process, ρ_f and ρ_s are the mean dimensionless densities of the fluid and rigid phases, respectively, scaled with the mean density of water, and ρ_0 under the atmospheric pressure and the zero temperature at the Celsius scale of temperature, g is the value of acceleration of gravity, $\varkappa_f$ and $\varkappa_s$ are heat conductivities in the fluid and solid components respectively, and, finally, c_{pf} and c_{ps} are specific heat capacities in the fluid and solid components respectively.

The problem is endowed with the initial and boundary conditions

$$\boldsymbol{w}|_{t=0}=0, \quad \frac{\partial \boldsymbol{w}}{\partial t}\bigg|_{t=0}=0, \quad \theta|_{t=0}=0, \quad \boldsymbol{x}\in\Omega, \tag{0.9}$$

$$\boldsymbol{w}=0, \quad \theta=0, \quad \boldsymbol{x}\in S=\partial\Omega,\ t\geq 0. \tag{0.10}$$

The corresponding mathematical model, described by (0.1)–(0.10), contains the natural small parameter ε which is the pore characteristic size l divided by the characteristic size L of the domain under consideration: $\varepsilon = l/L$. Therefore it is very natural to find the limiting regimes as the small parameter tends to zero. This approximation significantly simplifies the original problem and at the same time preserves all of its main features. However, the problem remains too difficult even under the presence of the small parameter, and some additional simplifying assumptions are necessary. In terms of the geometrical properties of the medium, it is most expedient to simplify the problem by postulating that the pore space is periodic.

Assumption 1. *The domain $\Omega = (0,1)^3$ is a periodic repetition of the elementary cell $Y^\varepsilon = \varepsilon Y$, where $Y = (0,1)^3$ and the quantity $1/\varepsilon$ is integer so that Ω always contains an integer number of copies of the elementary cell Y^ε. Let Y_s be the "solid part" of Y, and let the "fluid part" Y_f be the open complement of Y_s in Y, while the boundary $\gamma = \partial Y_f \cap \partial Y_s$ between the "fluid" and "solid" components is a Lipschitz continuous surface.*

The pore space Ω_f^ε is a periodic repetition of the elementary cell εY_f, the solid skeleton Ω_s^ε is a periodic repetition of the elementary cell εY_s, and the Lipschitz continuous boundary $\Gamma^\varepsilon = \partial\Omega_s^\varepsilon \cap \partial\Omega_f^\varepsilon$ is a periodic repetition in Ω of the boundary $\varepsilon\gamma$.

The solid skeleton Ω_s^ε and the pore space Ω_f^ε are connected domains.

Under these assumptions

$$\bar{\chi}(\boldsymbol{x}) = \chi^\varepsilon(\boldsymbol{x}) = \chi(\boldsymbol{x}/\varepsilon), \quad \bar{\rho} = \rho^\varepsilon(\boldsymbol{x}) = \chi^\varepsilon(\boldsymbol{x})\rho_f + (1 - \chi^\varepsilon(\boldsymbol{x}))\rho_s,$$
$$\bar{c}_p = c_p^\varepsilon(\boldsymbol{x}) = \chi^\varepsilon(\boldsymbol{x}) + (1 - \chi^\varepsilon(\boldsymbol{x}))c_p^0, \quad \bar{\rho} = \rho^\varepsilon(\boldsymbol{x}) = \chi^\varepsilon(\boldsymbol{x})\rho_f + (1 - \chi^\varepsilon(\boldsymbol{x}))\rho_s,$$
$$\bar{\alpha}_\varkappa = \alpha_\varkappa^\varepsilon(\boldsymbol{x}) = \chi^\varepsilon(\boldsymbol{x})\alpha_{\varkappa f} + (1 - \chi^\varepsilon(\boldsymbol{x}))\alpha_{\varkappa s}, \; \bar{\alpha}_\theta = \alpha_\theta^\varepsilon(\boldsymbol{x}) = \chi^\varepsilon(\boldsymbol{x})\alpha_{\theta f} + (1 - \chi^\varepsilon(\boldsymbol{x}))\alpha_{\theta s},$$

where $\chi(\boldsymbol{y})$ is the characteristic function of Y_f in Y which defines the pore space. In the present model $\chi(\boldsymbol{y})$ is a given function.

We assume that all listed dimensionless parameters depend on the small parameter ε and the following finite or infinite limits exist:

$$\lim_{\varepsilon\searrow 0} \alpha_\mu(\varepsilon) = \mu_0, \quad \lim_{\varepsilon\searrow 0} \alpha_\lambda(\varepsilon) = \lambda_0, \quad \lim_{\varepsilon\searrow 0} \alpha_p(\varepsilon) = p_*,$$
$$\lim_{\varepsilon\searrow 0} \alpha_\eta(\varepsilon) = \eta_0, \quad \lim_{\varepsilon\searrow 0} \alpha_{\varkappa f}(\varepsilon) = \varkappa_{0f}, \quad \lim_{\varepsilon\searrow 0} \alpha_{\varkappa s}(\varepsilon) = \varkappa_{0s},$$
$$\lim_{\varepsilon\searrow 0} \frac{\alpha_\mu}{\varepsilon^2} = \mu_1, \quad \lim_{\varepsilon\searrow 0} \frac{\alpha_\lambda}{\varepsilon^2} = \lambda_1, \quad \lim_{\varepsilon\searrow 0} \frac{\alpha_{\varkappa i}}{\varepsilon^2} = \varkappa_{1i}, \; i = f, s.$$

In this article we consider the processes of seismic acoustic wave propagation when the characteristic time of the process τ is about some seconds (some dozens of seconds), while the characteristic size L of the domain under study is about thousands of meters (dozens of thousands of meters). Let us recall that the speed of sound in fluid and solid media vary between two and seven thousands meters per second. For example, if we choose the size L of the domain, then it is naturally to choose the characteristic time of the process τ such that the acoustic wave covers this distance L during the time τ, that is $\tau = L/c_s$. As a rule, for this kind of processes the quantities α_μ and α_λ are proportional to some positive powers of the parameter ε, and the quantities α_p and α_η are about unity.

Note that the purely mathematical task of finding all limiting regimes of the problem, depending on some small parameter, does not answer how to choose the most appropriate approximate model for a given physical process. In practice there is a given physical medium with a given set of physical parameters: the viscosity and Lame's coefficients, the densities of different components, the characteristic size of pores, the characteristic size L of the physical domain under study, and, finally, the characteristic time τ of the physical process. The smallness of the dimensionless parameter ε does not mean that we may change the size of pores and make it smaller. The size of pores is fixed. The only parameters that can be varied are L and τ. However, even these parameters have the natural bounds (obviously, L is limited by diameter of the Earth).

Thus, in a given physical situation we may find some rules for combining dimensionless criteria, which would suggest the choice of the form of the limiting regimes. As we already noted, this choice may fail to be unique. Therefore, to find all possible limiting regimes (homogenized systems) is very important from both mathematical and practical standpoints.

The most complete results for isothermal motion are obtained in [2, 3]. In this article we continue the efforts that were begun in [2–7], and consider the not studied yet case $\lambda_0 = 0$; namely, a situation when $\mu_0 = \lambda_0 = 0$.

We show that the homogenized equations for the exact model (0.1)–(0.10) are either different systems of nonisothermal acoustics for a one-velocity continuum ($\lambda_1 = \mu_1 = \infty$ or $\lambda_1 < \infty$ and $\mu_1 < \infty$) or the different systems of nonisothermal acoustics for a two-velocity continuum ($\lambda_1 < \infty$ and $\mu_1 = \infty$ or $\lambda_1 = \infty$ and $\mu_1 < \infty$).

This is a very interesting fact: initially a one-velocity continuum becomes a two-velocity continuum after the homogenization procedure, which appears to be the result of different smoothness of the solution in the solid and fluid components:

$$\int_{\Omega} \alpha_\mu(\varepsilon)\chi^\varepsilon |\nabla \boldsymbol{w}^\varepsilon|^2 dx \le C_0, \quad \int_{\Omega} \alpha_\lambda(\varepsilon)(1-\chi^\varepsilon)|\nabla \boldsymbol{w}^\varepsilon|^2\, dx \le C_0,$$

where C_0 is a constant independent of the small parameter ε. To preserve the best properties of the solution, we must use the well-known extension lemma [6, 7] and extend the solution from the solid part to the fluid part and conversely. At this stage, the dimensionless criteria μ_1 and λ_1 become crucial. Namely, let $\boldsymbol{w}^\varepsilon_f$ ($\boldsymbol{w}^\varepsilon_s$) be an extension of the fluid (solid) displacements to the solid (fluid) part, and let $\mu_1 = \lambda_1 = \infty$. Then the limiting (homogenized) system describes a one-velocity continuum. This is because of the fact that each of the sequences $\{\boldsymbol{w}^\varepsilon\}$, $\{\boldsymbol{w}^\varepsilon_f\}$, and $\{\boldsymbol{w}^\varepsilon_s\}$ two-scale converges to the same function independent of the fast variable. This statement follows easily from Nguetseng's theorem [8]. If $\mu_1 < \infty$ and $\lambda_1 = \infty$ ($\mu_1 = \infty$ and $\lambda_1 < \infty$) then the homogenized systems describe a two-velocity continuum because the sequences $\{\boldsymbol{w}^\varepsilon_f\}$ and $\{\boldsymbol{w}^\varepsilon_s\}$ may converge to different limits. If $\mu_1 < \infty$ and $\lambda_1 < \infty$ then the homogenizes systems again describe a one-velocity continuum.

In this paper we restrict ourself with the simplest model of viscous fluid with one viscosity coefficient. Accounting for the second viscosity does not essentially change both the structure of homogenized equations (see [2], where the complete model was considered) and the corresponding proofs. The summary of the papers and some preliminary results may be found in [2, 9, 10], and the basic notation of function spaces, in [10].

§ 1. The Main Results

There are various forms of representation of (0.1), (0.2) with (0.7), (0.8) equivalent in the sense of distributions. In what follows, it is convenient to write them as integral identities.

DEFINITION 1. We say that the functions $(\boldsymbol{w}^\varepsilon, \theta^\varepsilon, p^\varepsilon_f, p^\varepsilon_s)$ are a *generalized solution* of (0.1)–(0.10) if they satisfy the regularity conditions

$$\frac{\partial \boldsymbol{w}^\varepsilon}{\partial t},\ \chi^\varepsilon \nabla \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t},\ \nabla \boldsymbol{w}^\varepsilon,\ \nabla\theta^\varepsilon,\ \theta^\varepsilon,\ p^\varepsilon_f,\ p^\varepsilon_s \in L^2(\Omega_T)$$

in the domain $\Omega_T = \Omega \times (0, T)$, the boundary conditions (0.10), the equations

$$\frac{1}{\alpha_p} p^\varepsilon_f = -\chi^\varepsilon \operatorname{div} \boldsymbol{w}^\varepsilon, \tag{1.1}$$

$$\frac{1}{\alpha_\eta} p^\varepsilon_s = -(1-\chi^\varepsilon) \operatorname{div} \boldsymbol{w}^\varepsilon \tag{1.2}$$

almost everywhere in Ω_T, the integral identity

$$\int_{\Omega_T} \Big(\rho^\varepsilon \boldsymbol{w}^\varepsilon \cdot \frac{\partial^2 \boldsymbol{\varphi}}{\partial t^2} + \chi^\varepsilon \alpha_\mu \mathbb{D}\Big(x, \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}\Big) : \mathbb{D}(x, \boldsymbol{\varphi}) - \rho^\varepsilon \boldsymbol{F} \cdot \boldsymbol{\varphi}$$
$$+\big\{(1-\chi^\varepsilon)\alpha_\lambda \mathbb{D}(x, \boldsymbol{w}^\varepsilon) - \big(p^\varepsilon_f + p^\varepsilon_s + \alpha^\varepsilon_\theta \theta^\varepsilon\big)\mathbb{I}\big\} : \mathbb{D}(x, \boldsymbol{\varphi})\Big)\, dxdt = 0 \tag{1.3}$$

for all smooth vector-functions $\boldsymbol{\varphi}=\boldsymbol{\varphi}(\boldsymbol{x},t)$ such that

$$\boldsymbol{\varphi}(\boldsymbol{x},t)=0,\ \boldsymbol{x}\in S,\ t>0;\quad \boldsymbol{\varphi}(\boldsymbol{x},T)=\frac{\partial\boldsymbol{\varphi}}{\partial t}(\boldsymbol{x},T)=0,\ \boldsymbol{x}\in\Omega,$$

and the integral identity

$$\int_{\Omega_T}\Big(\big(c_p^{\varepsilon}\theta^{\varepsilon}+\alpha_{\theta}^{\varepsilon}\operatorname{div}\boldsymbol{w}^{\varepsilon}\big)\frac{\partial\xi}{\partial t}-\alpha_{\varkappa}^{\varepsilon}\nabla\theta^{\varepsilon}\cdot\nabla\xi+\Psi\xi\Big)\,dxdt=0 \tag{1.4}$$

for all smooth functions $\xi=\xi(\boldsymbol{x},t)$ such that

$$\xi(\boldsymbol{x},t)=0,\ \boldsymbol{x}\in S,\ t>0;\quad \xi(\boldsymbol{x},T)=0,\ \boldsymbol{x}\in\Omega.$$

We additionally introduced the new unknown function p_s^{ε} which we call the *solid pressure* by analogy with p_f^{ε}, and so we regard (1.2) as the *continuity equation in the solid skeleton.*

In (2.4), by $A:B$ we denote the convolution (or, equivalently, the inner tensor product) of the two second-rank tensors along both indices, i.e.,

$$\mathbb{A}:\mathbb{B}=\operatorname{tr}(\mathbb{B}^{*}\cdot\mathbb{A})=\sum_{i,j=1}^{3}A_{ij}B_{ji}.$$

We impose the following constraints.

Assumption 2. 1. *The functions Ψ and $|\boldsymbol{F}|$ belong to $L^2(\Omega_T)$.*
2. *The dimensionless parameters satisfy the conditions: p_*^{-1}, η_0^{-1}, p_*, η_0, $\varkappa_{0f}$, $\varkappa_{0s}<\infty$, $\lambda_0=\mu_0=0$.*

The following Theorems 1–9 are the main results of the paper.

Theorem 1. *For all $\varepsilon>0$ on an arbitrary time interval $[0,T]$ there exists a unique generalized solution of (0.1)–(0.10) and*

$$\max_{0\leq t\leq T}\Big(\Big\|\Big(\Big|\frac{\partial\boldsymbol{w}^{\varepsilon}}{\partial t}\Big|+|\theta^{\varepsilon}|\Big)(.,t)\Big\|_{2,\Omega}+\sqrt{\alpha_\lambda}\|\nabla\boldsymbol{w}^{\varepsilon}(.,t)\|_{2,\Omega}\Big)\leq C_0, \tag{1.5}$$

$$\sqrt{\alpha_\mu}\Big\|\chi^{\varepsilon}\nabla\frac{\partial\boldsymbol{w}^{\varepsilon}}{\partial t}\Big\|_{2,\Omega_T}+\|\sqrt{\alpha_{\varkappa}^{\varepsilon}}\nabla\theta^{\varepsilon}\|_{2,\Omega_T}\leq C_0, \tag{1.6}$$

$$\max_{0\leq t\leq T}\big(\|p_f^{\varepsilon}(.,t)\|_{2,\Omega}+\|p_s^{\varepsilon}(.,t)\|_{2,\Omega}\big)\leq C_0, \tag{1.7}$$

where C_0 is a constant independent of ε.

Theorem 2. *Let one of the conditions be valid*

$$0<\varkappa_{0f},\quad \varkappa_{0s}, \tag{1.8}$$

or

$$\varkappa_{0f}=\varkappa_{0s}=0,\quad \varkappa_{1f}=\varkappa_{1s}=\infty. \tag{1.9}$$

There exists a subsequence of small parameters $\{\varepsilon>0\}$ and functions $\boldsymbol{w}_f^{\varepsilon},\boldsymbol{w}_s^{\varepsilon}\in L^{\infty}\big(0,T;W_2^1(\Omega)\big)$ such that

$$\boldsymbol{w}_f^{\varepsilon}=\boldsymbol{w}^{\varepsilon}\ \text{in}\ \Omega_f^{\varepsilon}\times(0,T),\quad \boldsymbol{w}_s^{\varepsilon}=\boldsymbol{w}^{\varepsilon}\ \text{in}\ \Omega_s^{\varepsilon}\times(0,T)$$

and the sequences $\{\theta^{\varepsilon}\}$, $\{p_f^{\varepsilon}\}$, $\{p_s^{\varepsilon}\}$, $\{\boldsymbol{w}^{\varepsilon}\}$, $\{\chi^{\varepsilon}\boldsymbol{w}^{\varepsilon}\}$, $\{(1-\chi^{\varepsilon})\boldsymbol{w}^{\varepsilon}\}$, $\{\boldsymbol{w}_f^{\varepsilon}\}$, and $\{\boldsymbol{w}_s^{\varepsilon}\}$ converge as $\varepsilon\searrow 0$ weakly in $L^2(\Omega_T)$ to θ, p_f, p_s, $\boldsymbol{w}$, $\boldsymbol{w}^f$, $\boldsymbol{w}^s$, $\boldsymbol{w}_f$, and $\boldsymbol{w}_s$ respectively.

If (1.8) holds then $\{\theta^{\varepsilon}\}$ converges two-scale in $L^2(\Omega_T)$ and weakly in $L^2\big((0,T);\overset{\circ}{W}{}_2^1(\Omega)\big)$ to θ. If (1.9) holds then $\{\theta^{\varepsilon}\}$ converges two-scale in $L^2(\Omega_T)$ to θ.

Theorem 3. *Assume that the hypotheses in Theorem 2 hold and $\mu_1 = \lambda_1 = \infty$. Then $\boldsymbol{w}_f = \boldsymbol{w}_s = \boldsymbol{w}$ and θ, $\boldsymbol{w}$, p_f, and p_s satisfy in Ω_T the system of acoustic equations*

$$\hat{\rho}\frac{\partial^2 \boldsymbol{w}}{\partial t^2} = -\nabla\left(\frac{1}{m}p_f + \alpha_{\theta f}\theta\right) + \hat{\rho}\boldsymbol{F}, \tag{1.10}$$

$$\frac{1}{p_*}p_f + \frac{1}{\eta_0}p_s + \operatorname{div}\boldsymbol{w} = 0, \tag{1.11}$$

$$\alpha_{\theta f}\theta + \frac{1}{m}p_f = \alpha_{\theta s}\theta + \frac{1}{1-m}p_s, \tag{1.12}$$

the homogeneous initial conditions

$$\boldsymbol{w}(\boldsymbol{x},0) = \frac{\partial \boldsymbol{w}}{\partial t}(\boldsymbol{x},0) = 0, \quad \boldsymbol{x} \in \Omega, \tag{1.13}$$

and the homogeneous boundary condition

$$\boldsymbol{w}(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in S,\ t > 0, \tag{1.14}$$

where $m = \int_Y \chi\, dy$ is the porosity and $\hat{\rho} = m\rho_f + (1-m)\rho_s$ is the average density of the mixture.

Theorem 4. *Assume that the hypotheses in Theorem 2 hold and $\partial\boldsymbol{F}/\partial t \in L^2(\Omega_T)$, $\mu_1 = \infty$, and $0 < \lambda_1 < \infty$. Then θ, $\boldsymbol{w}^f = m\boldsymbol{w}_f$, $\boldsymbol{w}^s$, p_f and p_s satisfy in Ω_T the system of acoustic equations, consisting of the state equation (1.12) and the homogenized momentum balance equation in the form*

$$\rho_f m\frac{\partial^2 \boldsymbol{w}_f}{\partial t^2} + \rho_s\frac{\partial^2 \boldsymbol{w}^s}{\partial t^2} = -\nabla\left(\frac{1}{m}p_f + \alpha_{\theta f}\theta\right) + \hat{\rho}\boldsymbol{F} \tag{1.15}$$

for the fluid component, the continuity equation

$$\frac{1}{p_*}p_f + \frac{1}{\eta_0}p_s + m\operatorname{div}\boldsymbol{w}_f + \operatorname{div}\boldsymbol{w}^s = 0, \tag{1.16}$$

and the relation

$$\begin{aligned} \frac{\partial \boldsymbol{w}^s}{\partial t} &= (1-m)\frac{\partial \boldsymbol{w}_f}{\partial t} + \int_0^t \mathbb{B}_1^s(t-\tau)\cdot\boldsymbol{z}^s(\boldsymbol{x},\tau)d\tau, \\ \boldsymbol{z}^s(\boldsymbol{x},t) &= -\nabla\left(\frac{1}{m}p_f + \alpha_{\theta f}\theta\right) + \rho_s\boldsymbol{F} - \rho_s\frac{\partial^2 \boldsymbol{w}_f}{\partial t^2} \end{aligned} \tag{1.17}$$

for the solid component. Problem (1.12), (1.15)–(1.17) is supplemented with the homogeneous initial conditions (1.13) for displacements in the fluid and solid components and the homogeneous boundary condition (1.14) for the displacements $\boldsymbol{w} = m\boldsymbol{w}_f + \boldsymbol{w}^s$.

In (1.17) the matrix $\mathbb{B}_1^s(t)$ is defined below by (5.5).

Theorem 5. *Assume that the hypotheses in Theorem 2 hold and $\mu_1 = \infty$ and $\lambda_1 = 0$. Then θ, $\boldsymbol{w}^f = m\boldsymbol{w}_f$, $\boldsymbol{w}^s$, p_f, and p_s satisfy in Ω_T the system consisting of the acoustic equations (1.12), (1.15), (1.16) and the homogenized momentum balance equation for the solid component in the form*

$$\rho_s\frac{\partial^2 \boldsymbol{w}^s}{\partial t^2} = \rho_s\mathbb{B}_2^s\cdot\frac{\partial^2 \boldsymbol{w}_f}{\partial t^2} + \big((1-m)\mathbb{I} - \mathbb{B}_2^s\big)\cdot\left(-\nabla\left(\frac{1}{m}p_f + \alpha_{\theta f}\theta\right) + \rho_s\boldsymbol{F}\right). \tag{1.18}$$

The problem (1.12), (1.15), (1.16), and (1.18) is supplemented with the homogeneous initial conditions (1.13) for displacements in the solid and fluid components and the homogeneous boundary condition (1.14) for displacements $\boldsymbol{w} = m\boldsymbol{w}_f + \boldsymbol{w}^s$. In (1.18) the matrix $\mathbb{B}_2^s$ is given below by (5.7), where $\big((1-m)\mathbb{I} - \mathbb{B}_2^s\big)$ is symmetric and strictly positive definite.

Theorem 6. *Assume that the hypotheses in Theorem 2 hold and* $\partial \boldsymbol{F}/\partial t \in L^2(\Omega_T)$, $0 < \mu_1 < \infty$, *and* $\lambda_1 = \infty$. *Then* θ, $\boldsymbol{w}^f = m\boldsymbol{w}_f$, $\boldsymbol{w}^s$, p_f, *and* p_s *satisfy in* Ω_T *the system of acoustic equations, consisting of the state equation* (1.12) *and the homogenized momentum balance equation in the form*

$$\rho_f \frac{\partial^2 \boldsymbol{w}^f}{\partial t^2} + \rho_s(1-m)\frac{\partial^2 \boldsymbol{w}_s}{\partial t^2} = -\nabla\left(\frac{1}{m}p_f + \alpha_{\theta f}\theta\right) + \hat{\rho}\boldsymbol{F} \tag{1.19}$$

for the solid component, the continuity equation

$$\frac{1}{p_*}p_f + \frac{1}{\eta_0}p_s + \operatorname{div}\boldsymbol{w}^f + (1-m)\operatorname{div}\boldsymbol{w}_s = 0, \tag{1.20}$$

and the relation

$$\frac{\partial \boldsymbol{w}^f}{\partial t} = m\frac{\partial \boldsymbol{w}_s}{\partial t} + \int_0^t \mathbb{B}_1^f(t-\tau)\cdot \boldsymbol{z}^f(\boldsymbol{x},\tau)\,d\tau, \tag{1.21}$$

$$\boldsymbol{z}^f(\boldsymbol{x},t) = -\nabla\left(\frac{1}{m}p_f + \alpha_{\theta f}\theta\right) + \rho_f\boldsymbol{F} - \rho_f\frac{\partial^2 \boldsymbol{w}_s}{\partial t^2}$$

for the fluid component. The problem (1.12), (1.19)–(1.21) *is supplemented with the homogeneous initial conditions* (1.13) *for the displacements in the solid and fluid components and the homogeneous boundary condition* (1.14) *for the displacements* $\boldsymbol{w} = \boldsymbol{w}^f + (1-m)\boldsymbol{w}^s$.

In (1.21) *the matrix* $\mathbb{B}_1^f(t)$ *is defined below by* (6.1).

Theorem 7. *Assume that the hypotheses in Theorem 2 hold and* $\mu_1 = 0$ *and* $\lambda_1 = \infty$. *Then* θ, $\boldsymbol{w}^f = m\boldsymbol{w}_f$, $\boldsymbol{w}^s$, p_f, *and* p_s *satisfy in* Ω_T *the system of acoustic equations, consisting of* (1.12), (1.19), (1.20), *and the homogenized momentum balance equation in the form*

$$\rho_f \frac{\partial^2 \boldsymbol{w}^f}{\partial t^2} = \rho_f \mathbb{B}_2^f \cdot \frac{\partial^2 \boldsymbol{w}_s}{\partial t^2} + \left(m\mathbb{I} - \mathbb{B}_2^f\right)\cdot\left(-\nabla\left(\frac{1}{m}p_f + \alpha_{\theta f}\theta\right) + \rho_f\boldsymbol{F}\right) \tag{1.22}$$

for the fluid component. The problem (1.12), (1.19), (1.20), *and* (1.22) *is supplemented with the homogeneous initial conditions* (1.13) *for the displacements in the solid and fluid components and the homogeneous boundary condition* (1.14) *for the displacements* $\boldsymbol{w} = \boldsymbol{w}^f + (1-m)\boldsymbol{w}_s$. *In* (1.22) *the matrix* $\mathbb{B}_2^f$ *is defined below by* (6.2), *where* $\left(m\mathbb{I} - \mathbb{B}_2^f\right)$ *is symmetric and strictly positive definite.*

Theorem 8. *Assume that the hypotheses in Theorem 2 hold and* $\mu_1 < \infty$ *and* $\lambda_1 < \infty$. *Then functions* θ, $\boldsymbol{w}^f = m\boldsymbol{w}_f$, $\boldsymbol{w}^s$, p_f, *and* p_s *satisfy in* Ω_T *the system of acoustic equations, consisting of the continuity equation* (1.11), *the state equation* (1.12), *and the relation*

$$\frac{\partial \boldsymbol{w}}{\partial t} = \int_0^t \mathbb{B}(t-\tau)\cdot\left(\nabla\left(\frac{1}{m}p_f + \alpha_{\theta f}\theta\right)\right)(\boldsymbol{x},\tau)\,d\tau + \mathbf{f}(\boldsymbol{x},t), \tag{1.23}$$

where the matrix $\mathbb{B}(t)$ *and the vector* $\mathbf{f}(\boldsymbol{x},t)$ *are given below by* (7.5) *and* (7.6).

The problem (1.11), (1.12), *and* (1.23) *is supplemented with the homogeneous initial and boundary conditions* (1.13) *and* (1.14).

Theorem 9. *Assume that the hypotheses in Theorem 2 and condition* (1.8) *hold. Then each of the systems* (1.10)–(1.12), (1.15)–(1.18), (1.19)–(1.22), *and* (1.11), (1.12), *and* (1.23) *is supplemented with the homogenized heat equation*

$$\hat{c}_p\frac{\partial \theta}{\partial t} - \frac{\alpha_{\theta f}}{p_*}\frac{\partial p_f}{\partial t} - \frac{\alpha_{\theta s}}{\eta_0}\frac{\partial p_s}{\partial t} = \operatorname{div}\{\mathbb{B}^\theta\cdot\nabla\theta\} + \Psi \tag{1.24}$$

and the homogeneous initial and boundary conditions (0.9), (0.10) *for the limiting temperature* θ.

In (1.24) $\hat{c}_p = m + (1-m)c_p^0$, *the symmetric and strictly positive definite matrix* $\mathbb{B}^\theta$ *is given below by* (8.1).

If the condition (1.8) *holds instead of* (1.9) *then all systems above are supplemented with the equation*

$$\hat{c}_p\theta(\boldsymbol{x},t) = \frac{\alpha_{\theta f}}{p_*}p_f(\boldsymbol{x},t) + \frac{\alpha_{\theta s}}{\eta_0}p_s(\boldsymbol{x},t) + \int_0^t \Psi(\boldsymbol{x},\tau)\,d\tau. \tag{1.25}$$

§ 2. Proof of Theorem 1

The existence and uniqueness of a generalized solution to (0.1)–(0.10) are proved in [1].

For a formal derivation of (1.5) and (1.6) we consider the energy equality

$$\frac{d}{dt}\bigg\{\int_\Omega\Big(\rho^\varepsilon\Big(\frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}\Big)^2 + c_p^\varepsilon(\theta^\varepsilon)^2\Big)\,dx + \alpha_\lambda\int_\Omega(1-\chi^\varepsilon)\mathbb{D}(x,\boldsymbol{w}^\varepsilon) : \mathbb{D}(x,\boldsymbol{w}^\varepsilon)\,dx$$
$$+\alpha_p\int_\Omega\chi^\varepsilon(\operatorname{div}\boldsymbol{w}^\varepsilon)^2\,dx + \alpha_\eta\int_\Omega(1-\chi^\varepsilon)(\operatorname{div}\boldsymbol{w}^\varepsilon)^2\,dx\bigg\} + \int_\Omega\alpha_\varkappa^\varepsilon|\nabla\theta^\varepsilon|^2\,dx$$
$$+\alpha_\mu\int_\Omega\chi^\varepsilon\mathbb{D}\Big(x,\frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}\Big) : \mathbb{D}\Big(x,\frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}\Big)\,dx = \int_\Omega \boldsymbol{F}\cdot\frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}\,dx, \tag{2.1}$$

which is obtained in result of multiplication of the equation for $\boldsymbol{w}^\varepsilon$ by $\partial\boldsymbol{w}^\varepsilon/\partial t$, the equation for θ^ε by θ^ε, integration by parts, and summation.

The last identity (2.1) together with the Hölder and Gronwall inequalities implies the estimate

$$\max_{0<t<T}\bigg(\bigg\|\Big(\Big|\frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}\Big| + |\theta^\varepsilon|\Big)(.,t)\bigg\|_{2,\Omega} + \|(\sqrt{\alpha_\lambda}|\nabla\boldsymbol{w}^\varepsilon| + \sqrt{\alpha_\eta}|\operatorname{div}\boldsymbol{w}^\varepsilon|)(.,t)\|_{2,\Omega_s^\varepsilon}$$
$$+\sqrt{\alpha_p}\|\operatorname{div}\boldsymbol{w}^\varepsilon(.,t)\|_{2,\Omega_f^\varepsilon}\bigg) + \bigg\|\sqrt{\alpha_\varkappa^\varepsilon}|\nabla\theta^\varepsilon| + \sqrt{\alpha_\mu}\Big|\chi^\varepsilon\nabla\frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}\Big|\bigg\|_{2,\Omega_T} \leq C_0, \tag{2.2}$$

where C_0 is independent of ε. Estimates (1.5) and (1.6) are obvious from (2.2). Finally, (1.7) for pressures p_f^ε and p_s^ε follows from the continuity equations (1.1), (1.2), and (2.2).

The informal derivation of estimates is a little longer than the formal one and follows, for example, from the proof of existence of a generalized solution by Galerkin's method. The derivation of the energy identity for the approximate solutions is quite simple, and the estimate (2.2) for the limiting solution follows from the estimate (2.2) for the approximate solutions as a result of the limiting procedure.

Remark 2.1. All results of Theorem 1 are also valid for the inhomogeneous initial conditions

$$\boldsymbol{w}^\varepsilon(\boldsymbol{x},0) = \boldsymbol{w}_0^\varepsilon(\boldsymbol{x}), \quad \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}(\boldsymbol{x},0) = \boldsymbol{v}_0^\varepsilon(\boldsymbol{x}),$$

if the norms $\|\nabla\boldsymbol{w}_0^\varepsilon\|_{2,\Omega}$ and $\|\boldsymbol{v}_0^\varepsilon\|_{2,\Omega}$ are bounded uniformly in ε.

Remark 2.2. Let $\partial\boldsymbol{F}/\partial t \in L^2(\Omega_T)$. It is obvious that we can differentiate the equations and boundary conditions of (0.1)–(0.10). Therefore the time derivative $\boldsymbol{v}^\varepsilon = \partial\boldsymbol{w}^\varepsilon/\partial t$ of the generalized solution $\boldsymbol{w}^\varepsilon$ of the problem (0.1)–(0.10) is also a generalized solution to the same system with the right-hand side $\partial\boldsymbol{F}/\partial t$ and the nonhomogeneous initial conditions

$$\boldsymbol{v}^\varepsilon(\boldsymbol{x},0) = 0, \quad \frac{\partial \boldsymbol{v}^\varepsilon}{\partial t}(\boldsymbol{x},0) = \boldsymbol{F}(\boldsymbol{x},0).$$

Thus, by Remark 2.1

$$\max_{0<t<T}\bigg\|\frac{\partial^2 \boldsymbol{w}^\varepsilon}{\partial t^2}(.,t)\bigg\|_{2,\Omega} \leq C_0, \tag{2.3}$$

where C_0 is independent of ε.

§ 3. Proof of Theorem 2

By the extension results [11, 12] (also see [2]) there exist $\boldsymbol{w}_f^\varepsilon, \boldsymbol{w}_s^\varepsilon \in L^\infty\big((0,T); W_2^1(\Omega)\big)$ such that $\boldsymbol{w}_f^\varepsilon = \boldsymbol{w}^\varepsilon$ in $\Omega_f^\varepsilon \times (0,T)$, $\boldsymbol{w}_s^\varepsilon = \boldsymbol{w}^\varepsilon$ in $\Omega_s^\varepsilon \times (0,T)$, and $\|\boldsymbol{w}_i^\varepsilon\|_{2,\Omega} \le C\|\boldsymbol{w}^\varepsilon\|_{2,\Omega_i^\varepsilon}$, $\|\nabla \boldsymbol{w}_i^\varepsilon\|_{2,\Omega} \le C\|\nabla \boldsymbol{w}^\varepsilon\|_{2,\Omega_i^\varepsilon}$, $i = f, s$, where C is a constant independent of ε. By Theorem 1, the sequences $\{p_f^\varepsilon\}$, $\{p_s^\varepsilon\}$, $\{\boldsymbol{w}^\varepsilon\}$, $\{\boldsymbol{w}_f^\varepsilon\}$, $\{\boldsymbol{w}_s^\varepsilon\}$, $\{\sqrt{\alpha_\lambda}\nabla \boldsymbol{w}_s^\varepsilon\}$, and $\{\sqrt{\alpha_\mu}\nabla \boldsymbol{w}_f^\varepsilon\}$ are uniformly bounded in $L^2(\Omega_T)$ in ε. Hence there exists a subsequence of small parameters $\{\varepsilon > 0\}$ and functions p_f, p_s, $\boldsymbol{w}$, $\boldsymbol{w}_f$, and $\boldsymbol{w}_s$ such that

$$p_f^\varepsilon \rightharpoonup p_f, \quad p_s^\varepsilon \rightharpoonup p_s, \quad \boldsymbol{w}^\varepsilon \rightharpoonup \boldsymbol{w}, \quad \boldsymbol{w}_f^\varepsilon \rightharpoonup \boldsymbol{w}_f, \quad \boldsymbol{w}_s^\varepsilon \rightharpoonup \boldsymbol{w}_s \tag{3.1}$$

weakly in $L^2(\Omega_T)$ as $\varepsilon \searrow 0$.

Note also that

$$(1-\chi^\varepsilon)\alpha_\lambda D\big(x, \boldsymbol{w}_s^\varepsilon\big) \to 0, \quad \chi^\varepsilon \alpha_\mu D\big(x, \boldsymbol{w}_f^\varepsilon\big) \to 0 \tag{3.2}$$

strongly in $L^2(\Omega_T)$ and

$$\operatorname{div} \boldsymbol{w}^\varepsilon \rightharpoonup \operatorname{div} \boldsymbol{w}, \quad \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t} \rightharpoonup \frac{\partial \boldsymbol{w}}{\partial t}, \quad \frac{\partial \boldsymbol{w}_f^\varepsilon}{\partial t} \rightharpoonup \frac{\partial \boldsymbol{w}_f}{\partial t}, \quad \frac{\partial \boldsymbol{w}_s^\varepsilon}{\partial t} \rightharpoonup \frac{\partial \boldsymbol{w}_s}{\partial t}$$

weakly in $L^2(\Omega_T)$ as $\varepsilon \searrow 0$.

REMARK 3.1. If $\partial \boldsymbol{F}/\partial t \in L^2(\Omega_T)$ then by Remark 2.2 the second derivatives of $\boldsymbol{w}^\varepsilon$ with respect to time are uniformly bounded in ε and

$$\frac{\partial^2 \boldsymbol{w}^\varepsilon}{\partial t^2} \rightharpoonup \frac{\partial^2 \boldsymbol{w}}{\partial t^2}, \quad \frac{\partial^2 \boldsymbol{w}_f^\varepsilon}{\partial t^2} \rightharpoonup \frac{\partial^2 \boldsymbol{w}_f}{\partial t^2}, \quad \frac{\partial^2 \boldsymbol{w}_s^\varepsilon}{\partial t^2} \rightharpoonup \frac{\partial^2 \boldsymbol{w}_s}{\partial t^2}$$

weakly in $L^2(\Omega_T)$ as $\varepsilon \searrow 0$.

By Nguetseng's theorem [8] (also see [2]), there exist functions $P_f(\boldsymbol{x}, t, \boldsymbol{y})$, $P_s(\boldsymbol{x}, t, \boldsymbol{y})$, $\boldsymbol{W}(\boldsymbol{x}, t, \boldsymbol{y})$, $\boldsymbol{W}_f(\boldsymbol{x}, t, \boldsymbol{y})$, and $\boldsymbol{W}_s(\boldsymbol{x}, t, \boldsymbol{y})$ one-periodic in $\boldsymbol{y}$ and satisfying the condition that the sequences $\{p_f^\varepsilon\}$, $\{p_s^\varepsilon\}$, $\{\boldsymbol{w}^\varepsilon\}$, $\{\boldsymbol{w}_f^\varepsilon\}$, and $\{\boldsymbol{w}_s^\varepsilon\}$ two-scale converge in $L^2(\Omega_T)$ to $P_f(\boldsymbol{x}, t, \boldsymbol{y})$, $P_s(\boldsymbol{x}, t, \boldsymbol{y})$, $\boldsymbol{W}(\boldsymbol{x}, t, \boldsymbol{y})$, $\boldsymbol{W}_f(\boldsymbol{x}, t, \boldsymbol{y})$, and $\boldsymbol{W}_s(\boldsymbol{x}, t, \boldsymbol{y})$ respectively.

If (1.8) holds then by the boundedness of $\{\theta^\varepsilon\}$ in $L^2\big(0, T; W_2^1(\Omega)\big)$ there exist a subsequence of small parameters $\{\varepsilon > 0\}$, the functions $\theta \in L^2\big(0, T; \overset{\circ}{W}{}_2^1(\Omega)\big)$, and a one-periodic (in y) function $\Theta(\boldsymbol{x}, t, \boldsymbol{y})$ such that $\theta^\varepsilon \rightharpoonup \theta$ weakly in $L^2\big((0,T); \overset{\circ}{W}{}_2^1(\Omega)\big)$ as $\varepsilon \searrow 0$, and the sequences $\{\theta^\varepsilon\}$ and $\{\nabla\theta^\varepsilon\}$ two scale converge in $L^2(\Omega_T)$ respectively to θ, and $\nabla\theta + \nabla_y \Theta(\boldsymbol{x}, t, \boldsymbol{y})$. If (1.9) holds then the sequence $\{\theta^\varepsilon\}$ two-scale converges in $L^2(\Omega_T)$ to the proper weak limit θ. Namely, we have

Lemma 3.1. *Assuming* (1.8) *or* (1.9), *the sequence* $\{\theta^\varepsilon\}$ *two-scale converges in* $L^2(\Omega_T)$ (*up to some subsequence*) *to the proper weak limit* $\theta(\boldsymbol{x}, t)$.

PROOF. By Nguetseng's theorem [8] there exist a subsequence of $\{\varepsilon > 0\}$ and a one-periodic (in y) function $\Theta_0(\boldsymbol{x}, t, \boldsymbol{y})$ such that $\{\theta^\varepsilon\}$ two-scale converges to $\Theta_0(\boldsymbol{x}, t, \boldsymbol{y})$.

Let $\bar{\alpha}_\kappa^\varepsilon = \min\big(\alpha_{\varkappa f}^\varepsilon, \alpha_{\varkappa f}^\varepsilon\big)$ and let $\Psi(\boldsymbol{x}, t, \boldsymbol{y})$ be a smooth scalar function one-periodic in $\boldsymbol{y}$. The sequence $\{\sigma_j^\varepsilon\}$, where

$$\sigma_j^\varepsilon = \sqrt{\bar{\alpha}_\kappa^\varepsilon} \int_{\Omega_T} \frac{\partial \theta^\varepsilon}{\partial x_j}(\boldsymbol{x}, t) \Psi(\boldsymbol{x}, t, \boldsymbol{x}/\varepsilon)\, dxdt, \quad j = 1, 2, 3,$$

is bounded in ε. Therefore,

$$\int_{\Omega_T} \varepsilon \frac{\partial \theta^\varepsilon}{\partial x_j}(\boldsymbol{x}, t) \Psi(\boldsymbol{x}, t, \boldsymbol{x}/\varepsilon)\, dxdt = \frac{\varepsilon}{\sqrt{\bar{\alpha}_\kappa^\varepsilon}} \sigma_j^\varepsilon \to 0 \quad \text{as } \varepsilon \searrow 0,$$

which is equivalent to

$$\int\limits_{\Omega_T}\int\limits_{Y}\Theta_0(\boldsymbol{x},t,\boldsymbol{y})\frac{\partial\Psi}{\partial y_j}(\boldsymbol{x},t,\boldsymbol{y})\,dydxdt=0,\quad j=1,2,3,$$

or $\Theta_0(\boldsymbol{x},t,\boldsymbol{y})=\theta(\boldsymbol{x},t)$. $\square$

In the same way as in Lemma 3.1 we can prove

Lemma 3.2. *Let $\mu_1=\infty$ ($\lambda_1=\infty$). Then $\boldsymbol{W}_f(\boldsymbol{x},t,\boldsymbol{y})=\boldsymbol{w}_f(\boldsymbol{x},t)$, $\chi(\boldsymbol{y})\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})=\chi(\boldsymbol{y})\boldsymbol{w}_f(\boldsymbol{x},t)$, and $\boldsymbol{w}^f=\langle\boldsymbol{W}\rangle_{Y_f}=m\boldsymbol{w}_f$ ($\boldsymbol{W}_s(\boldsymbol{x},t,\boldsymbol{y})=\boldsymbol{w}_s(\boldsymbol{x},t)$, $(1-\chi(\boldsymbol{y}))\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})=(1-\chi(\boldsymbol{y}))\boldsymbol{w}_s(\boldsymbol{x},t)$, and $\boldsymbol{w}^s=\langle\boldsymbol{W}\rangle_{Y_s}=(1-m)\boldsymbol{w}_s$).*

§ 4. The Microscopic and Macroscopic Equations

Lemma 4.1. *For almost all $\boldsymbol{x}\in\Omega$ and $\boldsymbol{y}\in Y$ the weak and two-scale limits of the sequences $\{p_f^\varepsilon\}$, $\{p_s^\varepsilon\}$, $\{\boldsymbol{w}^\varepsilon\}$, $\{\boldsymbol{w}_f^\varepsilon\}$, and $\{\boldsymbol{w}_s^\varepsilon\}$ satisfy the relations*

$$P_f=\frac{\chi}{m}p_f,\quad P_s=\frac{1-\chi}{1-m}p_s,\tag{4.1}$$

$$\alpha_{\theta f}\theta+\frac{1}{m}p_f=\alpha_{\theta s}\theta+\frac{1}{1-m}p_s,\tag{4.2}$$

$$\frac{1}{p_*}p_f+\frac{1}{\eta_0}p_s+\operatorname{div}\,\boldsymbol{w}=0,\tag{4.3}$$

$$\boldsymbol{w}(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x})=0,\quad \boldsymbol{x}\in S,\tag{4.4}$$

$$\operatorname{div}_y\boldsymbol{W}=0,\tag{4.5}$$

$$\boldsymbol{W}=\chi\boldsymbol{W}_f+(1-\chi)\boldsymbol{W}_s,\tag{4.6}$$

where $\boldsymbol{n}(\boldsymbol{x})$ is the unit normal vector to S at $\boldsymbol{x}\in S$.

Proof. In order to prove (4.1), insert into (1.3) the test function in the form $\boldsymbol{\psi}^\varepsilon=\varepsilon\boldsymbol{\psi}(\boldsymbol{x},t,\boldsymbol{x}/\varepsilon)$, where $\boldsymbol{\psi}(\boldsymbol{x},t,\boldsymbol{y})$ is an arbitrary one-periodic function of $\boldsymbol{y}$ that is compactly supported in Y_f (or in Y_s, or in Y). Let, for example, $\boldsymbol{\psi}(\boldsymbol{x},t,\boldsymbol{y})$ be a compactly supported function in Y_f. Passing to the limit as $\varepsilon\searrow0$ and taking Lemma 3.1 into account, we obtain the integral identity

$$\int\limits_{\Omega_T}\int\limits_{Y}(\alpha_{\theta f}\theta(\boldsymbol{x},t)+P_f(\boldsymbol{x},t,\boldsymbol{y}))\operatorname{div}_y\,\boldsymbol{\psi}\,dydxdt=0$$

which is equivalent to the equality $\nabla_yP_f=0$, $\boldsymbol{y}\in Y_f$.

All other cases are considered in the same way. Gathering all together, we have

$$\nabla_yP_f=0,\ \boldsymbol{y}\in Y_f;\quad \nabla_yP_s=0,\ \boldsymbol{y}\in Y_s;\quad \nabla_y(\alpha_\theta(\boldsymbol{y})\theta+P_f+P_s)=0,\ \boldsymbol{y}\in Y,\tag{4.7}$$

where

$$\alpha_\theta(\boldsymbol{y})=\alpha_{\theta f}\chi(\boldsymbol{y})+\alpha_{\theta s}(1-\chi(\boldsymbol{y})).$$

Next fulfilling the two-scale passage to the limit in the equalities $(1-\chi^\varepsilon)p_f^\varepsilon=0$, $\chi^\varepsilon p_s^\varepsilon=0$, we arrive at the relations $(1-\chi)P_f=0$, $\chi P_s=0$ which, together with (4.7), prove (4.1).

Equation (4.2) follows from (4.1), Lemma 3.1, and the last relation in (4.7): the sequence $\{\alpha_\theta^\varepsilon\theta^\varepsilon+p_f^\varepsilon+p_s^\varepsilon\}$ two-scale converges to

$$\alpha_\theta(\boldsymbol{y})\theta(\boldsymbol{x},t)+\frac{\chi(\boldsymbol{y})}{m}p_f(\boldsymbol{x},t)+\frac{1-\chi(\boldsymbol{y})}{1-m}p_s(\boldsymbol{x},t)=(\hat{\alpha}_\theta\theta+p_f+p_s)(\boldsymbol{x},t),$$

where $\hat{\alpha}_\theta=m\alpha_{\theta f}+(1-m)\alpha_{\theta s}$.

Equations (4.3)–(4.5) appear as the result of the two-scale passage to the limit in the sum of the equations (1.1) and (1.2).

Finally, (4.6) is the result of the two-scale passage to the limit in the equality

$$\boldsymbol{w}^\varepsilon=\chi^\varepsilon\boldsymbol{w}^\varepsilon+(1-\chi^\varepsilon)\boldsymbol{w}^\varepsilon.\quad\square$$

Lemma 4.2. *For almost all* $(\boldsymbol{x},t)\in\Omega_T$ *and* $y\in Y$,

$$\rho_f\frac{\partial^2\boldsymbol{w}^f}{\partial t^2}+\rho_s\frac{\partial^2\boldsymbol{w}^s}{\partial t^2}=-\nabla\Big(\frac{1}{m}p_f+\alpha_{\theta f}\theta\Big)+\hat{\rho}\boldsymbol{F}. \tag{4.8}$$

PROOF. Inserting a test function of the form $\boldsymbol{\psi}=\boldsymbol{\psi}(\boldsymbol{x},t)$ into (1.3) and passing to the limit as $\varepsilon\searrow 0$, we arrive at the desired macroscopic equation (4.8). We took it into account that the sequence $\{\alpha_\theta^\varepsilon\theta^\varepsilon+p_f^\varepsilon+p_s^\varepsilon\}$ two-scale converges to

$$\hat{\alpha}_\theta\theta+p_f+p_s=\alpha_{\theta f}\theta+\frac{1}{m}p_f=\alpha_{\theta s}\theta+\frac{1}{1-m}p_s$$

on assuming each of the conditions (1.8) or (1.9). □

Lemma 4.3. *Let* $\mu_1=\infty$ *and* $\lambda_1<\infty$. *Then the functions* θ, $\boldsymbol{W}^s=(1-\chi)\boldsymbol{W}$, $\boldsymbol{w}_f$, *and* p_f *in* Y_s *satisfy the system of microscopic equations*

$$\begin{aligned}\rho_s\frac{\partial^2\boldsymbol{W}^s}{\partial t^2}&=\lambda_1\Delta_y\boldsymbol{W}^s-\nabla_yR^s-\nabla\Big(\frac{1}{m}p_f+\alpha_{\theta f}\theta\Big)+\rho_s\boldsymbol{F},\ \boldsymbol{y}\in Y_s,\\ \boldsymbol{W}^s&=\boldsymbol{w}_f,\quad \boldsymbol{y}\in\gamma,\end{aligned} \tag{4.9}$$

in the case $\lambda_1>0$ *or the microscopic relations*

$$\rho_s\frac{\partial^2\boldsymbol{W}^s}{\partial t^2}=-\nabla_yR^s-\nabla\Big(\frac{1}{m}p_f+\alpha_{\theta f}\theta\Big)+\rho_s\boldsymbol{F},\quad \boldsymbol{y}\in Y_s, \tag{4.10}$$

$$(\boldsymbol{W}^s-\boldsymbol{w}_f)\cdot\boldsymbol{n}=0,\quad \boldsymbol{y}\in\gamma, \tag{4.11}$$

in the case $\lambda_1=0$. *The problem is endowed with the homogeneous initial data*

$$\boldsymbol{W}^s(\boldsymbol{y},0)=\frac{\partial\boldsymbol{W}^s}{\partial t}(\boldsymbol{y},0)=0,\quad \boldsymbol{y}\in Y_s. \tag{4.12}$$

In the boundary condition (4.11) $\boldsymbol{n}$ *is the unit normal vector to* γ.

PROOF. The differential equations (4.9) and (4.10) and the initial conditions (4.12) follow as $\varepsilon\searrow 0$ from (1.3) with the test functions in the form $\boldsymbol{\psi}=\boldsymbol{\varphi}(x\varepsilon^{-1})\cdot h(\boldsymbol{x},t)$, where $\boldsymbol{\varphi}$ is the solenoidal function compactly supported in Y_s. The boundary condition in (4.9) is a consequence of the two-scale convergence of $\{\sqrt{\alpha_\lambda}\nabla_x\boldsymbol{w}^\varepsilon\}$ to $\sqrt{\lambda_1}\nabla_y\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$. By this convergence, $\nabla_y\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$ is L^2-integrable in Y. The boundary condition (3.13) follows from (4.5), (4.6), and the relation $\boldsymbol{W}_f=\boldsymbol{w}_f$.

Note that (4.9) and (4.10) are understood in the generalized sense (in the sense of distributions) as appropriate integral identities. Let, for example, $H(Y_s)$ be the space of all one-periodic functions from $W_2^1(Y)$, solenoidal in Y_s and vanishing at γ. Then $\boldsymbol{W}^s$ is a generalized solution to (4.9), if

$$\widetilde{\boldsymbol{W}}^s=(\boldsymbol{W}^s-\boldsymbol{w}_f)\in L^2((0,T);H(Y_s)),\qquad \frac{\partial\widetilde{\boldsymbol{W}}^s}{\partial t}\in L^2(Y_s\times(0,T))$$

and

$$\int_0^T\int_{Y_s}\Big(\rho_s\frac{\partial\widetilde{\boldsymbol{W}}^s}{\partial t}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial t}-\lambda_1\nabla\widetilde{\boldsymbol{W}}^s:\nabla\boldsymbol{\varphi}+\boldsymbol{z}^s\cdot\boldsymbol{\varphi}\Big)\,dydt=0,\qquad \widetilde{\boldsymbol{W}}^s(\boldsymbol{y},0)=0 \tag{4.13}$$

for each function $\boldsymbol{\varphi}$ in $L^2((0,T);H(Y_s))$ such that $\partial\boldsymbol{\varphi}/\partial t\in L^2(Y_s\times(0,T))$ and $\boldsymbol{\varphi}(\boldsymbol{y},T)=0$. In (4.13)

$$\boldsymbol{z}^s(\boldsymbol{x},t)=-\nabla\Big(\frac{1}{m}p_f+\alpha_{\theta f}\theta\Big)+\rho_s\boldsymbol{F}-\rho_s\frac{\partial^2\boldsymbol{w}_f}{\partial t^2}.\quad □$$

In the same way, we can prove the following

Lemma 4.4. *Let $\mu_1 < \infty$ and $\lambda_1 = \infty$. Then the functions θ, $\boldsymbol{W}^f = \chi \boldsymbol{W}$, $\boldsymbol{w}_s$, and p_f in Y_f satisfy the system of microscopic equations*

$$\rho_f \frac{\partial^2 \boldsymbol{W}^f}{\partial t^2} = \mu_1 \Delta_y \frac{\partial \boldsymbol{W}^f}{\partial t} - \nabla_y R^f - \nabla\Big(\frac{1}{m} p_f + \alpha_{\theta f}\theta\Big) + \rho_f \boldsymbol{F}, \quad \boldsymbol{y} \in Y_f, \tag{4.14}$$

$$\boldsymbol{W}^f = \boldsymbol{w}_s, \quad \boldsymbol{y} \in \gamma, \tag{4.15}$$

in the case $\mu_1 > 0$ or the microscopic relations

$$\rho_f \frac{\partial^2 \boldsymbol{W}^f}{\partial t^2} = -\nabla_y R^f - \nabla\Big(\frac{1}{m} p_f + \alpha_{\theta f}\theta\Big) + \rho_f \boldsymbol{F}, \quad \boldsymbol{y} \in Y_f, \tag{4.16}$$

$$(\boldsymbol{W}^f - \boldsymbol{w}_s) \cdot \boldsymbol{n} = 0, \quad \boldsymbol{y} \in \gamma, \tag{4.17}$$

in the case $\mu_1 = 0$. The problem is endowed with the homogeneous initial data

$$\boldsymbol{W}^f(\boldsymbol{y}, 0) = \frac{\partial \boldsymbol{W}^f}{\partial t}(\boldsymbol{y}, 0) = 0, \quad \boldsymbol{y} \in Y_f. \tag{4.18}$$

Lemma 4.5. *Let $\mu_1 < \infty$, $\lambda_1 < \infty$, and $\rho(\boldsymbol{y}) = \rho_f \chi(\boldsymbol{y}) + \rho_s(1 - \chi(\boldsymbol{y}))$.*
Then, in Y, the functions θ, $\boldsymbol{W}$, and p_f satisfy the system of microscopic equations

$$\rho(\boldsymbol{y}) \frac{\partial^2 \boldsymbol{W}}{\partial t^2} + \nabla\Big(\frac{1}{m} p_f + \alpha_{\theta f}\theta\Big) - \rho(\boldsymbol{y})\boldsymbol{F} = \operatorname{div}_y \Big\{ \mu_1 \chi \mathbb{D}\Big(y, \frac{\partial \boldsymbol{W}}{\partial t}\Big) + \lambda_1 (1 - \chi) \mathbb{D}(y, \boldsymbol{W}) - R\mathbb{I} \Big\} \tag{4.19}$$

and the homogeneous initial data

$$\boldsymbol{W}(\boldsymbol{y}, 0) = \frac{\partial \boldsymbol{W}}{\partial t}(\boldsymbol{y}, 0) = 0, \quad \boldsymbol{y} \in Y. \tag{4.20}$$

In the proof of this lemma we choose in the integral identity (1.3) the test functions in the form $\boldsymbol{\psi} = \boldsymbol{\varphi}(x\varepsilon^{-1}) \cdot h(\boldsymbol{x}, t)$, where $\boldsymbol{\varphi}$ is the solenoidal function compactly supported in Y, and additionally use Nguetseng's theorem which states that the sequence $\{\varepsilon \mathbb{D}(x, \boldsymbol{w}^{\varepsilon})\}$ two-scale converges to $\mathbb{D}(y, \boldsymbol{W})$.

Lemma 4.6. *Let (1.8) hold. Then for all $(\boldsymbol{x}, t) \in \Omega_T$ and $y \in Y$ strong and two-scale limits θ and Θ satisfy the microscopic equation*

$$\operatorname{div}_y \{\varkappa_0(\boldsymbol{y})\big(\nabla\theta + \nabla_y \Theta\big)\} = 0, \tag{4.21}$$

where $\varkappa_0(\boldsymbol{y}) = \chi(\boldsymbol{y})\varkappa_{0f} + (1 - \chi(\boldsymbol{y}))\varkappa_{0s}$.

To prove this lemma it suffices to consider (1.4) with the test functions in the form $\xi = \varepsilon \xi_0(\boldsymbol{x}, t, \boldsymbol{x}/\varepsilon)$ and pass to the limit as $\varepsilon \searrow 0$.

Lemma 4.7. *For all $(\boldsymbol{x}, t) \in \Omega_T$ the weak and strong limits θ, p_f, and p_s satisfy the macroscopic heat equation*

$$\hat{c}_p \frac{\partial \theta}{\partial t} - \frac{\alpha_{\theta f}}{p_*} \frac{\partial p_f}{\partial t} - \frac{\alpha_{\theta s}}{\eta_0} \frac{\partial p_s}{\partial t} = \operatorname{div}\{\hat{\varkappa}_0 \nabla\theta + \langle \varkappa_0 \nabla_y \Theta \rangle_Y\} + \Psi, \tag{4.22}$$

where $\hat{c}_p = m + (1 - m)c_p^0$ and $\hat{\varkappa}_0 = \langle \varkappa_0 \rangle_Y > 0$ if (1.8) holds and $\varkappa_0(\boldsymbol{y}) = 0$ in the case of (1.9).

The proof of this lemma repeats that of Lemma 4.2, if we express the term $\alpha_\theta^{\varepsilon} \operatorname{div} \boldsymbol{w}^{\varepsilon}$ in (1.4) in terms of pressures using (1.1) and (1.2).

Theorem 3 is a simple consequence of Lemmas 3.2, 4.1, and 4.2.

§ 5. Proofs of Theorems 4 and 5

Equation (1.15) is immediate from (4.8). The continuity equation (1.16) follows from (4.3), if we take it into account that $\boldsymbol{w} = m\boldsymbol{w}_f + \boldsymbol{w}^s$.

To find the last two equations (1.17) and (1.18) we just have to solve the system of microscopic equations (4.5), (4.9)–(4.13) and to use the formula $\boldsymbol{w}^s = \langle \boldsymbol{W} \rangle_{Y_s}$.

1. Let $\lambda_1 > 0$. Then the solution of (4.13) is given by the formula

$$\widetilde{\boldsymbol{W}}^s = \int_0^t \Big(\sum_{i=1}^3 \boldsymbol{W}^{s,i}(\boldsymbol{y}, t-\tau) z_i^s(\boldsymbol{x}, \tau) \Big) d\tau, \quad \boldsymbol{z}^s = \big(z_1^s, z_2^s, z_3^s\big), \tag{5.1}$$

where the one-periodic (in $\boldsymbol{y}$) functions $\boldsymbol{W}^{s,i}(\boldsymbol{y},t) \in L^2((0,T); H(Y_s))$ and $\partial \boldsymbol{W}^{s,i}/\partial t \in L^2(Y_s \times (0,T))$ are solutions to the integral identities

$$\int_0^T \int_{Y_s} \Big(\rho_s \frac{\partial \boldsymbol{W}^{s,i}}{\partial t} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial t} - \lambda_1 \nabla \boldsymbol{W}^{s,i} : \nabla \boldsymbol{\varphi} \Big)\, dydt + \int_{Y_s} \boldsymbol{e}_i \cdot \boldsymbol{\varphi}(\boldsymbol{y}, 0)\, dy = 0 \tag{5.2}$$

for every smooth function $\boldsymbol{\varphi} \in L^2((0,T); H(Y_s))$ such that $\partial \boldsymbol{\varphi}/\partial t \in L^2(Y_s \times (0,T))$ and $\boldsymbol{\varphi}(\boldsymbol{y}, T) = 0$. In (5.2) $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is the standard Cartesian orthogonal basis.

In fact, we may rewrite (5.2) as

$$\int_{Y_s} \int_{\tau}^{T} \Big(\rho_s \frac{\partial \boldsymbol{W}^{s,i}}{\partial t}(\boldsymbol{y}, t-\tau) \cdot \frac{\partial \boldsymbol{\varphi}}{\partial t}(\boldsymbol{y}, t) - \lambda_1 \nabla \boldsymbol{W}^{s,i}(\boldsymbol{y}, t-\tau) : \nabla \boldsymbol{\varphi}(\boldsymbol{y}, t) \Big)\, dydt + \int_{Y_s} \boldsymbol{e}_i \cdot \boldsymbol{\varphi}(\boldsymbol{y}, \tau)\, dy = 0,$$

if $\boldsymbol{\varphi}(\boldsymbol{y}, t) = 0$ at $t \geq T$. Inserting (5.1) into (5.2), changing the order of integration in the time integrals, and considering the last relation, we arrive at (4.13). To solve (5.2) we use Galerkin's method. Let $\{\boldsymbol{\psi}_k\}_{k=1}^{\infty}$ be a complete basis in $H(Y_s)$ orthogonal in the inner product of $L^2(Y_s)$. Then the approximate solutions $\boldsymbol{V}_i^N(\boldsymbol{y}, t) = \sum_{k=1}^N c_k(t) \boldsymbol{\psi}_k(\boldsymbol{y})$ of (5.2) are defined from the Cauchy problem

$$\begin{aligned} \rho_s \frac{d^2 c_k}{dt^2} &= -\lambda_1 \sum_{m=1}^N c_m \int_{Y_s} \nabla \boldsymbol{\psi}_m : \nabla \boldsymbol{\psi}_k \, dy, \\ c_k(0) = 0, \quad \frac{dc_k}{dt}(0) &= \int_{Y_s} \boldsymbol{e}_i \cdot \boldsymbol{\psi}_k \, dy, \quad k = 1, \ldots, N. \end{aligned} \tag{5.3}$$

The unique and infinitely smooth solution of the Cauchy problem (5.3) exists on an arbitrary interval $(0,T)$. If we multiply the kth equation in (5.3) by dc_k/dt, sum over all k, and integrate the result over time interval $(0,t)$ then we obtain the well-known energy estimate

$$\int_{Y_s} \Big(\rho_s \Big(\frac{\partial \boldsymbol{V}_i^N}{\partial t}(\boldsymbol{y}, t) \Big)^2 + \lambda_1 \big(\nabla \boldsymbol{V}_i^N(\boldsymbol{y}, t) \big)^2 \Big)\, dy = \sum_{k=1}^N \int_{Y_s} (\boldsymbol{e}_i \cdot \boldsymbol{\psi}_k)^2 \, dy \leq 1 - m.$$

Using this estimate and the standard methods (see [10]) we may prove that the sequence $\{\boldsymbol{V}_i^N\}$ contains some weakly subsequence convergent in $L^2((0,T); H(Y_s))$. This subsequence converges to the solution $\boldsymbol{W}^{s,i}$ of (5.2) such that $\partial \boldsymbol{V}_i^N/\partial t$ converge weakly in $L^2(Y_s \times (0,T))$ to $\partial \boldsymbol{W}^{s,i}/\partial t$ and

$$\int_{Y_s} \Big(\rho_s \Big(\frac{\partial \boldsymbol{W}^{s,i}}{\partial t}(\boldsymbol{y}, t) \Big)^2 + \lambda_1 (\nabla \boldsymbol{W}^{s,i}(\boldsymbol{y}, t))^2 \Big)\, dy \leq 1 - m. \tag{5.4}$$

Thus, we can differentiate expression (5.1) with respect to the time variable and $\partial \boldsymbol{w}^s/\partial t = \langle \partial \boldsymbol{W}/\partial t\rangle_{Y_s}$. Equation (1.17) follows from the last equality if we put

$$\mathbb{B}_1^s(t) = \Big\langle \sum_{i=1}^{3} \frac{\partial \boldsymbol{W}^{s,i}}{\partial t} \Big\rangle_{Y_s}(t) \otimes \boldsymbol{e}_i. \tag{5.5}$$

In (5.5) the matrix $\boldsymbol{a} \otimes \boldsymbol{b}$ for given vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined by the expression $(\boldsymbol{a} \otimes \boldsymbol{b}) \cdot \boldsymbol{c} = \boldsymbol{a}(\boldsymbol{b} \cdot \boldsymbol{c})$ for every vector $\boldsymbol{c}$.

REMARK 5.1. It is clear that the main difficulty in the proof of the lemma is in the properties of $H(Y_s)$. Namely, $H(Y_s)$ must be nonempty. To prove this fact it suffices to consider the small ball $G \subset Y_s$ and solve in G the Stokes problem

$$\Delta \boldsymbol{\psi} - \nabla q = \boldsymbol{g}(\boldsymbol{y}), \quad \operatorname{div} \boldsymbol{\psi} = 0, \quad \boldsymbol{\psi}|_{\partial G} = 0,$$

with a smooth function $\boldsymbol{g}$ compactly supported in G and extend $\boldsymbol{\psi}$ in Y_s by zero outside G. It is easy to see that such a function $\boldsymbol{\psi}$ belongs to $H(Y_s)$.

2. If $\lambda_1 = 0$ then in solving (4.5), (4.10)–(4.12) we first find the pressure $R^s(\boldsymbol{x}, t, \boldsymbol{y})$ by solving the Neumann problem for the Laplace equation in Y_s in the form

$$R^s(\boldsymbol{x}, t, \boldsymbol{y}) = \sum_{i=1}^{3} R_{s,i}(\boldsymbol{y}) z_i^s(\boldsymbol{x}, t),$$

where $R_{s,i}(\boldsymbol{y})$ is the solution of the problem

$$\Delta_y R_{s,i} = 0, \quad \boldsymbol{y} \in Y_s; \quad \nabla_y R_{s,i} \cdot \boldsymbol{n} = \boldsymbol{n} \cdot \boldsymbol{e}_i, \quad \boldsymbol{y} \in \gamma; \quad \langle R_{s,i}\rangle_{Y_s} = 0. \tag{5.6}$$

Formula (1.18) is the result of integration of (4.10) over Y_s and

$$\mathbb{B}_2^s = \sum_{i=1}^{3} \langle \nabla R_{s,i}(\boldsymbol{y})\rangle_{Y_s} \otimes \boldsymbol{e}_i, \tag{5.7}$$

where the matrix $\mathbb{B} = ((1-m)\mathbb{I} - \mathbb{B}_2^s)$ is symmetric and strictly positive definite. In fact, let $\widetilde{R} = \sum_{i=1}^{3} R_{s,i}\xi_i$ for every unit vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$. Then

$$(\mathbb{B} \cdot \boldsymbol{\xi}) \cdot \boldsymbol{\xi} = \langle (\boldsymbol{\xi} - \nabla \widetilde{R})^2 \rangle_{Y_f} = 0$$

if and only if $\widetilde{R}$ is a linear function in $\boldsymbol{y}$. On the other hand, it follows from the assumption about the geometry of Y_s that all linear periodic functions on Y_s are constant. Finally, the normalization condition $\langle R_{s,i}\rangle_{Y_s} = 0$ implies that $\widetilde{R} = 0$. However this is impossible, since the functions $R_{s,i}$ are linearly independent. $\square$

§ 6. Proofs of Theorems 6 and 7

The proofs of these theorems repeat those of the previous Theorems 4 and 5. Here we have to solve the system of microscopic equations (4.5), (4.14)–(4.18) and to use the formula

$$\frac{\partial \boldsymbol{w}^f}{\partial t} = \Big\langle \frac{\partial \boldsymbol{W}}{\partial t} \Big\rangle_{Y_f}.$$

Thus,

$$\mathbb{B}_1^f(t) = \Big\langle \sum_{i=1}^{3} \frac{\partial \boldsymbol{W}^{f,i}}{\partial t} \Big\rangle_{Y_f}(t) \otimes \boldsymbol{e}_i, \tag{6.1}$$

$$\mathbb{B}_2^f = \sum_{i=1}^{3} \langle \nabla R_{f,i}\rangle_{Y_f} \otimes \boldsymbol{e}_i, \tag{6.2}$$

where the one-periodic (in y) functions $\boldsymbol{W}^{f,i}(\boldsymbol{y},t)$ solve the periodic initial boundary-value problem

$$\rho_f \frac{\partial^2 \boldsymbol{W}^{f,i}}{\partial t^2} - \mu_1 \Delta \frac{\partial \boldsymbol{W}^{f,i}}{\partial t} + \nabla R^{f,i} = 0, \quad \boldsymbol{y} \in Y_f,\ t > 0, \tag{6.3}$$

$$\operatorname{div}_y \boldsymbol{W}^{f,i} = 0, \quad \boldsymbol{y} \in Y_f,\ t > 0, \tag{6.4}$$

$$\boldsymbol{W}^{f,i} = 0, \quad \boldsymbol{y} \in \gamma,\ t > 0, \tag{6.5}$$

$$\boldsymbol{W}^{f,i}(\boldsymbol{y},0) = 0, \quad \rho_f \frac{\partial \boldsymbol{W}^{f,i}}{\partial t}(\boldsymbol{y},0) = \boldsymbol{e}_i, \quad \boldsymbol{y} \in Y_f, \tag{6.6}$$

and the one-periodic (in y) functions $R_{f,i}(\boldsymbol{y})$ solve the periodic Neumann boundary-value problem for the Laplace equation

$$\Delta_y R_{f,i} = 0, \quad \boldsymbol{y} \in Y_f; \quad \nabla_y R_{f,i} \cdot \boldsymbol{n} = \boldsymbol{n} \cdot \boldsymbol{e}_i, \quad \boldsymbol{y} \in \gamma; \quad \langle R_{f,i} \rangle_{Y_f} = 0. \quad \square \tag{6.7}$$

§ 7. Proof of Theorem 8

To derive the momentum conservation law (1.23) we must solve the system of microscopic equations (4.5), (4.19), with the initial conditions (4.20), and use the formula $\boldsymbol{w} = \langle \boldsymbol{W} \rangle_Y$.

Let

$$\boldsymbol{W} = \sum_{i=1}^{3} \int_0^t (\boldsymbol{W}^{z,i}(\boldsymbol{y},t-\tau) z_i(\boldsymbol{x},\tau) + \boldsymbol{W}^{F,i}(\boldsymbol{y},t-\tau) F_i(\boldsymbol{x},\tau))\, d\tau,$$

$$R = \sum_{i=1}^{3} \int_0^t (R^{z,i}(\boldsymbol{y},t-\tau) z_i(\boldsymbol{x},\tau) + R^{F,i}(\boldsymbol{y},t-\tau) F_i(\boldsymbol{x},\tau))\, d\tau,$$

where

$$\boldsymbol{z}(x,t) = \sum_{i=1}^{3} z_i \boldsymbol{e}_i = \nabla \left(\frac{1}{m} p_f + \alpha_{\theta f} \theta \right), \quad \boldsymbol{F} = \sum_{i=1}^{3} F_i \boldsymbol{e}_i.$$

Then the one-periodic (in y) functions $\{\boldsymbol{W}^{j,i}(\boldsymbol{y},t),\, R^{j,i}(\boldsymbol{y},t)\}$ ($j = s, F,\ i = 1,2,3$) are found as solutions to the system of equations

$$\operatorname{div}_y \left(\mu_1 \chi \mathbb{D}\left(y, \frac{\partial \boldsymbol{W}^{j,i}}{\partial t} \right) + \lambda_1 (1-\chi) \mathbb{D}(y, \boldsymbol{W}^{j,i}) - R^{j,i} \mathbb{I} \right) = \rho(\boldsymbol{y}) \frac{\partial^2 \boldsymbol{W}^{j,i}}{\partial t^2}, \tag{7.1}$$

$$\operatorname{div}_y \boldsymbol{W}^{j,i} = 0 \tag{7.2}$$

in Y for $t > 0$, and satisfy the initial conditions:

$$\boldsymbol{W}^{z,i}(\boldsymbol{y},0) = 0, \quad \rho(\boldsymbol{y}) \frac{\partial \boldsymbol{W}^{p,i}}{\partial t}(\boldsymbol{y},0) = -\boldsymbol{e}_i, \quad \boldsymbol{x} \in Y, \tag{7.3}$$

$$\boldsymbol{W}^{F,i}(\boldsymbol{y},0) = 0, \quad \frac{\partial \boldsymbol{W}^{F,i}}{\partial t}(\boldsymbol{y},0) = \boldsymbol{e}_i, \quad \boldsymbol{x} \in Y. \tag{7.4}$$

Therefore,

$$\mathbb{B}(t) = \sum_{i=1}^{3} \left\langle \frac{\partial \boldsymbol{W}^{z,i}}{\partial t} \right\rangle_Y (t) \otimes \boldsymbol{e}_i, \tag{7.5}$$

$$\mathbf{f}(\boldsymbol{x},t) = \int_0^t \sum_{i=1}^{3} \left\langle \frac{\partial \boldsymbol{W}^{F,i}}{\partial t} \right\rangle_Y (t-\tau) F_i(\boldsymbol{x},\tau)\, d\tau. \tag{7.6}$$

The solvability and uniqueness of generalized solutions to the periodic boundary-value problems for $\{\boldsymbol{W}^{j,i}(\boldsymbol{y},t),\ R^{j,i}(\boldsymbol{y},t)\}$ $(j=s,F,\ i=1,2,3)$ follow directly from the corresponding energy identities

$$\frac{1}{2}\int\limits_Y\bigg(\rho(\boldsymbol{y})\Big(\frac{\partial\boldsymbol{W}^{j,i}}{\partial t}(\boldsymbol{y},t)\Big)^2+\lambda_1 D(y,\boldsymbol{W}^{j,i}(\boldsymbol{y},t)):D(y,\boldsymbol{W}^{j,i}(\boldsymbol{y},t))\bigg)\,dy$$

$$+\int\limits_0^t\int\limits_Y\mu_1 D\Big(y,\frac{\partial\boldsymbol{W}^{j,i}}{\partial\tau}(\boldsymbol{y},\tau)\Big):D\Big(y,\frac{\partial\boldsymbol{W}^{j,i}}{\partial\tau}(\boldsymbol{y},\tau)\Big)\,dyd\tau=\frac{1}{2}\gamma^{(j)}$$

for $i=1,2,3$ and $j=z,F$. Here $\gamma^{(z)}=\langle 1/\rho\rangle_Y$, $\gamma^{(F)}=\langle\rho\rangle_Y$. $\square$

§ 8. Proof of Theorem 9

The homogenized heat equation (1.24) is the macroscopic heat equation (4.22) where the expression $\langle\varkappa_0\nabla_y\Theta\rangle_Y$ is replaced with the term $\langle\varkappa_0\nabla_y\Theta\rangle_Y=\mathbb{B}_0^\theta\cdot\nabla\theta$. The last formula is the result of solving the microscopic heat equation (4.21) in the form

$$\Theta(\boldsymbol{x},t,\boldsymbol{y})=\sum_{i=1}^3\Theta_i(\boldsymbol{y})\frac{\partial\theta}{\partial x_i}(\boldsymbol{x},t),$$

where Θ_i, $i=1,2,3$, is a periodic solution to the equation $\mathrm{div}_y\{\varkappa_0(\nabla_y\Theta_i+\boldsymbol{e}_i)\}=0$ in Y and

$$\mathbb{B}^\theta=\hat{\varkappa}_0\mathbb{I}+\mathbb{B}_0^\theta,\quad \mathbb{B}_0^\theta=\sum_{i=1}^3\nabla_y\langle\Theta_i\rangle_Y\otimes\boldsymbol{e}_i.\quad\square \tag{8.1}$$

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