NGUETSENG'S TWO-SCALE CONVERGENCE METHOD FOR FILTRATION AND SEISMIC ACOUSTIC PROBLEMS IN ELASTIC POROUS MEDIA

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Abstract: A linear system is considered of the differential equations describing a joint motion of an elastic porous body and a fluid occupying a porous space. The problem is linear but very hard to tackle since its main differential equations involve some (big and small) nonsmooth oscillatory coefficients. Rigorous justification under various conditions on the physical parameters is fulfilled for the homogenization procedures as the dimensionless size of pores vanishes, while the porous body is geometrically periodic. In result, we derive Biot's equations of poroelasticity, the system consisting of the anisotropic Lamé equations for the solid component and the acoustic equations for the fluid component, the equations of viscoelasticity, or the decoupled system consisting of Darcy's system of filtration or the acoustic equations for the fluid component (first approximation) and the anisotropic Lamé equations for the solid component (second approximation) depending on the ratios between the physical parameters. The proofs are based on Nguetseng's two-scale convergence method of homogenization in periodic structures.

Keywords: Biot equations, Stokes equations, Lamé equations, two-scale convergence, homogenization of periodic structures, poroelasticity, viscoelasticity

Introduction

In this article we consider the problem of modeling small perturbations in an elastic deformable medium perforated by a system of channels (pores) filled with a fluid. Such media are called *elastic porous media* and they are a rather good approximation to real consolidated grounds. In the present-day literature, the field of study in mechanics corresponding to these media is called "poromechanics" [1]. The solid component of such a medium is the *skeleton*, and the domain, filled with a fluid, is a *porous space*. The exact mathematical model of an elastic porous medium consists of the classical momentum and mass balance equations in Euler variables; the equations of the stress fields in the solid and fluid phases; and an endowing relation determining the behavior of the interface of the fluid and solid components. This relation expresses the fact that the interface is a material surface, which amounts to the condition that it consists of the same material particles all the time. Denoting by ρ the density of the medium, by $\boldsymbol{v}$ the velocity, by $\mathbb{P}^f$ the stress tensor in the fluid component, by $\mathbb{P}^s$ the stress tensor in the solid skeleton, and by $\tilde{\chi}$ the characteristic (indicator) function of the porous space, we write the fundamental differential equations of the nonlinear model in the form

$$\rho\frac{d\boldsymbol{v}}{dt} = (\mathrm{div}_x)\{\tilde{\chi}\mathbb{P}^f + (1-\tilde{\chi})\mathbb{P}^s\} + \rho\boldsymbol{F}, \quad \frac{d\rho}{dt} + \rho\cdot\mathrm{div}_x\,\boldsymbol{v} = 0, \quad \frac{d\tilde{\chi}}{dt} = 0,$$

where d/dt stands for the material derivative with respect to the time variable.

Clearly the above original model has an unknown (free) boundary. A more precise formulation of the nonlinear problem is not in the focus of our present work. Instead, we aim to study the problem that is linearized at the rest state. In continuum mechanics the methods of linearization are developed

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rather deeply. The so-obtained linear model is commonly accepted and basic for description of filtration and seismic acoustics in elastic porous media (for example, see [2–4]). In this model the characteristic function of the porous space $\tilde{\chi}$ is a known function for $t > 0$. It is assumed that this function coincides with the characteristic function of the porous space $\bar{\chi}$ given at the initial moment. In the dimensionless variables (without primes)

$$\boldsymbol{x}' = L\boldsymbol{x}, \quad t' = \tau t, \quad \boldsymbol{w}' = L\boldsymbol{w}, \quad \rho_s' = \rho_0 \rho_s, \quad \rho_f' = \rho_0 \rho_f, \quad \boldsymbol{F}' = g\boldsymbol{F}$$

the differential equations of the problem in a domain $\Omega \in \mathbb{R}^3$ for $t > 0$ for small declinations of the dimensionless displacement vector $\boldsymbol{w}$ of the continuum medium have the form:

$$\alpha_\tau \bar{\rho} \frac{\partial^2 \boldsymbol{w}}{\partial t^2} = \operatorname{div}_x \left\{ \bar{\chi} \alpha_\mu \mathbb{D}\left(x, \frac{\partial \boldsymbol{w}}{\partial t} \right) + (1 - \bar{\chi}) \alpha_\lambda \mathbb{D}(x, \boldsymbol{w}) - (q + \pi)\mathbb{I} \right\} + \bar{\rho}\boldsymbol{F}, \tag{0.1}$$

$$p = -\alpha_p \bar{\chi} \operatorname{div}_x \boldsymbol{w}, \quad \pi = -\alpha_\eta (1 - \bar{\chi}) \operatorname{div}_x \boldsymbol{w}, \quad q = p + \frac{\alpha_\nu}{\alpha_p} \frac{\partial p}{\partial t}, \tag{0.2}$$

where $\bar{\rho} = \bar{\chi}\rho_f + (1 - \bar{\chi})\rho_s$, $\mathbb{D}(x, \boldsymbol{u}) := (1/2)(\nabla_x \boldsymbol{u} + (\nabla_x \boldsymbol{u})^T)$. The dimensionless constants α_i $(i = \tau, \nu, \dots)$ are defined by the formulas

$$\alpha_\tau = \frac{L}{g\tau^2}, \quad \alpha_\nu = \frac{\nu}{\tau L g \rho_0}, \quad \alpha_\mu = \frac{2\mu}{\tau L g \rho_0}, \quad \alpha_p = \frac{c^2 \rho_f}{L g}, \quad \alpha_\eta = \frac{\eta}{L g \rho_0}, \quad \alpha_\lambda = \frac{2\lambda}{L g \rho_0},$$

where μ is the viscosity of the fluid, ν is the bulk viscosity of the fluid, λ and η are the Lamé constants, c is the speed of sound in the fluid, L is a characteristic size of the domain under consideration, τ is a characteristic time of the process, ρ_f and ρ_s are respectively the mean dimensionless densities of the fluid and solid phases, correlated with the mean density of water, and g is the value of acceleration of gravity.

The problem is endowed with the homogeneous initial and boundary conditions

$$\boldsymbol{w}|_{t=0} = 0, \quad \left. \frac{\partial \boldsymbol{w}}{\partial t} \right|_{t=0} = 0, \quad \boldsymbol{x} \in \Omega, \tag{0.3}$$

$$\boldsymbol{w} = 0, \quad \boldsymbol{x} \in S = \partial\Omega, \quad t \geq 0. \tag{0.4}$$

The corresponding mathematical model, described by the system (0.1)–(0.4), contains a natural small parameter ε that is the characteristic size of pores l divided by the characteristic size L of the entire porous body: $\varepsilon = l/L$. Our aim is to derive all possible limiting regimes (homogenized equations) as a small parameter vanishes. Such an approximation significantly simplifies the original problem and at the same time preserves all main features of the latter. But even this approach is too hard to implement, and some additional simplifying assumptions are necessary. In terms of geometrical properties of the medium, the most appropriate way is to simplify the problem by postulating that the porous structure is periodic. We accept the following constraints:

Assumption 1. *The domain $\Omega = (0, 1)^3$ is a periodic replication of the elementary cell $Y^\varepsilon = \varepsilon Y$, where $Y = (0, 1)^3$ and the quantity $1/\varepsilon$ is integer, so that Ω always contains an integer number of elementary cells Y_i^ε. Let Y_s be the "solid part" of Y, and let the "fluid part" Y_f be its open complement. We put $\gamma = \partial Y_f \cap \partial Y_s$ and assume that γ is a C^1-surface. The porous space Ω_f^ε is a periodic replication of the elementary cell εY_f, and the solid skeleton Ω_s^ε is a periodic replication of the elementary cell εY_s. The boundary $\Gamma^\varepsilon = \partial\Omega_s^\varepsilon \cap \partial\Omega_f^\varepsilon$ is a periodic replication in Ω of the boundary $\varepsilon\gamma$. The "solid skeleton" Ω_s is a connected domain.*

Under these assumptions

$$\bar{\chi}(\boldsymbol{x}) = \chi^\varepsilon(\boldsymbol{x}) = \chi\left(\boldsymbol{x}/\varepsilon\right), \quad \bar{\rho} = \rho^\varepsilon(\boldsymbol{x}) = \chi^\varepsilon(\boldsymbol{x})\rho_f + (1 - \chi^\varepsilon(\boldsymbol{x}))\rho_s,$$

where $\chi(\boldsymbol{y})$ is the characteristic function of Y_f in Y.

The first attempt at deriving the limiting regimes in the case when the skeleton was assumed an absolutely rigid body was carried out by E. Sanchez-Palencia and L. Tartar. E. Sanchez-Palencia [3, Chapter 7.2] formally obtained Darcy's law of filtration by the method of two-scale asymptotic expansions, and L. Tartar (see [3, the Appendix]) justified homogenization with a mathematical rigor. Using the same method of two-scale expansions, J. Keller and R. Burridge [2] formally derived the system of Biot's equations [5] from the system (0.1)–(0.4) in the case when the parameter α_μ was of order ε^2 and the remaining coefficients were fixed independently of ε. Under the same assumptions as in the article [2], the rigorous justification of Biot's model was given by G. Nguetseng [6] and later by Th. Clopeau in cooperation with J. L. Ferrin, R. P. Gilbert, and A. Mikelić in [7]. Also R. P. Gilbert and A. Mikelić [4] derived a system of equations of viscoelasticity, when all physical parameters were fixed independently of ε. In these works, Nguetseng's two-scale convergence method [8, 9] was the main method of investigation.

Suppose that all dimensionless parameters depend on the small parameter ε of the model and the following (finite or infinite) limits exist:

$$\lim_{\varepsilon\searrow 0}\alpha_\mu(\varepsilon)=\mu_0,\quad \lim_{\varepsilon\searrow 0}\alpha_\lambda(\varepsilon)=\lambda_0,\quad \lim_{\varepsilon\searrow 0}\alpha_\tau(\varepsilon)=\tau_0.$$

We restrict consideration to the cases when $\tau_0<\infty$ and either of the following situations takes place:

(I) $\mu_0=0,\ 0<\lambda_0<\infty$;

(II) $0\le\mu_0<\infty,\ \lambda_0=\infty$;

(III) $0<\mu_0,\ \lambda_0<\infty$.

If $\tau_0=\infty$ then renormalizing the displacement vector by putting $\boldsymbol{w}\to\alpha_\tau\boldsymbol{w}$ we reduce the problem to one of the cases (I)–(III).

In the present article we show that in the case (I) the homogenized equations are Biot's system of equations of poroelasticity for the two-velocity continuum media, or a similar system consisting of the anisotropic Lamé equations for the solid component coupled with the acoustic equations for the velocity of the fluid component, or the anisotropic Lamé system of equations of the one-velocity media (for example, for the case of a disconnected porous space; Theorem 2). In the case (II) the homogenized equations are Darcy's system of equations of filtration or the acoustic equations for the velocity of the fluid component (moreover, in the first approximation the solid component is viewed as an absolutely rigid body) and, in the second approximation, the anisotropic Lamé system of equations for the renormalized displacements of the solid component or the system of equations, described by Theorem 2, for the renormalized displacements of the fluid and solid components (Theorem 3). Finally, in the case (III) they are the nonlocal viscoelasticity equations or the nonlocal anisotropic Lamé system of equations of the one-velocity media (Theorem 4).

§ 1. Formulation of the Main Results

As usual, the equations (0.1) are understood in the sense of distributions. They involve the proper equations (0.1), (0.2) in a usual sense in the domains Ω^ε_f and Ω^ε_s and the boundary conditions

$$[\boldsymbol{w}]=0,\quad [\mathbb{P}\cdot\mathbf{n}]=0,\quad \boldsymbol{x}_0\in\Gamma^\varepsilon,\ t\ge 0. \tag{1.1}$$

On the boundary Γ^ε, where

$$\mathbb{P}=\chi^\varepsilon\alpha_\mu\mathbb{D}\Big(x,\frac{\partial\boldsymbol{w}}{\partial t}\Big)+(1-\chi^\varepsilon)\alpha_\lambda\mathbb{D}(x,\boldsymbol{w})-(q+\pi)\mathbb{I},$$

$\mathbf{n}$ is a unit normal to the boundary and

$$[\varphi](\boldsymbol{x}_0)=\varphi_{(s)}(\boldsymbol{x}_0)-\varphi_{(f)}(\boldsymbol{x}_0),\quad \varphi_{(s)}(\boldsymbol{x}_0)=\lim_{\boldsymbol{x}\to\boldsymbol{x}_0,\boldsymbol{x}\in\Omega^\varepsilon_s}\varphi(\boldsymbol{x}),\quad \varphi_{(f)}(\boldsymbol{x}_0)=\lim_{\boldsymbol{x}\to\boldsymbol{x}_0,\boldsymbol{x}\in\Omega^\varepsilon_f}\varphi(\boldsymbol{x}).$$

There are various forms of representation of equations (0.1)–(0.4) and boundary conditions (1.1) which are equivalent in the sense of distributions. In what follows, it is convenient to write them in the form of the integral equalities.

Definition 1. The four functions $(\boldsymbol{w}^\varepsilon, p^\varepsilon, q^\varepsilon, \pi^\varepsilon)$ are called a *weak solution of the problem* (0.1)–(0.4), if they satisfy the regularity conditions $\boldsymbol{w}^\varepsilon, \mathbb{D}(x, \boldsymbol{w}^\varepsilon), \operatorname{div}_x \boldsymbol{w}^\varepsilon, q^\varepsilon, p^\varepsilon, \frac{\partial p^\varepsilon}{\partial t}, \pi^\varepsilon \in L^2(\Omega_T)$ in the domain $\Omega_T = \Omega \times (0, T)$, the boundary conditions (0.4) and equations (0.2) a.e. in Ω_T, and the integral identity

$$\int_{\Omega_T} \Big(\alpha_\tau \rho^\varepsilon \boldsymbol{w}^\varepsilon \cdot \frac{\partial^2 \boldsymbol{\varphi}}{\partial t^2} - \chi^\varepsilon \alpha_\mu \mathbb{D}(x, \boldsymbol{w}^\varepsilon) : \mathbb{D}\Big(x, \frac{\partial \boldsymbol{\varphi}}{\partial t}\Big) - \rho^\varepsilon \boldsymbol{F} \cdot \boldsymbol{\varphi}$$

$$+ \{(1 - \chi^\varepsilon)\alpha_\lambda \mathbb{D}(x, \boldsymbol{w}^\varepsilon) - (q^\varepsilon + \pi^\varepsilon)\mathbb{I}\} : \mathbb{D}(x, \boldsymbol{\varphi}) \Big) \, dxdt = 0 \tag{1.2}$$

for all smooth vector-functions $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\boldsymbol{x}, t)$, such that $\boldsymbol{\varphi}|_{\partial\Omega} = \boldsymbol{\varphi}|_{t=T} = \partial\boldsymbol{\varphi}/\partial t|_{t=T} = 0$.

In (1.2) by $A : B$ we denote the convolution (or, equivalently, the inner tensor product) of two second-rank tensors with respect to the two indices, i.e., $A : B = \operatorname{tr}(B^* \circ A) = \sum_{i,j=1}^3 A_{ij} B_{ji}$.

Suppose additionally that the following (finite or infinite) limits exist:

$$\lim_{\varepsilon \searrow 0} \alpha_\nu(\varepsilon) = \nu_0, \quad \lim_{\varepsilon \searrow 0} \alpha_p(\varepsilon) = p_*, \quad \lim_{\varepsilon \searrow 0} \alpha_\eta(\varepsilon) = \eta_0, \quad \lim_{\varepsilon \searrow 0} \frac{\alpha_\mu}{\varepsilon^2} = \mu_1,$$

$$\lim_{\varepsilon \searrow 0} \frac{\varepsilon^2 \alpha_p}{\alpha_\mu} = p_1, \quad \lim_{\varepsilon \searrow 0} \frac{\alpha_\lambda \varepsilon^2}{\alpha_\mu} = \lambda_1, \quad \lim_{\varepsilon \searrow 0} \frac{\alpha_\eta \varepsilon^2}{\alpha_\mu} = \eta_1, \quad \lim_{\varepsilon \searrow 0} \frac{\alpha_\eta}{\alpha_\lambda} = \eta_2, \quad \lim_{\varepsilon \searrow 0} \frac{\alpha_p}{\alpha_\lambda} = p_2.$$

In the sequel, we use the following

Assumption 2. 1. *$\boldsymbol{F}, \partial\boldsymbol{F}/\partial t \in L^2(\Omega_T)$.*

2. *The dimensionless parameters satisfy the restrictions:*

$$p_*^{-1}, \quad \mu_0, \quad \nu_0, \quad \lambda_0^{-1}, \quad \eta_0^{-1} < \infty; \quad 0 < \tau_0 + \mu_1. \tag{1.3}$$

Note that all parameters may take all permitted values. For example, if $\tau_0 = 0$ or $p_*^{-1} = 0$ then all terms disappear of the final equations with these parameters.

The following Theorems 1–4 are the main results of the paper.

Theorem 1. *For all $\varepsilon > 0$ on the arbitrary time interval $[0, T]$ there exists a unique weak solution of the problem* (0.1)–(0.4).

(I) *If $\lambda_0 < \infty$ then*

$$\max_{0 \le t \le T} \| |\boldsymbol{w}^\varepsilon(t)| + \sqrt{\alpha_\mu} \chi^\varepsilon |\nabla_x \boldsymbol{w}^\varepsilon(t)| + \sqrt{\alpha_\lambda}(1 - \chi^\varepsilon)|\nabla_x \boldsymbol{w}^\varepsilon(t)| \|_{2,\Omega} \le I_F, \tag{1.4}$$

$$\|q^\varepsilon\|_{2,\Omega_T} + \|p^\varepsilon\|_{2,\Omega_T} + \|\pi^\varepsilon\|_{2,\Omega_T} \le I_F, \tag{1.5}$$

where $I_F = C\| |\boldsymbol{F}| + |\partial\boldsymbol{F}/\partial t| \|_{2,\Omega_T}$ and C is a constant independent of ε.

(II) *If $\lambda_0 = \infty$, $\mu_1 = \infty$, $0 < \lambda_1 < \infty$, then* (1.4) *and* (1.5) *hold for renormalized displacements $w^\varepsilon \to \varepsilon^{-2}\alpha_\mu w^\varepsilon$, with renormalized parameters*

$$\alpha_\mu \to \varepsilon^2, \quad \alpha_\lambda \to \varepsilon^2 \frac{\alpha_\lambda}{\alpha_\mu}, \quad \alpha_\tau \to \varepsilon^2 \frac{\alpha_\tau}{\alpha_\mu}, \quad \alpha_\nu \to \varepsilon^2 \frac{\alpha_\nu}{\alpha_\mu}, \quad \alpha_p \to \varepsilon^2 \frac{\alpha_p}{\alpha_\mu}.$$

(III) *If $\lambda_0 = \infty$, $\mu_1 < \infty$ then estimates* (1.4) *hold for the displacements $\boldsymbol{w}^\varepsilon$, and under the condition*

$$p_* < \infty \tag{1.6}$$

(1.5) *hold for the pressures q^ε and p^ε in the fluid component.*

If the conditions

$$0 < p_2; \quad \boldsymbol{F} = \nabla\Phi, \quad \frac{\partial\Phi}{\partial t}, \quad \left|\frac{\partial\boldsymbol{F}}{\partial t}\right| \in L^2(\Omega_T) \tag{1.7}$$

hold instead of (1.6) *then*

$$\max_{0 \le t \le T} (\|(1 - \chi^\varepsilon)\nabla_x(\alpha_\lambda \boldsymbol{w}^\varepsilon(t))\|_{2,\Omega} + \|\chi^\varepsilon \operatorname{div}_x(\alpha_\lambda \boldsymbol{w}^\varepsilon(t))\|_{2,\Omega}) \le I_F^{(1)}, \tag{1.8}$$

where $I_F^{(1)} = C\| |\boldsymbol{F}| + |\partial\Phi/\partial t| + |\partial\boldsymbol{F}/\partial t| \|_{2,\Omega_T}$ and C is a constant independent of ε. These last estimates (1.8) *imply* (1.5).

Theorem 2. *Let $\lambda_0 < \infty$, $\mu_0 = 0$. Then the functions $\boldsymbol{w}^\varepsilon$ admit extensions $\boldsymbol{u}^\varepsilon$ from the domain $\Omega_s^\varepsilon \times (0,T)$ to the domain Ω_T, such that the sequence $\{\boldsymbol{u}^\varepsilon\}$ converges strongly in $L^2(\Omega_T)$ and weakly in $L^2\big[(0,T); W_2^1(\Omega)\big]$ to a function $\boldsymbol{u}$. At the same time, the sequences $\{\boldsymbol{w}^\varepsilon\}$, $\{p^\varepsilon\}$, $\{q^\varepsilon\}$, and $\{\pi^\varepsilon\}$ converge weakly in $L^2(\Omega_T)$ to $\boldsymbol{w}$, p, q, and π respectively.*

(I) *If $\mu_1 = \infty$ or the porous space is disconnected (the case of isolated pores for $\gamma \cap \partial Y = \varnothing$) then $\boldsymbol{w} = \boldsymbol{u}$ and the functions $\boldsymbol{u}$, p, q, and π satisfy the following initial-boundary value problem in the domain Ω_T:*

$$\tau_0 \hat{\rho} \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \operatorname{div}_x \big\{ \lambda_0 \mathbb{A}_0^s : \mathbb{D}(x, \boldsymbol{u}) + B_0^s \operatorname{div}_x \boldsymbol{u} + B_1^s q - (q + \pi) \cdot \mathbb{I} \big\} + \hat{\rho} \boldsymbol{F}, \tag{1.9}$$

$$\frac{1}{\eta_0} \pi + C_0^s : \mathbb{D}(x, \boldsymbol{u}) + a_0^s \operatorname{div}_x \boldsymbol{u} + a_1^s q = 0, \tag{1.10}$$

$$\frac{1}{p_*} p + \frac{1}{\eta_0} \pi + \operatorname{div}_x \boldsymbol{u} = 0, \quad p + \nu_0 p_*^{-1} \frac{\partial p}{\partial t} = q, \tag{1.11}$$

where $\hat{\rho} = m\rho_f + (1-m)\rho_s$, $m = \int_Y \chi(\boldsymbol{y})\, dy$. The strictly positive definite constant fourth-rank symmetric tensor $\mathbb{A}_0^s$, the matrices C_0^s, B_0^s, and B_1^s, and the constants a_0^s and a_1^s are defined below by (4.28), (4.30), and (4.31).

The differential equations (1.9) are endowed with the homogeneous initial and boundary conditions

$$\tau_0 \boldsymbol{u}(\boldsymbol{x}, 0) = \tau_0 \frac{\partial \boldsymbol{u}}{\partial t}(\boldsymbol{x}, 0) = 0, \ \boldsymbol{x} \in \Omega, \quad \boldsymbol{u}(\boldsymbol{x}, t) = 0, \ \boldsymbol{x} \in S, \ t > 0. \tag{1.12}$$

(II) *If $\mu_1 < \infty$ then the weak limits $\boldsymbol{u}$, $\boldsymbol{w}^f$, p, q, π of the sequences $\{\boldsymbol{u}^\varepsilon\}$, $\{\chi^\varepsilon \boldsymbol{w}^\varepsilon\}$, $\{p^\varepsilon\}$, $\{q^\varepsilon\}$, $\{\pi^\varepsilon\}$ satisfy in Ω_T the initial-boundary value problem consisting of the momentum balance equation:*

$$\tau_0 \rho_f \frac{\partial \boldsymbol{v}}{\partial t} + \tau_0 \rho_s (1 - m) \frac{\partial^2 \boldsymbol{u}}{\partial t^2} + \nabla(q + \pi) - \hat{\rho} \boldsymbol{F} = \operatorname{div}_x \big\{ \lambda_0 \mathbb{A}_0^s : \mathbb{D}(x, \boldsymbol{u}) + B_0^s \operatorname{div}_x \boldsymbol{u} + B_1^s q \big\}, \tag{1.13}$$

and the continuity equations (1.10) for the solid component, the continuity equation and the state equation:

$$\frac{1}{p_*} p + \frac{1}{\eta_0} \pi + \operatorname{div}_x \boldsymbol{w}^f = (m - 1) \operatorname{div}_x \boldsymbol{u}, \quad p + \nu_0 p_*^{-1} \frac{\partial p}{\partial t} = q \tag{1.14}$$

for the fluid component and the relations

$$\boldsymbol{v} = m \frac{\partial \boldsymbol{u}}{\partial t} + \int_0^t B_1(\mu_1, t - \tau) \cdot \boldsymbol{z}(\boldsymbol{x}, \tau)\, d\tau, \tag{1.15}$$

$$\boldsymbol{z}(\boldsymbol{x}, t) = -\frac{1}{m} \nabla q(\boldsymbol{x}, t) + \rho_f \boldsymbol{F}(\boldsymbol{x}, t) - \tau_0 \rho_f \frac{\partial^2 \boldsymbol{u}}{\partial t^2}(\boldsymbol{x}, t),$$

in the case of $\tau_0 > 0$ and $\mu_1 > 0$, or Darcy's law in the form

$$\boldsymbol{v} = m \frac{\partial \boldsymbol{u}}{\partial t} + B_2(\mu_1) \cdot \Big(-\frac{1}{m} \nabla q + \rho_f \boldsymbol{F} \Big), \tag{1.16}$$

in the case of $\tau_0 = 0$ and $\mu_1 > 0$, or, finally, the momentum balance equation for the fluid component in the form

$$\tau_0 \rho_f \frac{\partial \boldsymbol{v}}{\partial t} = \tau_0 \rho_f B_3 \cdot \frac{\partial^2 \boldsymbol{u}}{\partial t^2} + (m\mathbb{I} - B_3) \cdot \Big(-\frac{1}{m} \nabla q + \rho_f \boldsymbol{F} \Big), \tag{1.17}$$

in the case of $\tau_0 > 0$ and $\mu_1 = 0$.

Here $\boldsymbol{v} = \partial \boldsymbol{w}^f / \partial t$ and $\mathbb{A}_0^s$, B_0^s, and B_1^s are the same as in (1.9).

This problem is endowed with initial and boundary conditions (1.12) for the displacements of the solid component and the homogeneous initial condition and boundary condition

$$\boldsymbol{v}(\boldsymbol{x}, t) \cdot n(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in S, \ t > 0, \tag{1.18}$$

for the velocity $\boldsymbol{v}$ of the fluid component.

In (1.15)–(1.18) $n(\boldsymbol{x})$ is the unit normal vector to S at a point $\boldsymbol{x} \in S$, and the matrix $B_1(\mu_1, t)$ and strictly positive definite symmetric matrices $B_2(\mu_1)$ and $(m\mathbb{I} - B_3)$ are defined below by (4.39), (4.41), and (4.43).

Theorem 3. *Let $\lambda_0 = \infty$.*

(I) If $\mu_1 < \infty$ and one of the conditions (1.6) or (1.7) holds then the sequences $\{\chi^\varepsilon \boldsymbol{w}^\varepsilon\}$, $\{p^\varepsilon\}$, and $\{q^\varepsilon\}$ converge weakly in $L^2(\Omega_T)$ to $\boldsymbol{w}^f$, p, and q respectively. The functions $\boldsymbol{w}^\varepsilon$ admit extensions $\boldsymbol{u}^\varepsilon$ from the domain $\Omega^\varepsilon_s \times (0,T)$ to the domain Ω_T such that the sequence $\{\boldsymbol{u}^\varepsilon\}$ converges strongly in $L^2(\Omega_T)$ and weakly in $L^2\big[(0,T); W^1_2(\Omega)\big]$ to zero and

(1) if $\tau_0 > 0$ and $\mu_1 > 0$ then the functions $\boldsymbol{v} = \partial \boldsymbol{w}^f/\partial t$, p, and q solve the problem (F_1) in the domain Ω_T, where

$$\boldsymbol{v} = \int\limits_0^t B_1(\mu_1, t-\tau)\cdot \boldsymbol{z}_0(\boldsymbol{x},\tau)\, d\tau, \quad \boldsymbol{z}_0 = -\frac{1}{m}\nabla q + \rho_f \boldsymbol{F}, \tag{1.19}$$

$$p + \nu_0 p_*^{-1}\frac{\partial p}{\partial t} = q, \quad \frac{1}{p_*}\frac{\partial p}{\partial t} + \operatorname{div}_x \boldsymbol{v} = 0; \tag{1.20}$$

(2) if $\tau_0 = 0$ and $\mu_1 > 0$ then the functions $\boldsymbol{v}$, p, and q solve the problem (F_2) in the domain Ω_T, where $\boldsymbol{v}$ satisfies Darcy's law in the form

$$\boldsymbol{v} = B_2(\mu_1)\cdot\Big(-\frac{1}{m}\nabla q + \rho_f \boldsymbol{F}\Big), \tag{1.21}$$

and the pressures p and q satisfy (1.20);

(3) if $\tau_0 > 0$ and $\mu_1 = 0$ then the functions $\boldsymbol{v}$, p, and q solve the problem (F_3) in the domain Ω_T, where $\boldsymbol{v}$ satisfies the momentum balance equation for the fluid component in the form

$$\tau_0 \rho_f \frac{\partial \boldsymbol{v}}{\partial t} = (m\mathbb{I} - B_3)\cdot\Big(-\frac{1}{m}\nabla q + \rho_f \boldsymbol{F}\Big), \tag{1.22}$$

and the pressures p and q satisfy (1.20).

All these problems are endowed with homogeneous initial conditions and boundary condition (1.18).

In (1.19), (1.21), and (1.22) the matrices $B_1(\mu_1, t)$, $B_2(\mu_1)$, and B_3 are the same as in Theorem 2.

(II) If $\mu_1 < \infty$ and (1.7) hold then the sequence $\{\alpha_\lambda \boldsymbol{u}^\varepsilon\}$ converges strongly in $L^2(\Omega_T)$ and weakly in $L^2((0,T); W^1_2(\Omega))$ to a function $\boldsymbol{u}$, and the sequence $\{\pi^\varepsilon\}$ converges weakly in $L^2(\Omega_T)$ to a function π. The limiting functions satisfy the boundary value problem

$$0 = \operatorname{div}_x\big\{\mathbb{A}^s_0 : \mathbb{D}(x,\boldsymbol{u}) + B^s_0 \operatorname{div}_x \boldsymbol{u} + B^s_1 q - (q+\pi)\cdot\mathbb{I}\big\} + \hat{\rho}\boldsymbol{F}, \quad \boldsymbol{x}\in\Omega, \tag{1.23}$$

$$\frac{1}{\eta_2}\pi + C^s_0 : \mathbb{D}(x,\boldsymbol{u}) + a^s_0 \operatorname{div}_x \boldsymbol{u} + a^s_1 q = 0, \quad \boldsymbol{x}\in\Omega, \tag{1.24}$$

where the function q is referred to as given. It is defined from the corresponding data of Problems F_1–F_3 (the choice of the problem depends on τ_0 and μ_1). The strictly positive definite constant fourth-rank symmetric tensor $\mathbb{A}^s_0$, the matrices C^s_0, B^s_0, and B^s_1, and the constants a^s_0 and a^s_1 are defined below by (4.28), (4.30), and (4.31) in which we have $\eta_0 = \eta_2$ and $\lambda_0 = 1$.

This problem is endowed with the homogeneous boundary conditions.

(III) If $\mu_1 = \infty$, $p_1^{-1}, \eta_1^{-1} < \infty$, and $0 < \lambda_1 < \infty$ then there exist weak limits $\boldsymbol{w}^f$, p and π of the sequences $\{\alpha_\mu \varepsilon^{-2}\chi^\varepsilon \boldsymbol{w}^\varepsilon\}$, $\{p^\varepsilon\}$ and $\{\pi^\varepsilon\}$ and a strong limit $\boldsymbol{u}$ of the sequence $\{\alpha_\mu\varepsilon^{-2}\boldsymbol{u}^\varepsilon\}$, in $L^2(\Omega_T)$ which satisfy the following initial-boundary value problem in Ω_T:

$$\begin{gathered}\operatorname{div}_x\big\{\lambda_1 \mathbb{A}^s_0 : \mathbb{D}(x,\boldsymbol{u}) + B^s_0 \operatorname{div}_x \boldsymbol{u} + B^s_1 p - (p+\pi)\cdot\mathbb{I}\big\} + \hat{\rho}\boldsymbol{F} = 0,\\ \frac{\partial \boldsymbol{w}^f}{\partial t} = \frac{\partial \boldsymbol{u}}{\partial t} + B_2(1)\cdot\Big(-\frac{1}{m}\nabla p + \rho_f\boldsymbol{F}\Big),\\ \frac{1}{p_1}p + \frac{1}{\eta_1}\pi + \operatorname{div}_x \boldsymbol{w}^f = (m-1)\operatorname{div}_x \boldsymbol{u}, \quad \frac{1}{\eta_1}\pi + C^s_0 : \mathbb{D}(x,\boldsymbol{u}) + a^s_0\operatorname{div}_x\boldsymbol{u} + a^s_1 p = 0.\end{gathered} \tag{1.25}$$

Here the strictly positive definite constant fourth-rank symmetric tensor $\mathbb{A}^s_0$, *matrices* C^s_0, B^s_0, B^s_1, *and constants* a^s_0, a^s_1 *are defined below by* (4.28), (4.30), (4.31), *in which we have* $\eta_0 = \eta_1$ *and* $\lambda_0 = \lambda_1$.

This problem is endowed with the homogeneous initial and boundary conditions.

(IV) *If* $\mu_1 = \infty$ *and* $\lambda_1 = \infty$ *then the corresponding problem for displacements* $\{\alpha_\mu \varepsilon^{-2} \boldsymbol{w}^\varepsilon\}$ *was considered in parts* (I), (II) *of the present theorem.*

Theorem 4. *Let* $0 < \mu_0; \lambda_0 < \infty$. *Then the weak limits* $\boldsymbol{w}$, p, q, *and* π *of the sequences* $\{\boldsymbol{w}^\varepsilon\}$, $\{p^\varepsilon\}$, $\{\pi^\varepsilon\}$, *and* $\{q^\varepsilon\}$ *satisfy in* Ω_T *the following system of differential equations:*

$$\tau_0 \hat{\rho} \frac{\partial^2 \boldsymbol{w}}{\partial t^2} + \nabla(q+\pi) - \hat{\rho}\boldsymbol{F} = \operatorname{div}_x \Big(\mathbb{A}_2 : \mathbb{D}\Big(x, \frac{\partial \boldsymbol{w}}{\partial t}\Big) + \mathbb{A}_3 : \mathbb{D}(x, \boldsymbol{w}) + B_4 \operatorname{div}_x \boldsymbol{w}$$
$$+ \int_0^t \big(\mathbb{A}_4(t-\tau) : \mathbb{D}(x, \boldsymbol{w}(\boldsymbol{x},\tau)) + B_5(t-\tau) \operatorname{div}_x \boldsymbol{w}(\boldsymbol{x},\tau)\big)\, d\tau \Big), \tag{1.26}$$

$$\frac{1}{p_*} p + m \operatorname{div}_x \boldsymbol{w} = -\int_0^t \big(C_2(t-\tau) : \mathbb{D}(x, \boldsymbol{w}(\boldsymbol{x},\tau)) + a_2(t-\tau) \operatorname{div}_x \boldsymbol{w}(\boldsymbol{x},\tau)\big)\, d\tau, \tag{1.27}$$

$$\frac{1}{\eta_0} \pi + (1-m) \operatorname{div}_x \boldsymbol{w} = -\int_0^t \big(C_3(t-\tau) : \mathbb{D}(x, \boldsymbol{w}(\boldsymbol{x},\tau)) + a_3(t-\tau) \operatorname{div}_x \boldsymbol{w}(\boldsymbol{x},\tau)\big) d\tau, \tag{1.28}$$

$$q = p + \frac{\nu_0}{p_*} \frac{\partial p}{\partial t}. \tag{1.29}$$

Here $\mathbb{A}_2$–$\mathbb{A}_4$ *are fourth-rank tensors;* B_4, B_5, C_2, C_3 *are matrices; and* a_2, a_3 *are scalars. The exact expressions for these objects are given below by* (6.25)–(6.30).

The problem is supplemented by the homogeneous initial and boundary conditions. If the porous space is connected then $\mathbb{A}_2$ *is a strictly positive definite symmetric tensor. If the porous space is disconnected* (*isolated pores*) *then* $\mathbb{A}_2$ *is equal to zero and the system* (1.26) *degenerates into the nonlocal anisotropic Lamé system with strictly positive definite symmetric tensor* $\mathbb{A}_3$.

§ 2. Preliminaries

2.1. Two-scale convergence. Justification of Theorems 2–4 relies on the systematic use of the two-scale convergence method proposed by G. Nguetseng [8] and applied recently to a wide range of homogenization problems (for example, see the survey [9]).

Definition 2. A sequence $\{\varphi^\varepsilon\} \subset L^2(\Omega_T)$ is said to be *two-scale convergent* to a limit $\varphi \in L^2(\Omega_T \times Y)$ if and only if for every function $\sigma = \sigma(\boldsymbol{x}, t, \boldsymbol{y})$ 1-periodic in $\boldsymbol{y}$ the limit relation holds:

$$\lim_{\varepsilon \searrow 0} \int_{\Omega_T} \varphi^\varepsilon(\boldsymbol{x},t)\sigma(\boldsymbol{x},t,\boldsymbol{x}/\varepsilon)\, d\boldsymbol{x}dt = \int_{\Omega_T} \int_Y \varphi(\boldsymbol{x},t,\boldsymbol{y})\sigma(\boldsymbol{x},t,\boldsymbol{y})\, d\boldsymbol{y}d\boldsymbol{x}dt. \tag{2.1}$$

Existence and main properties of weakly convergent sequences are established by the following fundamental theorem [8, 9].

Theorem 5 (Nguetseng's theorem). 1. *Each bounded sequence in* $L^2(\Omega_T)$ *contains a subsequence two-scale convergent to some limit* $\varphi \in L^2(\Omega_T \times Y)$.

2. *Let the sequences* $\{\varphi^\varepsilon\}$ *and* $\{\varepsilon \nabla_x \varphi^\varepsilon\}$ *be uniformly bounded in* $L^2(\Omega_T)$. *Then there exist a 1-periodic function* $\varphi = \varphi(\boldsymbol{x}, t, \boldsymbol{y})$ *in* $\boldsymbol{y}$ *and a subsequence* $\{\varphi^\varepsilon\}$ *such that* $\varphi, \nabla_y \varphi \in L^2(\Omega_T \times Y)$, *and* φ^ε *and* $\varepsilon \nabla_x \varphi^\varepsilon$ *two-scale converge to* φ *and* $\nabla_y \varphi$ *respectively.*

3. *Let* $\{\varphi^\varepsilon\}$ *and* $\{\nabla_x \varphi^\varepsilon\}$ *be bounded sequences in* $L^2(\Omega_T)$. *Then there exist functions* $\varphi \in L^2(\Omega_T)$, $\psi \in L^2(\Omega_T \times Y)$ *and a subsequence from* $\{\varphi^\varepsilon\}$ *such that* ψ *is 1-periodic in* $\boldsymbol{y}$, $\nabla_y \psi \in L^2(\Omega_T \times Y)$, φ^ε *and* $\nabla_x \varphi^\varepsilon$ *two-scale converge to* φ *and* $\nabla_x \varphi(\boldsymbol{x},t) + \nabla_y \psi(\boldsymbol{x},t,\boldsymbol{y})$, *respectively.*

Corollary 2.1. *Let $\sigma \in L^2(Y)$ and $\sigma^\varepsilon(\boldsymbol{x}) = \sigma(\boldsymbol{x}/\varepsilon)$. Assume that a sequence $\{\varphi^\varepsilon\} \subset L^2(\Omega_T)$ two-scale converges to $\varphi \in L^2(\Omega_T \times Y)$. Then the sequence $\sigma^\varepsilon\varphi^\varepsilon$ two-scale converges to $\sigma\varphi$.*

2.2. An extension lemma. A typical difficulty in homogenization problems, like (0.1)–(0.4), arises because of the fact that the bounds on the gradient of displacement $\nabla_x \boldsymbol{w}^\varepsilon$ may be distinct in the domains Ω_s and Ω_f (the fluid and solid phases), which do not permit us to use more strong estimates. The classical approach to overcoming this difficulty consists of extension to the whole Ω of the displacement field defined merely on Ω_s with preserving the bounds on the gradient in Ω_s. We formulate the following lemma in appropriate form:

Lemma 2.1 [10, 11]. *Let Assumption 1 on the geometry of the domain Ω_s^ε hold, $\psi^\varepsilon \in W_2^1(\Omega_s^\varepsilon)$, and $\psi^\varepsilon = 0$ on the boundary $S_s^\varepsilon = \partial\Omega_s^\varepsilon \cap \partial\Omega$. Then there exists a function $\sigma^\varepsilon \in W_2^1(\Omega)$ whose restriction on the subdomain Ω_s^ε coincides with ψ^ε; i.e.,*

$$(1 - \chi^\varepsilon(\boldsymbol{x}))(\sigma^\varepsilon(\boldsymbol{x}) - \psi^\varepsilon(\boldsymbol{x})) = 0, \quad \boldsymbol{x} \in \Omega. \tag{2.2}$$

Moreover, the estimates hold:

$$\|\sigma^\varepsilon\|_{2,\Omega} \leq C\|\psi^\varepsilon\|_{2,\Omega_s^\varepsilon}, \quad \|\nabla_x\sigma^\varepsilon\|_{2,\Omega} \leq C\|\nabla_x\psi^\varepsilon\|_{2,\Omega_s^\varepsilon}, \tag{2.3}$$

where the constant C depends only on the geometry of Y and does not depend on ε.

2.3. The Friedrichs–Poincaré inequality in a periodic structure. The following lemma was proved by L. Tartar in [3, the Appendix]. It specifies the constant in the Friedrichs–Poincaré inequality for an ε-periodic geometrical structure.

Lemma 2.2. *Let the assumptions on the geometry of the domain Ω_f^ε be satisfied. Then for every function $\varphi \in \overset{\circ}{W}{}_2^1(\Omega_f^\varepsilon)$ the inequality holds:*

$$\int\limits_{\Omega_f^\varepsilon} |\varphi|^2\, dx \leq C\varepsilon^2 \int\limits_{\Omega_f^\varepsilon} |\nabla_x\varphi|^2\, dx$$

with some constant C independent of ε.

We further denote $\langle\Phi\rangle_Y = \int_Y \Phi\, dy$, $\langle\Phi\rangle_{Y_f} = \int_Y \chi\Phi\, dy$, $\langle\Phi\rangle_{Y_s} = \int_Y (1-\chi)\Phi\, dy$, $\langle\varphi\rangle_\Omega = \int_\Omega \varphi\, dx$, $\langle\varphi\rangle_{\Omega_T} = \int_{\Omega_T} \varphi\, dxdt$; if $\mathbf{a}$ and $\mathbf{b}$ are two vectors then the matrix $\mathbf{a}\otimes\mathbf{b}$ is defined by the formula $(\mathbf{a}\otimes\mathbf{b})\cdot\mathbf{c} = \mathbf{a}(\mathbf{b}\cdot\mathbf{c})$ for every vector $\mathbf{c}$; if B and C are two matrices then $B \otimes C$ is a forth-rank tensor whose convolution with each matrix A is defined by the formula $(B \otimes C) : A = B(C : A)$; by $\mathbb{I}^{ij}$ we denote the matrix with a sole nonvanishing entry equal to one and standing in the ith row and jth column; finally, $J^{ij} = \frac{1}{2}(\mathbb{I}^{ij} + \mathbb{I}^{ji}) = \frac{1}{2}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j + \boldsymbol{e}_j \otimes \boldsymbol{e}_i)$, where $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ are the standard Cartesian basis vectors.

§ 3. Proof of Theorem 1

I. Let $\lambda_0 < \infty$. If $\tau_0 > 0$ then estimates (1.4) follow from the inequality

$$\max_{0<t<T}\Big(\sqrt{\alpha_\eta}\Big\|\operatorname{div}_x \frac{\partial\boldsymbol{w}^\varepsilon}{\partial t}(t)\Big\|_{2,\Omega_s^\varepsilon} + \sqrt{\alpha_\lambda}\Big\|\nabla_x\frac{\partial\boldsymbol{w}^\varepsilon}{\partial t}(t)\Big\|_{2,\Omega_s^\varepsilon} + \sqrt{\alpha_\tau}\Big\|\frac{\partial^2\boldsymbol{w}^\varepsilon}{\partial t^2}(t)\Big\|_{2,\Omega}$$
$$+\sqrt{\alpha_p}\Big\|\operatorname{div}_x \frac{\partial\boldsymbol{w}^\varepsilon}{\partial t}(t)\Big\|_{2,\Omega_f^\varepsilon}\Big) + \sqrt{\alpha_\mu}\Big\|\chi^\varepsilon\nabla_x\frac{\partial^2\boldsymbol{w}^\varepsilon}{\partial t^2}\Big\|_{2,\Omega_T} + \sqrt{\alpha_\nu}\Big\|\chi^\varepsilon\operatorname{div}_x\frac{\partial^2\boldsymbol{w}^\varepsilon}{\partial t^2}\Big\|_{2,\Omega_T} \leq \frac{C_0}{\sqrt{\alpha_\tau}}, \tag{3.1}$$

where C_0 is independent of ε. The last estimates appear if we differentiate the equation for $\boldsymbol{w}^\varepsilon$ with respect to time, multiply the results by $\partial^2\boldsymbol{w}^\varepsilon/\partial t^2$, and integrate by parts. The same estimates guarantee the existence and uniqueness of a weak solution for (0.1)–(0.4).

If $p_* < \infty$ and $\eta_0 < \infty$ then the pressures p^ε and π^ε are bounded from (0.2) with the help of (3.1). The pressure q^ε is bounded from the equality

$$q^\varepsilon = p^\varepsilon - \alpha_\nu \chi^\varepsilon \operatorname{div}_x \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t} \tag{3.2}$$

which follows from (3.1) and the corresponding bound for the pressure p^ε.

If $p_* = \infty$ then (1.5) for the sum of the pressures $(q^\varepsilon + \pi^\varepsilon)$ follows from the basic integral identity (1.2) and (3.1) as an estimate for the corresponding functional, if we renormalize the pressures as

$$\int_\Omega \big(q^\varepsilon(\boldsymbol{x},t) + \pi^\varepsilon(\boldsymbol{x},t)\big)\, d\boldsymbol{x} = 0.$$

Indeed, the basic integral identity (1.2) and (3.1) imply

$$\Big| \int_\Omega (q^\varepsilon + \pi^\varepsilon) \operatorname{div}_x \boldsymbol{\psi}\, d\boldsymbol{x} \Big| \leq C \|\nabla \boldsymbol{\psi}\|_{2,\Omega}.$$

Choosing $\boldsymbol{\psi}$ now so that $(q^\varepsilon + \pi^\varepsilon) = \operatorname{div}_x \boldsymbol{\psi}$, we obtain the desired estimate for the sum of the pressures $(q^\varepsilon + \pi^\varepsilon)$. Such a choice is always possible (see [12]), if we put

$$\boldsymbol{\psi} = \nabla \varphi + \boldsymbol{\psi}_0, \quad \operatorname{div}_x \boldsymbol{\psi}_0 = 0, \quad \triangle \varphi = q^\varepsilon + \pi^\varepsilon, \quad \varphi|_{\partial\Omega} = 0, \quad (\nabla\varphi + \boldsymbol{\psi}_0)|_{\partial\Omega} = 0.$$

Note that the renormalization of the pressures transforms the continuity equations (0.2) for the pressures into

$$\frac{1}{\alpha_p} p^\varepsilon + \chi^\varepsilon \operatorname{div}_x \boldsymbol{w}^\varepsilon = -\frac{1}{m} \beta^\varepsilon \chi^\varepsilon, \tag{3.3}$$

$$\frac{1}{\alpha_\eta} \pi^\varepsilon + (1-\chi^\varepsilon) \operatorname{div}_x \boldsymbol{w}^\varepsilon = \frac{1}{(1-m)} (1-\chi^\varepsilon) \beta^\varepsilon, \tag{3.4}$$

where $\beta^\varepsilon = \langle (1-\chi^\varepsilon) \operatorname{div}_x \boldsymbol{w}^\varepsilon \rangle_\Omega$. The case $\eta_0 = \infty$ is considered in the same way. Note that for all situations the basic integral identity (1.2) permits us to estimate only the sum $(q^\varepsilon + \pi^\varepsilon)$. However, since the product of these two functions q^ε and π^ε is equal to zero, it suffices to find bounds for each of these functions. The pressure p^ε is bounded due to the state equation (0.2), if in this equation we replace the term $(\alpha_\nu/\alpha_p)\partial p^\varepsilon/\partial t$ from the continuity equation (3.3) by $-\chi^\varepsilon\big(\operatorname{div}_x \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t} + \frac{1}{m}\frac{\partial \beta^\varepsilon}{\partial t}\big)$ and use (3.1).

Estimation of $\boldsymbol{w}^\varepsilon$ in the case $\tau_0 = 0$ is not simple, and we outline it in more detail.

Let $\mu_1 > 0$ and $\tau_0 = 0$. As usual, we find the basic estimates if we multiply the equations for $\boldsymbol{w}^\varepsilon$ by $\partial \boldsymbol{w}^\varepsilon/\partial t$ and then integrate all obtained terms by parts. The only one term $\rho^\varepsilon \boldsymbol{F} \cdot \partial \boldsymbol{w}^\varepsilon/\partial t$ requires additional consideration here. First of all, by Lemma 2.1 we construct an extension $\boldsymbol{u}^\varepsilon$ of $\boldsymbol{w}^\varepsilon$ from Ω_s^ε to Ω_f^ε so that $\boldsymbol{u}^\varepsilon = \boldsymbol{w}^\varepsilon$ in Ω_s^ε, $\boldsymbol{u}^\varepsilon \in W_2^1(\Omega)$, and

$$\|\boldsymbol{u}^\varepsilon\|_{2,\Omega} \leq C \|\nabla_x \boldsymbol{u}^\varepsilon\|_{2,\Omega} \leq \frac{C}{\sqrt{\alpha_\lambda}} \|(1-\chi^\varepsilon)\sqrt{\alpha_\lambda} \nabla_x \boldsymbol{w}^\varepsilon\|_{2,\Omega}.$$

We then estimate $\|\boldsymbol{w}^\varepsilon\|_{2,\Omega}$ with the Friedrichs–Poincaré inequality in a periodic structure (Lemma 2.2) for the difference $(\boldsymbol{u}^\varepsilon - \boldsymbol{w}^\varepsilon)$:

$$\begin{aligned}
\|\boldsymbol{w}^\varepsilon\|_{2,\Omega} &\leq \|\boldsymbol{u}^\varepsilon\|_{2,\Omega} + \|\boldsymbol{u}^\varepsilon - \boldsymbol{w}^\varepsilon\|_{2,\Omega} \leq \|\boldsymbol{u}^\varepsilon\|_{2,\Omega} + C\varepsilon \|\chi^\varepsilon \nabla_x (\boldsymbol{u}^\varepsilon - \boldsymbol{w}^\varepsilon)\|_{2,\Omega} \\
&\leq \|\boldsymbol{u}^\varepsilon\|_{2,\Omega} + C\varepsilon \|\nabla_x \boldsymbol{u}^\varepsilon\|_{2,\Omega} + C\big(\varepsilon \alpha_\mu^{-\frac{1}{2}}\big) \|\chi^\varepsilon \sqrt{\alpha_\mu} \nabla_x \boldsymbol{w}^\varepsilon\|_{2,\Omega} \\
&\leq \frac{C}{\sqrt{\alpha_\lambda}} \|(1-\chi^\varepsilon)\sqrt{\alpha_\lambda} \nabla_x \boldsymbol{w}^\varepsilon\|_{2,\Omega} + C\big(\varepsilon \alpha_\mu^{-\frac{1}{2}}\big) \|\chi^\varepsilon \sqrt{\alpha_\mu} \nabla_x \boldsymbol{w}^\varepsilon\|_{2,\Omega}.
\end{aligned}$$

Next we pass the derivative with respect to time from $\partial \boldsymbol{w}^\varepsilon/\partial t$ to $\rho^\varepsilon \boldsymbol{F}$ and estimate all positive terms (including the term $\alpha_\nu \chi^\varepsilon \operatorname{div}_x \partial \boldsymbol{w}^\varepsilon/\partial t$) in a usual way with the help of the Hölder and Gronwall inequalities. The rest of the proof is the same as for the case $\tau_0 > 0$ if we use a consequence of (3.1):

$$\max_{0<t<T} \alpha_\tau \left\| \frac{\partial^2 \boldsymbol{w}^\varepsilon}{\partial t^2}(t) \right\|_{2,\Omega} \leq C.$$

II. The proof of this part of the theorem is obvious, because the renormalization reduces this case to the case of $\mu_1 = 1$ and $\tau_0 = 0$, which was already considered in the first part of the theorem.

III. Let $\lambda_0 = \infty$, $\mu_1 < \infty$ and let (1.7) hold. It is obvious that (1.4) are still valid. Estimates (1.8) follow from the basic integral identity for $\alpha_\lambda \boldsymbol{w}^\varepsilon$ in the same way as in the case of (1.4). The main difference here is in the term $\rho^\varepsilon \boldsymbol{F} \cdot \alpha_\lambda \partial \boldsymbol{w}^\varepsilon/\partial t$ which transforms now to

$$\Upsilon \equiv \rho_f \boldsymbol{F} \cdot \alpha_\lambda \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t} + (\rho_f - \rho_f)(1 - \chi^\varepsilon) \boldsymbol{F} \cdot \alpha_\lambda \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}.$$

The integral of the first term in Υ transforms as follows:

$$\rho_f \int_0^t \int_\Omega \nabla \Phi \cdot \alpha_\lambda \frac{\partial \boldsymbol{w}^\varepsilon}{\partial \tau}\, d\boldsymbol{x} d\tau = -\rho_f \int_0^t \int_\Omega \Phi \alpha_\lambda \operatorname{div}_x \frac{\partial \boldsymbol{w}^\varepsilon}{\partial \tau}\, d\boldsymbol{x} d\tau$$

$$= -\rho_f \int_0^t \int_\Omega \Big(\chi^\varepsilon \cdot \Phi \alpha_\lambda \operatorname{div}_x \frac{\partial \boldsymbol{w}^\varepsilon}{\partial \tau} + (1 - \chi^\varepsilon) \cdot \Phi \alpha_\lambda \operatorname{div}_x \frac{\partial \boldsymbol{w}^\varepsilon}{\partial \tau} \Big)\, d\boldsymbol{x} d\tau$$

$$= -\rho_f \int_\Omega \Big(\chi^\varepsilon \cdot \Phi \alpha_\lambda \operatorname{div}_x \boldsymbol{w}^\varepsilon + (1 - \chi^\varepsilon) \cdot \Phi \alpha_\lambda \operatorname{div}_x \boldsymbol{u}^\varepsilon \Big)\, d\boldsymbol{x}$$

$$+\rho_f \int_0^t \int_\Omega \Big(\chi^\varepsilon \cdot \Phi_\tau \alpha_\lambda \operatorname{div}_x \boldsymbol{w}^\varepsilon + (1 - \chi^\varepsilon) \cdot \Phi_\tau \alpha_\lambda \operatorname{div}_x \boldsymbol{u}^\varepsilon \Big)\, d\boldsymbol{x} d\tau,$$

and it is bounded by the terms

$$\int_\Omega \left(\chi^\varepsilon (\alpha_p \alpha_\lambda^{-1})(\alpha_\lambda \operatorname{div}_x \boldsymbol{w}^\varepsilon)^2 + (1 - \chi^\varepsilon) |\alpha_\lambda \nabla_x \boldsymbol{u}^\varepsilon|^2 \right) d\boldsymbol{x},$$

which appear in the basic identity on using the continuity equation (0.2) for the fluid component.

The integral of the second term in Υ is bounded by the positive term $\langle (1 - \chi^\varepsilon) |\alpha_\lambda \nabla_x \boldsymbol{u}^\varepsilon|^2 \rangle_\Omega$ just as before.

Estimates (1.5) follow now from (1.8). Here, as before, the sum of pressures $(q^\varepsilon + \pi^\varepsilon)$ is bounded from the basic integral identity (1.2) as a corresponding functional, and the pressure p^ε is bounded from (3.2) due to the bound (3.1) for the divergence of the velocity of the fluid component $\chi^\varepsilon \operatorname{div}_x(\partial \boldsymbol{w}^\varepsilon/\partial t)$ and bound for the pressure q^ε.

If instead of (1.7) we have condition (1.6) then the estimates (1.5) for the pressures p^ε and q^ε follow from (0.2), (3.2), and (3.1). Note that in this case β^ε is equal to 0. $\square$

§ 4. Proof of Theorem 2

4.1. Weak and two-scale limits of the sequences of displacement and pressures. All conditions of Theorem 1 hold under the assumptions of Theorem 2. Therefore, the sequences $\{p^\varepsilon\}$, $\{q^\varepsilon\}$, $\{\pi^\varepsilon\}$, and $\{\boldsymbol{w}^\varepsilon\}$ are bounded in $L^2(\Omega_T)$ uniformly in ε. Hence there exist a subsequence of the small parameters $\{\varepsilon > 0\}$ and the functions p, q, π, and $\boldsymbol{w}$ such that

$$p^\varepsilon \to p, \quad \pi^\varepsilon \to \pi, \quad q^\varepsilon \to q, \quad \boldsymbol{w}^\varepsilon \to \boldsymbol{w} \quad \text{weakly in } L^2(\Omega_T) \text{ as } \varepsilon \searrow 0. \tag{4.1}$$

Moreover, the sequence $\{(1-\chi^\varepsilon)\nabla \boldsymbol{w}^\varepsilon\}$ is bounded in $L^2(\Omega_T)$ uniformly in ε. By Lemma 2.1 (an extension lemma) there is a function $\boldsymbol{u}^\varepsilon \in L^\infty\big(0,T;W^1_2(\Omega)\big)$ such that $\boldsymbol{u}^\varepsilon = \boldsymbol{w}^\varepsilon$ in Ω_s and a family $\{\boldsymbol{u}^\varepsilon\}$ is bounded in $L^\infty\big(0,T;W^1_2(\Omega)\big)$ uniformly in ε. Therefore, it is possible to extract a subsequence of $\{\varepsilon > 0\}$ such that

$$\boldsymbol{u}^\varepsilon \to \boldsymbol{u} \text{ weakly in } L^2\big(0,T;W^1_2(\Omega)\big). \tag{4.2}$$

Note also that

$$\chi^\varepsilon \alpha_\mu \mathbb{D}(x,\boldsymbol{w}^\varepsilon) \to 0 \text{ strongly in } L^2(\Omega_T) \text{ as } \varepsilon \searrow 0. \tag{4.3}$$

Relabeling if necessary, we assume that the sequences converge themselves.

By the Nguetseng theorem, there are the functions $P(\boldsymbol{x},t,\boldsymbol{y})$, $\Pi(\boldsymbol{x},t,\boldsymbol{y})$, $Q(\boldsymbol{x},t,\boldsymbol{y})$, $\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$, and $\boldsymbol{U}(\boldsymbol{x},t,\boldsymbol{y})$ 1-periodic in $\boldsymbol{y}$ such that the sequences $\{p^\varepsilon\}$, $\{\pi^\varepsilon\}$, $\{q^\varepsilon\}$, $\{\boldsymbol{w}^\varepsilon\}$, and $\{\nabla \boldsymbol{u}^\varepsilon\}$ two-scale converge to $P(\boldsymbol{x},t,\boldsymbol{y})$, $\Pi(\boldsymbol{x},t,\boldsymbol{y})$, $Q(\boldsymbol{x},t,\boldsymbol{y})$, $\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$, and $\nabla_x \boldsymbol{u} + \nabla_y \boldsymbol{U}(\boldsymbol{x},t,\boldsymbol{y})$, respectively.

Moreover, the sequence $\{\operatorname{div}_x \boldsymbol{w}^\varepsilon\}$ converges weakly to the function $\operatorname{div}_x \boldsymbol{w}$ and $\boldsymbol{u} \in L^2\big(0,T;\overset{\circ}{W}{}^1_2(\Omega)\big)$. The last assertion for a disconnected porous space follows from the containment $\boldsymbol{u}^\varepsilon \in L^2\big(0,T;\overset{\circ}{W}{}^1_2(\Omega)\big)$. For the connected porous space it follows from the Friedrichs–Poincaré inequality for $\boldsymbol{u}^\varepsilon$ in the ε-layer of the boundary S and from the convergence of the sequence $\{\boldsymbol{u}^\varepsilon\}$ to $\boldsymbol{u}$ strong in $L^2(\Omega_T)$ and weak in $L^2\big((0,T);W^1_2(\Omega)\big)$.

4.2. The microscopic and macroscopic equations I.

Lemma 4.1. *For all $\boldsymbol{x} \in \Omega$ and $\boldsymbol{y} \in Y$ the weak and two-scale limits of the sequences $\{p^\varepsilon\}$, $\{\pi^\varepsilon\}$, $\{q^\varepsilon\}$, $\{\boldsymbol{w}^\varepsilon\}$, and $\{\boldsymbol{u}^\varepsilon\}$ satisfy the relations*

$$P = p\chi/m, \quad Q = q\chi/m; \tag{4.4}$$

$$\Pi/\eta_0 + (1-\chi)(\operatorname{div}_x \boldsymbol{u} + \operatorname{div}_y \boldsymbol{U}) = \beta(1-\chi)/(1-m); \tag{4.5}$$

$$\operatorname{div}_y \boldsymbol{W} = 0; \tag{4.6}$$

$$\boldsymbol{W} = \chi(\boldsymbol{y})\boldsymbol{W} + (1-\chi)\boldsymbol{u}; \tag{4.7}$$

$$q = p + \nu_0 p_*^{-1} \partial p/\partial t; \tag{4.8}$$

$$p/p_* + \operatorname{div}_x \boldsymbol{w} = (1-m)\operatorname{div}_x \boldsymbol{u} + \langle \operatorname{div}_y \boldsymbol{U}\rangle_{Y_s} - \beta; \tag{4.9}$$

$$\operatorname{div}_x \boldsymbol{u} + \langle \operatorname{div}_y \boldsymbol{U}\rangle_{Y_s} = \beta, \tag{4.10}$$

where $\beta = \int_\Omega \langle \operatorname{div}_y \boldsymbol{U}\rangle_{Y_s} d\boldsymbol{x}$ in the case $p_* + \eta_0 = \infty$; and $\beta = 0$ in the case $p_* + \eta_0 < \infty$.

Proof. In order to prove (4.4), insert into (1.2) a test function in the form $\boldsymbol{\psi}^\varepsilon = \varepsilon \boldsymbol{\psi}(\boldsymbol{x},t,\boldsymbol{x}/\varepsilon)$, where $\boldsymbol{\psi}(\boldsymbol{x},t,\boldsymbol{y})$ is an arbitrary function 1-periodic in $\boldsymbol{y}$ and compactly supported on Y_f. Passing to the limit as $\varepsilon \searrow 0$, we infer

$$\nabla_y Q(\boldsymbol{x},t,\boldsymbol{y}) = 0, \quad \boldsymbol{y} \in Y_f. \tag{4.11}$$

Passage to the weak and two-scale limits in the state equation (0.2) yields (4.8) and

$$Q = P + \frac{\nu_0}{p_*}\frac{\partial P}{\partial t}. \tag{4.12}$$

Taking into account (4.11) and (4.12), we obtain $\nabla_y P(\boldsymbol{x},t,\boldsymbol{y}) = 0$, $\boldsymbol{y} \in Y_f$. Next, passing the two-scale limit in the equalities $(1-\chi^\varepsilon)p^\varepsilon = 0$, $(1-\chi^\varepsilon)q^\varepsilon = 0$, we arrive at $(1-\chi)P = 0$ and $(1-\chi)Q = 0$, which justifies (4.4) along with (4.11) and (4.12).

Equations (4.5), (4.6), (4.9), and (4.10) appear as the results of passage to the two-scale limit in (3.3), (3.4) with the proper test functions. Thus, for example, (4.9) arises if we represent (3.3) in the form

$$\frac{1}{\alpha_p} p^\varepsilon + \operatorname{div}_x \boldsymbol{w}^\varepsilon = (1-\chi^\varepsilon)\operatorname{div}_x \boldsymbol{u}^\varepsilon - \frac{1}{m}\beta^\varepsilon \chi^\varepsilon, \tag{4.13}$$

multiply by an arbitrary function, independent of the "fast" variable $\boldsymbol{y} = \boldsymbol{x}/\varepsilon$, and then pass to the limit as $\varepsilon \searrow 0$. Equation (4.10) is derived rather similarly. In order to prove (4.6) it is sufficient to consider the two-scale limit relations in (4.13) with the test functions in the form $\varepsilon\psi(\boldsymbol{x}/\varepsilon)\, h(\boldsymbol{x},t)$, where ψ and h are arbitrary smooth test functions. In order to prove (4.7), it is sufficient to consider the two-scale limit relations in $(1-\chi^\varepsilon)(\boldsymbol{w}^\varepsilon - \boldsymbol{u}^\varepsilon) = 0$. □

Corollary 4.1. *If $p_* + \eta_0 = \infty$ then the weak limits p, π, and q satisfy the relations*

$$\langle p\rangle_\Omega = \langle \pi\rangle_\Omega = \langle q\rangle_\Omega = 0. \tag{4.14}$$

Lemma 4.2. *For all $(\boldsymbol{x},t)\in\Omega_T$ the relation*

$$\operatorname{div}_y\Big\{\lambda_0(1-\chi)(\mathbb{D}(y,\boldsymbol{U})+\mathbb{D}(x,\boldsymbol{u}))-\Big(\Pi+\frac{1}{m}q\chi\Big)\cdot\mathbb{I}\Big\}=0 \tag{4.15}$$

holds.

PROOF. Inserting a test function of the form $\boldsymbol{\psi}^\varepsilon = \varepsilon\boldsymbol{\psi}(\boldsymbol{x},t,\boldsymbol{x}/\varepsilon)$, where $\boldsymbol{\psi}(\boldsymbol{x},t,\boldsymbol{y})$ is an arbitrary function 1-periodic in $\boldsymbol{y}$ and vanishing on the boundary S, into (1.2) and passing to the limit as $\varepsilon\searrow 0$, we arrive at the desired microscopic relation on the cell Y. □

Lemma 4.3. *Let $\hat\rho = m\rho_f+(1-m)\rho_s$, $\boldsymbol{V}=\chi\partial\boldsymbol{w}/\partial t$, and $\boldsymbol{v}=\langle\boldsymbol{V}\rangle_Y$. Then the functions $\{\boldsymbol{u},\boldsymbol{v},q,\pi\}$ satisfy in Ω_T the system of macroscopic equations:*

$$\tau_0\rho_f\frac{\partial\boldsymbol{v}}{\partial t}+\tau_0\rho_s(1-m)\frac{\partial^2\boldsymbol{u}}{\partial t^2}=\operatorname{div}_x\{\lambda_0((1-m)\mathbb{D}(x,\boldsymbol{u})+\langle\mathbb{D}(y,\boldsymbol{U})\rangle_{Y_s})-(q+\pi)\cdot\mathbb{I}\}+\hat\rho\boldsymbol{F}. \tag{4.16}$$

PROOF. The equations arise as the limit of (1.2) with the test functions compactly supported in Ω_T and independent of the "fast" variable $\boldsymbol{y}=\boldsymbol{x}/\varepsilon$. □

4.3. Microscopic and macroscopic equations II.

Lemma 4.4. *If $\mu_1=\infty$ then the weak and two-scale limits of the sequences $\{\boldsymbol{u}^\varepsilon\}$ and $\{\boldsymbol{w}^\varepsilon\}$ coincide.*

PROOF. In order to verify, it is sufficient to consider the difference $(\boldsymbol{u}^\varepsilon-\boldsymbol{w}^\varepsilon)$ and apply the Friedrichs–Poincaré inequality (Lemma 2.2), just like in the proof of Theorem 1. □

Lemma 4.5. *Let $\mu_1<\infty$. Then the weak and two-scale limits of the sequences $\{q^\varepsilon\}$ and $\{\boldsymbol{w}^\varepsilon\}$ satisfy the microscopic relations*

$$\tau_0\rho_f\frac{\partial\boldsymbol{V}}{\partial t}=\mu_1\triangle_y\boldsymbol{V}-\nabla_yR-\frac{1}{m}\nabla_xq+\rho_f\boldsymbol{F},\quad \boldsymbol{y}\in Y_f, \tag{4.17}$$

$$\boldsymbol{V}=\frac{\partial\boldsymbol{u}}{\partial t},\quad \boldsymbol{y}\in\gamma, \tag{4.18}$$

in the case of $\mu_1>0$, and the relations

$$\tau_0\rho_f\frac{\partial\boldsymbol{V}}{\partial t}=-\nabla_yR-\frac{1}{m}\nabla_xq+\rho_f\boldsymbol{F},\quad \boldsymbol{y}\in Y_f, \tag{4.19}$$

$$\Big(\boldsymbol{V}-\frac{\partial\boldsymbol{u}}{\partial t}\Big)\cdot\boldsymbol{n}=0,\quad \boldsymbol{y}\in\gamma, \tag{4.20}$$

in the case of $\mu_1=0$. In (4.20) $\boldsymbol{n}$ is the unit normal to γ.

PROOF. The differential equations (4.17) and (4.19) follow as $\varepsilon\searrow 0$ from the integral equality (1.2), with the test function $\boldsymbol{\psi}=\boldsymbol{\varphi}(x\varepsilon^{-1})\cdot h(\boldsymbol{x},t)$ where $\boldsymbol{\varphi}$ is solenoidal and compactly supported in Y_f.

The boundary conditions (4.18) are the consequences of the two-scale convergence of $\{\alpha_\mu^{\frac12}\nabla\boldsymbol{w}^\varepsilon\}$ to $\mu_1^{\frac12}\nabla\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$. By this convergence, the function $\nabla\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$ is L^2-integrable in Y. The boundary conditions (4.20) follow from (4.6). □

Lemma 4.6. *If the porous space is disconnected (isolated pores) then the weak and two-scale limits of the sequences $\{\boldsymbol{u}^\varepsilon\}$ and $\{\boldsymbol{w}^\varepsilon\}$ coincide.*

PROOF. Indeed, in each of the cases $0<\mu_1<\infty$ or $\mu_1=0$ the systems of equations (4.6), (4.17), (4.18) or (4.6), (4.19), (4.20) have the unique solution $\boldsymbol{V}=\partial\boldsymbol{u}/\partial t$. □

4.4. Homogenized equations I.

Lemma 4.7. *If $\mu_1 = \infty$ or the porous space is disconnected (isolated pores) then $\boldsymbol{w} = \boldsymbol{u}$ and the weak limits $\boldsymbol{u}$, p, q, and π of the sequences $\{\boldsymbol{u}^\varepsilon\}$, $\{p^\varepsilon\}$, $\{q^\varepsilon\}$, and $\{\pi^\varepsilon\}$ satisfy in Ω_T the initial-boundary value problem*

$$\tau_0\hat{\rho}\frac{\partial^2\boldsymbol{u}}{\partial t^2} = \operatorname{div}_x\big\{\lambda_0\mathbb{A}^s_0 : \mathbb{D}(x,\boldsymbol{u}) + B^s_0\operatorname{div}_x\boldsymbol{u} + B^s_1 q - (q+\pi)\cdot\mathbb{I}\big\} + \hat{\rho}\boldsymbol{F}, \tag{4.21}$$

$$\frac{1}{\eta_0}\pi + C^s_0 : \mathbb{D}(x,\boldsymbol{u}) + a^s_0\operatorname{div}_x\boldsymbol{u} + a^s_1 q = 0, \tag{4.22}$$

$$q = p + \nu_0 p_*^{-1}\frac{\partial p}{\partial t},\quad \frac{1}{p_*}p + \frac{1}{\eta_0}\pi + \operatorname{div}_x\boldsymbol{u} = 0, \tag{4.23}$$

where the strictly positive definite constant fourth-rank symmetric tensor $\mathbb{A}^s_0$, the matrices C^s_0, B^s_0, and B^s_1, and the constants a^s_0 and a^s_1 are defined below by (4.28), (4.30), and (4.31).

The differential equations (4.21) are endowed with the homogeneous initial and boundary conditions

$$\tau_0\boldsymbol{u}(\boldsymbol{x},0) = \tau_0\frac{\partial\boldsymbol{u}}{\partial t}(\boldsymbol{x},0) = 0,\ \boldsymbol{x}\in\Omega,\quad \boldsymbol{u}(\boldsymbol{x},t) = 0,\ \boldsymbol{x}\in S,\ t>0. \tag{4.24}$$

Proof. Note first that $\boldsymbol{u} = \boldsymbol{w}$ by Lemmas 4.4 and 4.6.

The homogenized equations (4.21) follow from the macroscopic equations (4.16) on inserting the expression

$$\lambda_0\langle\mathbb{D}(y,\boldsymbol{U})\rangle_{Y_s} = \lambda_0\mathbb{A}^s_1 : \mathbb{D}(x,\boldsymbol{u}) + B^s_0\operatorname{div}_x\boldsymbol{u} + B^s_1 q.$$

In turn, this expression ensues from the solutions of (4.5) and (4.15) on the pattern cell Y_s. Indeed, putting

$$\boldsymbol{U} = \sum_{i,j=1}^{3}\boldsymbol{U}^{ij}(\boldsymbol{y})D_{ij} + \boldsymbol{U}_0(\boldsymbol{y})\operatorname{div}_x\boldsymbol{u} + \frac{1}{m}\boldsymbol{U}_1(\boldsymbol{y})q,$$

$$\Pi = \lambda_0\sum_{i,j=1}^{3}\Pi^{ij}(\boldsymbol{y})D_{ij} + \Pi_0(\boldsymbol{y})\operatorname{div}_x\boldsymbol{u} + \frac{1}{m}\Pi_1(\boldsymbol{y})q,$$

where $D_{ij}(\boldsymbol{x},t) = \frac{1}{2}\big(\frac{\partial u_i}{\partial x_j}(\boldsymbol{x},t) + \frac{\partial u_j}{\partial x_i}(\boldsymbol{x},t)\big)$, we arrive at the periodic-boundary value problems in Y_s:

$$\operatorname{div}_y\{(1-\chi)(\mathbb{D}(y,\boldsymbol{U}^{ij}) + J^{ij}) - \Pi^{ij}\cdot\mathbb{I}\} = 0,\quad \frac{\lambda_0}{\eta_0}\Pi^{ij} + (1-\chi)\operatorname{div}_y\boldsymbol{U}^{ij} = 0; \tag{4.25}$$

$$\operatorname{div}_y\{\lambda_0(1-\chi)\mathbb{D}(y,\boldsymbol{U}_0) - \Pi_0\cdot\mathbb{I}\} = 0,\quad \frac{1}{\eta_0}\Pi_0 + (1-\chi)(\operatorname{div}_y\boldsymbol{U}_0 + 1) = 0; \tag{4.26}$$

$$\operatorname{div}_y\{\lambda_0(1-\chi)\mathbb{D}(y,\boldsymbol{U}_1) - (\Pi_1+\chi)\cdot\mathbb{I}\} = 0,\quad \frac{1}{\eta_0}\Pi_1 + (1-\chi)\operatorname{div}_y\boldsymbol{U}_1 = 0. \tag{4.27}$$

Note that $\beta = 0$ even if $p_* + \eta_0 = \infty$ by the homogeneous boundary condition for $\boldsymbol{u}(\boldsymbol{x},t)$ and (4.14).

By the assumptions on the geometry of the pattern "solid" cell Y_s, problems (4.25)–(4.27) have a unique solution up to an arbitrary constant vector. In order to discard the arbitrary constant vectors, we demand $\langle\boldsymbol{U}^{ij}\rangle_{Y_s} = \langle\boldsymbol{U}_0\rangle_{Y_s} = \langle\boldsymbol{U}_1\rangle_{Y_s} = 0$. Thus,

$$\mathbb{A}^s_0 = (1-m)\sum_{i,j=1}^{3}J^{ij}\otimes J^{ij} + \mathbb{A}^s_1,\quad \mathbb{A}^s_1 = \sum_{i,j=1}^{3}\langle(1-\chi)\mathbb{D}(y,\boldsymbol{U}^{ij})\rangle_Y\otimes J^{ij}. \tag{4.28}$$

The symmetry of the tensor $\mathbb{A}^s_0$ follows from the symmetry of $\mathbb{A}^s_1$. And the symmetry of the latter follows from the equality

$$\langle\mathbb{D}(y,\boldsymbol{U}^{ij})\rangle_{Y_s} : J^{kl} = -\langle\mathbb{D}(y,\boldsymbol{U}^{ij}) : \mathbb{D}(y,\boldsymbol{U}^{kl})\rangle_{Y_s} - \frac{\lambda_0}{\eta_0}\Pi^{ij}\Pi^{kl}, \tag{4.29}$$

which appears on multiplication of (4.25) for $\boldsymbol{U}^{ij}$ by $\boldsymbol{U}^{kl}$ and integration by parts. This equality also implies the positive definiteness of $\mathbb{A}_0^s$. Indeed, let $\boldsymbol{\zeta}$ be an arbitrary symmetric matrix. Putting

$$\mathbb{Z} = \sum_{i,j=1}^{3} \boldsymbol{U}^{ij}\boldsymbol{\zeta}_{ij}, \quad \widetilde{\Pi} = \sum_{i,j=1}^{3} \Pi^{ij}\boldsymbol{\zeta}_{ij}$$

and taking into account (4.29), we obtain

$$\langle \mathbb{D}(y,\mathbb{Z})\rangle_{Y_s} : \boldsymbol{\zeta} = -\langle \mathbb{D}(y,\mathbb{Z}) : \mathbb{D}(y,\mathbb{Z})\rangle_{Y_s} - \frac{\lambda_0}{\eta_0}\widetilde{\Pi}^2.$$

This equality and the definition of $\mathbb{A}_0^s$ yield

$$(\mathbb{A}_0^s : \boldsymbol{\zeta}) : \boldsymbol{\zeta} = \langle (\mathbb{D}(y,\mathbb{Z}) + \boldsymbol{\zeta}) : (\mathbb{D}(y,\mathbb{Z}) + \boldsymbol{\zeta})\rangle_{Y_s} + \frac{\lambda_0}{\eta_0}\widetilde{\Pi}^2.$$

The strict positive definiteness of $\mathbb{A}_0^s$ is immediate from the equality above and the geometry of the elementary cell Y_s. Indeed, suppose that $(\mathbb{A}_0^s : \boldsymbol{\zeta}) : \boldsymbol{\zeta} = 0$ for some $\boldsymbol{\zeta}$ such that $\boldsymbol{\zeta} : \boldsymbol{\zeta} = 1$. Then $(\mathbb{D}(y,\mathbb{Z}) + \boldsymbol{\zeta}) = 0$, which is possible if and only if $\mathbb{Z}$ is a linear function in $\boldsymbol{y}$. On the other hand, all linear periodic functions on Y_s are constant. Finally, the normalization condition $\langle \boldsymbol{U}^{ij}\rangle_{Y_s} = 0$ yields that $\mathbb{Z} = 0$. However, this is impossible.

Finally, the equations for pressures (4.22) and (4.23) follow from (4.8)–(4.10) and

$$B_0^s = \lambda_0\langle \mathbb{D}(y,\boldsymbol{U}_0)\rangle_{Y_s}, \quad B_1^s = \frac{\lambda_0}{m}\langle \mathbb{D}(y,\boldsymbol{U}_1)\rangle_{Y_s}, \quad a_1^s = \frac{1}{m}\langle \operatorname{div}_y \boldsymbol{U}_1\rangle_{Y_s}, \tag{4.30}$$

$$C_0^s = \sum_{i,j=1}^{3} \langle \operatorname{div}_y \boldsymbol{U}^{ij}\rangle_{Y_s} J^{ij}, \quad a_0^s = 1 - m + \langle \operatorname{div}_y \boldsymbol{U}_0\rangle_{Y_s}. \tag{4.31}$$

4.5. Homogenized equations II. We now let $\mu_1 < \infty$. In the same manner as above, we verify that the limit $\boldsymbol{u}$ of the sequence $\{\boldsymbol{u}^\varepsilon\}$ satisfies the initial-boundary value problem like (4.21)–(4.23). The main difference here that, in general, the weak limit $\boldsymbol{w}$ of $\{\boldsymbol{w}^\varepsilon\}$ differs from $\boldsymbol{u}$. More precisely, the following is true:

Lemma 4.8. *If $\mu_1 < \infty$ then the weak limits $\boldsymbol{u}$, $\boldsymbol{w}^f$, p, q, and π of the sequences $\{\boldsymbol{u}^\varepsilon\}$, $\{\chi^\varepsilon \boldsymbol{w}^\varepsilon\}$, $\{p^\varepsilon\}$, $\{q^\varepsilon\}$, and $\{\pi^\varepsilon\}$ satisfy the initial-boundary value problem in Ω_T consisting of the momentum balance equation:*

$$\tau_0\Big(\rho_f\frac{\partial \boldsymbol{v}}{\partial t} + \rho_s(1-m)\frac{\partial^2 \boldsymbol{u}}{\partial t^2}\Big) + \nabla(q+\pi) - \hat{\rho}\boldsymbol{F} = \operatorname{div}_x\{\lambda_0\mathbb{A}_0^s : \mathbb{D}(x,\boldsymbol{u}) + B_0^s \operatorname{div}_x \boldsymbol{u} + B_1^s q\}, \tag{4.32}$$

and the continuity equation (4.22) for the solid component, where $\boldsymbol{v} = \partial \boldsymbol{w}^f/\partial t$ and $\mathbb{A}_0^s$, B_0^s, and B_1^s are the same as in (4.21), the state and continuity equations:

$$p + \nu_0 p_*^{-1}\frac{\partial p}{\partial t} = q, \quad \frac{1}{p_*}p + \frac{1}{\eta_0}\pi + \operatorname{div}_x \boldsymbol{w}^f = (m-1)\operatorname{div}_x \boldsymbol{u} \tag{4.33}$$

for the fluid component and the relations

$$\boldsymbol{v} = m\frac{\partial \boldsymbol{u}}{\partial t} + \int_0^t B_1(\mu_1, t-\tau)\cdot \boldsymbol{z}(\boldsymbol{x},\tau)\,d\tau, \tag{4.34}$$

$$\boldsymbol{z}(\boldsymbol{x},t) = -\frac{1}{m}\nabla q(\boldsymbol{x},t) + \rho_f \boldsymbol{F}(\boldsymbol{x},t) - \tau_0\rho_f\frac{\partial^2 \boldsymbol{u}}{\partial t^2}(\boldsymbol{x},t),$$

in the case of $\tau_0 > 0$ and $\mu_1 > 0$, or Darcy's law in the form

$$\boldsymbol{v} = m\frac{\partial \boldsymbol{u}}{\partial t} + B_2(\mu_1)\cdot\Big(-\frac{1}{m}\nabla q + \rho_f \boldsymbol{F}\Big), \tag{4.35}$$

in the case of $\tau_0 = 0$ *and* $\mu_1 > 0$, *or, finally, the momentum balance equation for the fluid component in the form*

$$\tau_0\rho_f\frac{\partial \boldsymbol{v}}{\partial t} = \tau_0\rho_f B_3\cdot\frac{\partial^2 \boldsymbol{u}}{\partial t^2} + (m\mathbb{I} - B_3)\cdot\Big(-\frac{1}{m}\nabla q + \rho_f \boldsymbol{F}\Big), \tag{4.36}$$

in the case of $\tau_0 > 0$ *and* $\mu_1 = 0$. *The problem is supplemented by boundary and initial conditions* (4.24) *for the displacement* $\boldsymbol{u}$ *of the solid component and the boundary condition*

$$\boldsymbol{v}(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x}) = 0, \quad \boldsymbol{x}\in S, \quad t>0, \tag{4.37}$$

for the velocity $\boldsymbol{v}$ *of the fluid component. In* (4.34)–(4.37) $\boldsymbol{n}(\boldsymbol{x})$ *is the unit normal vector to* S *at a point* $\boldsymbol{x}\in S$, *and the matrices* $B_1(\mu_1,t)$, $B_2(\mu_1)$, *and* B_3 *are given below by* (4.39)–(4.43).

PROOF. The homogenized equations of the balance of momentum and the balance of mass are derived exactly as (4.21), (4.22). For example, to obtain (4.33) we just expressed $\mathrm{div}_x\,\boldsymbol{w}$ in a sum of (4.9) and (4.10) using (4.7) after homogenization: $\boldsymbol{w} = \boldsymbol{w}^f + (1-m)\boldsymbol{u}$. Therefore, we omit the relevant proofs now and focus only on the derivation of the homogenized equations for the velocity $\boldsymbol{v}$ of the fluid component. The derivation of the boundary condition (4.37) is standard [3].

(a) If $\mu_1 > 0$ and $\tau_0 > 0$ then the solution of the system of microscopic equations (4.6), (4.17), (4.18), provided with the homogeneous initial data, is given as

$$\boldsymbol{V} = \frac{\partial \boldsymbol{u}}{\partial t} + \int_0^t \mathbf{B}_1^f(\boldsymbol{y},t-\tau)\cdot\boldsymbol{z}(\boldsymbol{x},\tau)\,d\tau, \quad R = \int_0^t R_f(\boldsymbol{y},t-\tau)\cdot\boldsymbol{z}(\boldsymbol{x},\tau)\,d\tau,$$

where $\mathbf{B}_1^f(\boldsymbol{y},t) = \sum_{i=1}^3 \boldsymbol{V}^i(\boldsymbol{y},t)\otimes\boldsymbol{e}_i$, $R_f(\boldsymbol{y},t) = \sum_{i=1}^3 R^i(\boldsymbol{y},t)\boldsymbol{e}_i$, and the functions $\boldsymbol{V}^i(\boldsymbol{y},t)$ and $R^i(\boldsymbol{y},t)$ are defined by the periodic initial-boundary value problem

$$\begin{gathered}\tau_0\rho_f\frac{\partial \boldsymbol{V}^i}{\partial t} - \mu_1\triangle\boldsymbol{V}^i + \nabla R^i = 0, \quad \mathrm{div}_y\,\boldsymbol{V}^i = 0, \quad \boldsymbol{y}\in Y_f,\ t>0,\\ \boldsymbol{V}^i = 0, \quad \boldsymbol{y}\in\gamma, t>0; \quad \tau_0\rho_f\boldsymbol{V}^i(\boldsymbol{y},0) = \boldsymbol{e}_i, \quad \boldsymbol{y}\in Y_f.\end{gathered} \tag{4.38}$$

In (4.38) $\boldsymbol{e}_i$ is the standard Cartesian basis vector of the coordinate axis x_i. Therefore,

$$B_1(\mu_1,t) = \langle \mathbf{B}_1^f\rangle_{Y_f}(t). \tag{4.39}$$

(b) If $\tau_0 = 0$ and $\mu_1 > 0$ then the solution of the stationary microscopic equations (4.6), (4.17), (4.18) is given by the formula

$$\boldsymbol{V} = \frac{\partial \boldsymbol{u}}{\partial t} + \mathbf{B}_2^f(\boldsymbol{y})\cdot(-\nabla q + \rho_f\boldsymbol{F}),$$

in which $\mathbf{B}_2^f(\boldsymbol{y}) = \sum_{i=1}^3 \boldsymbol{U}^i(\boldsymbol{y})\otimes\boldsymbol{e}_i$, and the functions $\boldsymbol{U}^i(\boldsymbol{y})$ are defined from the periodic-boundary value problem

$$-\mu_1\triangle\boldsymbol{U}^i + \nabla R^i = \boldsymbol{e}_i, \quad \mathrm{div}_y\,\boldsymbol{U}^i = 0, \quad \boldsymbol{y}\in Y_f, \quad \boldsymbol{U}^i = 0, \quad \boldsymbol{y}\in\gamma. \tag{4.40}$$

Thus,

$$B_2(\mu_1) = \langle \mathbf{B}_2^f(\boldsymbol{y})\rangle_{Y_f}. \tag{4.41}$$

The matrix $B_2(\mu_1)$ is symmetric and positive definite [3, Chapter 8].

(c) Finally, if $\tau_0 > 0$ and $\mu_1 = 0$ then in the process of solving the system (4.6), (4.19), (4.20) we firstly find the pressure $R(\boldsymbol{x},t,\boldsymbol{y})$ by solving the Neumann problem for Laplace's equation in Y_f. If $R(\boldsymbol{x},t,\boldsymbol{y}) = \sum_{i=1}^3 R_i(\boldsymbol{y})\boldsymbol{e}_i\cdot\boldsymbol{z}(\boldsymbol{x},t)$, where $R^i(\boldsymbol{y})$ is the solution of the problem

$$\Delta R_i = 0, \quad \boldsymbol{y}\in Y_f; \quad \nabla R_i\cdot\boldsymbol{n} = \boldsymbol{n}\cdot\boldsymbol{e}_i, \quad \boldsymbol{y}\in\gamma, \tag{4.42}$$

then (4.36) appears as the result of homogenization of (4.19) and

$$B_3 = \sum_{i=1}^3 \langle\nabla R_i(\boldsymbol{y})\rangle_{Y_f}\otimes\boldsymbol{e}_i, \tag{4.43}$$

where the matrix $B = (m\mathbb{I} - B_3)$ is symmetric and positive definite. In fact, let $\widetilde{R} = \sum_{i=1}^3 R_i\xi_i$ for any unit vector ξ. Then $(B\cdot\xi)\cdot\xi = \langle(\xi - \nabla\widetilde{R})^2\rangle_{Y_f} > 0$ due to the same reasons as in Lemma 4.7. □

§ 5. Proof of Theorem 3

5.1. Weak and two-scale limits of the sequences of displacement and pressures.

I. Let $\mu_1 < \infty$ and let one of the conditions (1.6) or (1.7) hold. Then by Theorems 1 and 5 we conclude that the sequences $\{\boldsymbol{w}^\varepsilon\}$, $\{p^\varepsilon\}$, and $\{q^\varepsilon\}$ two-scale converge to the functions $\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$, $P(\boldsymbol{x},t,\boldsymbol{y})$, and $Q(\boldsymbol{x},t,\boldsymbol{y})$ and converge weakly in $L^2(\Omega_T)$ to the functions $\boldsymbol{w}$, p, and q respectively, and the sequence $\{\boldsymbol{u}^\varepsilon(\boldsymbol{x},t)\}$, where $\boldsymbol{u}^\varepsilon(\boldsymbol{x},t)$ is an extension of $\boldsymbol{w}^\varepsilon(\boldsymbol{x},t)$ from the domain Ω_s^ε to the domain Ω, vanishes strongly in $L^2(\Omega_T)$ and weakly in $L^2\big((0,T);W_2^1(\Omega)\big)$.

II. If $\mu_1 < \infty$ and conditions (1.7) hold then by (1.5) and (1.8) the sequence $\{\alpha_\lambda \boldsymbol{u}^\varepsilon\}$ converges strongly in $L^2(\Omega_T)$ and weakly in $L^2\big((0,T);W_2^1(\Omega)\big)$ to a function $\boldsymbol{u}$, and the sequence $\{\pi^\varepsilon\}$ converges weakly in $L^2(\Omega_T)$ to π.

III. If $\mu_1 = \infty$, $p_1^{-1},\eta_1^{-1} < \infty$ and $0 < \lambda_1 < \infty$ then by part (II) of Theorem 1 the sequences $\{\alpha_\mu\varepsilon^{-2}\chi^\varepsilon\boldsymbol{w}^\varepsilon\}$, $\{p^\varepsilon\}$, $\{\pi^\varepsilon\}$, $\{q^\varepsilon\}$ two-scale converge to the functions $\chi(\boldsymbol{y})\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$, $P(\boldsymbol{x},t,\boldsymbol{y})$, $\Pi(\boldsymbol{x},t,\boldsymbol{y})$, $Q(\boldsymbol{x},t,\boldsymbol{y})$ and converge weakly in $L^2(\Omega_T)$ to the functions $\boldsymbol{w}^f$, p, π, q, respectively; and the sequence $\{\alpha_\mu\varepsilon^{-2}\boldsymbol{u}^\varepsilon\}$ converges strongly in $L^2(\Omega_T)$ and weakly in $L^2\big((0,T);W_2^1(\Omega)\big)$ to $\boldsymbol{u}$.

As before in § 4, we conclude that $\boldsymbol{u} \in L^2\big(0,T;\overset{\circ}{W}{}_2^1(\Omega)\big)$.

5.2. Homogenized equations. I. If $\mu_1 < \infty$ and one of the conditions (1.6) or (1.7) holds then, as in the proof of Theorem 2, we construct a closed system of equations for the velocity $\boldsymbol{v} = \partial\boldsymbol{w}^f/\partial t$ in the fluid component and for the pressures p and q. We entitle the above-described systems as *Problem* (F_1)–(F_3) depending on the forms of the matrices B_1, B_2, or B_3. Each system consists of one of the momentum balance equations (4.34)–(4.36), the boundary condition (4.37) in which $\boldsymbol{u}(\boldsymbol{x},t) = 0$, and the equations

$$p + \nu_0 p_*^{-1}\frac{\partial p}{\partial t} = q, \quad \frac{1}{p_*}\frac{\partial p}{\partial t} + \operatorname{div}_x \boldsymbol{v} = 0, \quad \boldsymbol{x} \in \Omega,\ t > 0. \tag{5.1}$$

II. Let $\mu_1 < \infty$ and let (1.7) hold. We observe that the limiting displacements in the solid skeleton are equal to zero. In order to find a more accurate asymptotic of the solution of the original model, we use again the renormalization. Namely, let $\boldsymbol{w}^\varepsilon \to \alpha_\lambda\boldsymbol{w}^\varepsilon$. Then the new displacements satisfy the same problem as the displacements before renormalization, but with the new parameters $\alpha_\eta \to \alpha_\eta\alpha_\lambda^{-1}$, $\alpha_\lambda \to 1$, $\alpha_\tau \to \alpha_\tau\alpha_\lambda^{-1}$. Thus we arrive at the assumptions of Theorem 2. Namely, let the weak and two-scale limits $\boldsymbol{u}(\boldsymbol{x},t)$, π, $\Pi(\boldsymbol{x},t,\boldsymbol{y})$, and $\boldsymbol{U}(\boldsymbol{x},t,\boldsymbol{y})$ satisfy the same system of microscopic and macroscopic equations for the corresponding functions, defining the behavior of the solid component, in which the pressure q is given by one of the problems F_1–F_3. The only difference from the already considered case is in the microscopic and macroscopic continuity equations which coincide with (4.5) and (4.10) if we insert η_2 instead of η_0.

Hence, for $\boldsymbol{u}(\boldsymbol{x},t)$ and $\pi(\boldsymbol{x},t)$ we have the homogenized momentum equation in the form

$$0 = \operatorname{div}_x\big\{\mathbb{A}_0^s : \mathbb{D}(x,\boldsymbol{u}) + B_0^s \operatorname{div}_x \boldsymbol{u} + B_1^s q - (q+\pi)\cdot\mathbb{I}\big\} + \hat{\rho}\boldsymbol{F}, \quad \boldsymbol{x} \in \Omega, \tag{5.2}$$

the macroscopic continuity equation (4.22) in which $\eta_0 = \eta_2$, and the boundary condition (1.12). The tensor $\mathbb{A}_0^s$, the matrices C_0^s, B_0^s, and B_1^s, and the scalars a_0^s and a_1^s are defined from (4.28), (4.30), and (4.31) in which we have $\eta_0 = \eta_2$ and $\lambda_0 = 1$.

III. If $\mu_1 = \infty$, $p_1^{-1},\eta_1^{-1} < \infty$ and $0 < \lambda_1 < \infty$ renormalizing by $\boldsymbol{w}^\varepsilon \to \alpha_\mu\varepsilon^{-2}\boldsymbol{w}^\varepsilon$ we arrive at the assumptions of Theorem 2, when $\mu_0 = 0$, $\mu_1 = 1$, $\tau_0 = 0$ and $\lambda_0 = \lambda_1$, $\nu_0 = 0$, $p_* = p_1$ and $\eta_0 = \eta_1$. Namely, the functions $\boldsymbol{w}^f$, p, π, and $\boldsymbol{u}$ satisfy the following initial-boundary value problem in Ω_T:

$$\begin{gathered}
\operatorname{div}_x\big\{\lambda_1\mathbb{A}_0^s : \mathbb{D}(x,\boldsymbol{u}) + B_0^s \operatorname{div}_x \boldsymbol{u} + B_1^s p - (p+\pi)\cdot\mathbb{I}\big\} + \hat{\rho}\boldsymbol{F} = 0, \\
\frac{\partial\boldsymbol{w}^f}{\partial t} = \frac{\partial\boldsymbol{u}}{\partial t} + B_2(1)\cdot\Big(-\frac{1}{m}\nabla p + \rho_f\boldsymbol{F}\Big), \\
\frac{1}{p_1}p + \frac{1}{\eta_1}\pi + \operatorname{div}_x \boldsymbol{w}^f = (m-1)\operatorname{div}_x \boldsymbol{u}, \quad \frac{1}{\eta_1}\pi + C_0^s : \mathbb{D}(x,\boldsymbol{u}) + a_0^s \operatorname{div}_x \boldsymbol{u} + a_1^s p = 0.
\end{gathered} \tag{5.3}$$

As before, the tensor $\mathbb{A}^s_0$, the matrices C^s_0, B^s_0, and B^s_1, and the scalars a^s_0 and a^s_1 are defined by (4.28), (4.30), and (4.31) in which we have $\eta_0 = \eta_1$ and $\lambda_0 = \lambda_1$.

Note that here $\nu_0 = 0$. Therefore the state equation $p + \nu_0 p_*^{-1}\frac{\partial p}{\partial t} = q$ becomes $p = q$.

The problem is endowed with the corresponding homogeneous initial and boundary conditions.

§ 6. Proof of Theorem 4

6.1. Weak and two-scale limits of the sequences of displacement and pressures. By Theorem 1, the sequences $\{p^\varepsilon\}$, $\{q^\varepsilon\}$, $\{\pi^\varepsilon\}$, and $\{\boldsymbol{w}^\varepsilon\}$ are uniformly bounded in $L^2(\Omega_T)$ in ε. Then there exist a subsequence from $\{\varepsilon > 0\}$ and functions p, π, q, and $\boldsymbol{w}$ such that

$$p^\varepsilon \to p, \quad q^\varepsilon \to q, \quad \pi^\varepsilon \to \pi, \quad \boldsymbol{w}^\varepsilon \to \boldsymbol{w} \quad \text{weakly in } L^2(\Omega_T) \text{ as } \varepsilon \searrow 0. \tag{6.1}$$

Moreover, the bounds (3.1) imply

$$\nabla_x \boldsymbol{w}^\varepsilon \underset{\varepsilon\searrow 0}{\longrightarrow} \nabla_x \boldsymbol{w} \quad \text{weakly in } L^2(\Omega_T). \tag{6.2}$$

Owing to Nguetseng's theorem, there exist one more subsequence from $\{\varepsilon > 0\}$ and 1-periodic functions $P(\boldsymbol{x},t,\boldsymbol{y})$, $\Pi(\boldsymbol{x},t,\boldsymbol{y})$, $Q(\boldsymbol{x},t,\boldsymbol{y})$, and $\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$ in $\boldsymbol{y}$ such that the sequences $\{p^\varepsilon\}$, $\{\pi^\varepsilon\}$, $\{q^\varepsilon\}$, and $\{\nabla \boldsymbol{w}^\varepsilon\}$ two-scale converge as $\varepsilon \searrow 0$ respectively to P, Π, Q, and $\nabla_x \boldsymbol{w} + \nabla_y \boldsymbol{W}$.

6.2. Microscopic and macroscopic equations. In the present section we do not consider the functions of time t which renormalize pressures. As shown before, all these functions are equal eventually to zero.

Lemma 6.1. *The two-scale limits of the sequences $\{p^\varepsilon\}$, $\{\pi^\varepsilon\}$, $\{q^\varepsilon\}$, and $\{\nabla \boldsymbol{w}^\varepsilon\}$ satisfy in $Y_T = Y \times (0,T)$ the following microscopic equations:*

$$\frac{1}{\eta_0}\Pi + (1-\chi)(\operatorname{div}_x \boldsymbol{w} + \operatorname{div}_y \boldsymbol{W}) = 0; \tag{6.3}$$

$$\frac{1}{p_*}P + \chi(\operatorname{div}_x \boldsymbol{w} + \operatorname{div}_y \boldsymbol{W}) = 0, \quad Q = P + \frac{\nu_0}{p_*}\frac{\partial P}{\partial t}; \tag{6.4}$$

$$\operatorname{div}_y \Big(\chi\mu_0 \Big(\mathbb{D}\Big(x, \frac{\partial \boldsymbol{w}}{\partial t}\Big) + \mathbb{D}\Big(y, \frac{\partial \boldsymbol{W}}{\partial t}\Big)\Big) + (1-\chi)\lambda_0(\mathbb{D}(x,\boldsymbol{w}) + \mathbb{D}(y,\boldsymbol{W}))\Big) - \nabla_y(Q + \Pi) = 0. \tag{6.5}$$

Lemma 6.2. *The weak limits p, π, q, and $\boldsymbol{w}$ satisfy in Ω_T the system of macroscopic equations:*

$$\frac{1}{\eta_0}\pi + (1-m)\operatorname{div}_x \boldsymbol{w} + \langle \operatorname{div}_y \boldsymbol{W}\rangle_{Y_s} = 0; \tag{6.6}$$

$$\frac{1}{p_*}p + m\operatorname{div}_x \boldsymbol{w} + \langle \operatorname{div}_y \boldsymbol{W}\rangle_{Y_f} = 0, \quad q = p + \frac{\nu_0}{p_*}\frac{\partial p}{\partial t}; \tag{6.7}$$

$$\tau_0\hat{\rho}\frac{\partial^2 \boldsymbol{w}}{\partial t^2} = \operatorname{div}_x \Big(\mu_0 \Big(m\mathbb{D}\Big(x, \frac{\partial \boldsymbol{w}}{\partial t}\Big) + \Big\langle \mathbb{D}\Big(y, \frac{\partial \boldsymbol{W}}{\partial t}\Big)\Big\rangle_{Y_f}\Big) + \lambda_0((1-m)\mathbb{D}(x,\boldsymbol{w}) + \langle \mathbb{D}(y,\boldsymbol{W})\rangle_{Y_s} - (q+\pi)\mathbb{I}\Big) + \hat{\rho}\boldsymbol{F}. \tag{6.8}$$

The proofs of these statements are the same as in Lemmas 4.1–4.3.

6.3. Homogenized equations.

Lemma 6.3. *The weak limits* p, π, q, *and* $\boldsymbol{w}$ *satisfy in* Ω_T *the next system of homogenized equations:*

$$\tau_0\hat{\rho}\frac{\partial^2\boldsymbol{w}}{\partial t^2}+\nabla(q+\pi)-\hat{\rho}\boldsymbol{F}=\operatorname{div}_x\Big(\mathbb{A}_2:\mathbb{D}\Big(x,\frac{\partial\boldsymbol{w}}{\partial t}\Big)+\mathbb{A}_3:\mathbb{D}(x,\boldsymbol{w})+B_4\operatorname{div}_x\boldsymbol{w}$$
$$+\int_0^t(\mathbb{A}_4(t-\tau):\mathbb{D}(x,\boldsymbol{w}(\boldsymbol{x},\tau))+B_5(t-\tau)\operatorname{div}_x\boldsymbol{w}(\boldsymbol{x},\tau))\,d\tau\Big),\tag{6.9}$$

$$\frac{1}{p_*}p+m\operatorname{div}_x\boldsymbol{w}=-\int_0^t(C_2(t-\tau):\mathbb{D}(x,\boldsymbol{w}(\boldsymbol{x},\tau))+a_2(t-\tau)\operatorname{div}_x\boldsymbol{w}(\boldsymbol{x},\tau))\,d\tau,\tag{6.10}$$

$$\frac{1}{\eta_0}\pi+(1-m)\operatorname{div}_x\boldsymbol{w}=-\int_0^t(C_3(t-\tau):\mathbb{D}(x,\boldsymbol{w}(\boldsymbol{x},\tau))+a_3(t-\tau)\operatorname{div}_x\boldsymbol{w}(\boldsymbol{x},\tau))\,d\tau,\tag{6.11}$$

$$q=p+\frac{\nu_0}{p_*}\frac{\partial p}{\partial t}.\tag{6.12}$$

Here $\mathbb{A}_2$–$\mathbb{A}_4$ *are fourth-rank tensors;* B_4, B_5, C_2, C_3 *are matrices; and* a_2, a_3 *are scalars. The exact expressions for these objects are given below in* (6.25)–(6.30).

Proof. Put $Z(\boldsymbol{x},t)=\mu_0\mathbb{D}(x,\frac{\partial\boldsymbol{w}}{\partial t})-\lambda_0\mathbb{D}(x,\boldsymbol{w})$, $Z_{ij}=\mathbf{e}_i\cdot(Z\cdot\mathbf{e}_j)$, and $z(\boldsymbol{x},t)=\operatorname{div}_x\boldsymbol{w}$. As usual we look for the solution of the system of microscopic equations (6.3)–(6.5) in the form

$$\boldsymbol{W}=\int_0^t\Big[\boldsymbol{W}^0(y,t-\tau)z(\boldsymbol{x},\tau)+\sum_{i,j=1}^3\boldsymbol{W}^{ij}(\boldsymbol{y},t-\tau)Z_{ij}(\boldsymbol{x},\tau)\Big]\,d\tau,\tag{6.13}$$

$$P=\chi\int_0^t\Big[P^0(\boldsymbol{y},t-\tau)z(\boldsymbol{x},\tau)+\sum_{i,j=1}^3P^{ij}(\boldsymbol{y},t-\tau)Z_{ij}(\boldsymbol{x},\tau)\Big]d\tau,\tag{6.14}$$

$$Q=\chi\Big(Q_0(y)\cdot z(\boldsymbol{x},t)+\sum_{i,j=1}^3Q_0^{ij}(\boldsymbol{y})\cdot Z_{ij}(\boldsymbol{x},t)$$
$$+\int_0^t\Big[Q^0(\boldsymbol{y},t-\tau)z(\boldsymbol{x},\tau)+\sum_{i,j=1}^3Q^{ij}(\boldsymbol{y},t-\tau)Z_{ij}(\boldsymbol{x},\tau)\Big]\,d\tau\Big),\tag{6.15}$$

$$\Pi=(1-\chi)\Big(\int_0^t\Big[\Pi^0(\boldsymbol{y},t-\tau)z(\boldsymbol{x},\tau)+\sum_{i,j=1}^3\Pi^{ij}(\boldsymbol{y},t-\tau)Z_{ij}(\boldsymbol{x},\tau)\Big]\,d\tau\Big),\tag{6.16}$$

where the functions $\boldsymbol{W}^0$, $\boldsymbol{W}^{ij}$, P^0, P^{ij}, Q_0, Q^0, Q^{ij}, Q_0^{ij}, Π^0, and Π^{ij} 1-periodic in $\boldsymbol{y}$ satisfy the following periodic initial-boundary value problems in the elementary cell Y:

Problem (I).

$$\operatorname{div}_y\Big(\chi\Big(\mu_0\mathbb{D}\Big(y,\frac{\partial\boldsymbol{W}^{ij}}{\partial t}\Big)+(1-\chi)\lambda_0\mathbb{D}(y,\boldsymbol{W}^{ij})-((1-\chi)\Pi^{ij}+\chi Q^{ij})\mathbb{I}\Big)\Big)=0;\tag{6.17}$$

$$\frac{1}{p_*}P^{ij}+\chi\operatorname{div}_y\boldsymbol{W}^{ij}=0,\quad Q^{ij}=P^{ij}+\frac{\nu_0}{p_*}\frac{\partial P^{ij}}{\partial t};\tag{6.18}$$

$$\frac{1}{\eta_0}\Pi^{ij} + (1-\chi)\,\mathrm{div}_y\,\boldsymbol{W}^{ij} = 0, \quad \boldsymbol{W}^{ij}(\boldsymbol{y},0) = \boldsymbol{W}_0^{ij}(\boldsymbol{y}); \tag{6.19}$$

$$\mathrm{div}_y\big(\chi\big(\mu_0\mathbb{D}(y,\boldsymbol{W}_0^{ij}) + J^{ij} - Q_0^{ij}\mathbb{I}\big)\big) = 0, \quad \chi\big(Q_0^{ij} + \nu_0\,\mathrm{div}_y\,\boldsymbol{W}_0^{ij}\big) = 0. \tag{6.20}$$

Problem (II).

$$\mathrm{div}_y\Bigg(\chi\Bigg(\mu_0\mathbb{D}\Big(y,\frac{\partial\boldsymbol{W}^0}{\partial t}\Big) + (1-\chi)\lambda_0\mathbb{D}(y,\mathbf{W}^0) - ((1-\chi)\Pi^0 + \chi Q^0)\mathbb{I}\Bigg)\Bigg) = 0; \tag{6.21}$$

$$\frac{1}{p_*}P^0 + \chi(\mathrm{div}_y\,\mathbf{W}^0 + 1) = 0, \quad Q^0 = P^0 + \frac{\nu_0}{p_*}\frac{\partial P^0}{\partial t}; \tag{6.22}$$

$$\frac{1}{\eta_0}\Pi^0 + (1-\chi)(\mathrm{div}_y\,\mathbf{W}^0 + 1) = 0, \quad \mathbf{W}^0(\boldsymbol{y},0) = \mathbf{W}_0^0(\boldsymbol{y}); \tag{6.23}$$

$$\mathrm{div}_y\big(\chi\big(\mu_0\mathbb{D}(y,\boldsymbol{W}_0^0) - Q_0\mathbb{I}\big)\big) = 0, \quad \chi\big(Q_0 + \nu_0\big(\mathrm{div}_y\,\mathbf{W}_0^0 + 1\big)\big) = 0. \tag{6.24}$$

Therefore,

$$\mathbb{A}_2 = \mu_0 m\sum_{i,j=1}^{3} J^{ij}\otimes J^{ij} + \mu_0\mathbb{A}_0^f, \quad \mathbb{A}_0^f = \sum_{i,j=1}^{3}\big\langle\mu_0\mathbb{D}(y,\mathbf{W}_0^{ij})\big\rangle_{Y_f}\otimes J^{ij}; \tag{6.25}$$

$$\mathbb{A}_3 = \lambda_0(1-m)\sum_{i,j=1}^{3} J^{ij}\otimes J^{ij} - \lambda_0\mathbb{A}_0^f + \mu_0\mathbb{A}_1^f(0), \quad \mathbb{A}_4(t) = \mu_0\frac{d}{dt}\mathbb{A}_1^f(t) - \lambda_0\mathbb{A}_1^f(t); \tag{6.26}$$

$$\mathbb{A}_1^f(t) = \sum_{i,j=1}^{3}\Bigg(\Bigg\langle\mu_0\mathbb{D}\Big(y,\frac{\partial\,W^{ij}}{\partial t}(\boldsymbol{y},t)\Big)\Bigg\rangle_{Y_f} + \langle\lambda_0\mathbb{D}(y,\boldsymbol{W}^{ij}(\boldsymbol{y},t))\rangle_{Y_s}\Bigg)\otimes J^{ij}; \tag{6.27}$$

$$B_5(t) = \Bigg\langle\chi\mu_0\mathbb{D}\Big(y,\frac{\partial\boldsymbol{W}^0}{\partial t}(\boldsymbol{y},t)\Big) + (1-\chi)\lambda_0\mathbb{D}(y,\boldsymbol{W}^0(\boldsymbol{y},t))\Bigg\rangle_Y; \tag{6.28}$$

$$C_2(t) = -C_3(t) = \sum_{i,j=1}^{3}\langle\chi\,\mathrm{div}_y\,\boldsymbol{W}^{ij}(\boldsymbol{y},t)\rangle_Y J^{ij}; \tag{6.29}$$

$$a_2(t) = -a_3(t) = \langle\chi\,\mathrm{div}_y\,\boldsymbol{W}^0(\boldsymbol{y},t)\rangle_Y, \quad B_4 = \big\langle\chi\mu_0\mathbb{D}\big(y,\boldsymbol{W}_0^0(\boldsymbol{y})\big)\big\rangle_Y. \qquad \square \tag{6.30}$$

Lemma 6.4. *The tensors $\mathbb{A}_2$–$\mathbb{A}_4$, the matrices B_4, B_5, C_2, and C_3, and the scalars a_2 and a_3 are well defined and infinitely smooth in time.*

If a porous space is connected then the symmetric tensor $\mathbb{A}_2$ is strictly positive definite. In the case of a disconnected porous space, $\mathbb{A}_2 = 0$ and the symmetric tensor $\mathbb{A}_3$ becomes strictly positive definite.

Proof. All these objects are defined soundly if Problems (I) and (II) are well posed. The solvability of the above problems and smoothness with respect to time follow by linearity from the standard a priori estimates (multiplication of the equation for the solution by the proper solution and integration by parts). Note that all these problems have a unique solution up to an arbitrary constant vector. In order to discard the arbitrary constant vectors we demand that the average value of the solution over the domain Y should be equal to zero. The smoothness with respect to time follows from the estimates of the solution at an initial moment. Thus, for example, in Problem (I) first of all we estimate $\chi\boldsymbol{W}_0^{ij}$ as a solution to the problem (6.20). Further solving (6.17) together with the continuity equation (6.19) at $t = 0$ and using the continuity of displacements on the boundary γ we define and estimate $(1-\chi)\boldsymbol{W}_0^{ij}$. After that from (6.17) at $t = 0$ we define and estimate $\chi(\partial\boldsymbol{W}^{ij}/\partial t)(\boldsymbol{y},0)$. In the same way we estimate the second derivatives with respect to time after differentiating all equations with respect to time.

The symmetry and positive definiteness of the tensor $\mathbb{A}_2$ are showed in the same way as for the tensor $\mathbb{A}_0^s$. If the porous space is disconnected then the problem (6.20) has a unique solution linear in $\boldsymbol{y}$ such that

$$\chi\big(\mathbb{D}(y, \boldsymbol{W}_0^{ij}) + J^{ij}\big) = 0. \tag{6.31}$$

The last equality implies $\mathbb{A}_2 = 0$.

In this case the tensor $\mathbb{A}_3$ becomes strictly positive definite. Indeed,

$$\mathbb{A}_3 = \lambda_0 \sum_{i,j=1}^{3} J^{ij} \otimes J^{ij} + \mu_0 \mathbb{A}_1^f(0)$$

$$= \lambda_0 \sum_{i,j=1}^{3} J^{ij} \otimes J^{ij} + \sum_{i,j=1}^{3} \Big\langle \chi\mu_0 \mathbb{D}\Big(y, \frac{\partial \boldsymbol{W}^{ij}}{\partial t}(\boldsymbol{y}, 0)\Big) + \frac{\lambda_0}{\mu_0} J^{ij} \Big\rangle_Y \otimes J^{ij}.$$

On the other hand, coming back to (6.17) at the initial moment we see that

$$\Big\langle \chi\mu_0 \mathbb{D}\Big(y, \frac{\partial \boldsymbol{W}^{ij}}{\partial t}(\boldsymbol{y}, 0)\Big) : \mathbb{D}\big(y, \boldsymbol{W}_0^{kl}\big) \Big\rangle_Y = -\lambda_0 \big\langle \chi \mathbb{D}\big(y, \boldsymbol{W}_0^{ij}\big) : \mathbb{D}\big(y, \boldsymbol{W}_0^{kl}\big)\big\rangle - \Big\langle \frac{1}{\eta_0} \Pi^{ij} \cdot \Pi^{kl} \Big\rangle_Y \Big|_{t=0}.$$

Moreover, by (6.31)

$$\Big\langle \chi\mu_0 \mathbb{D}\Big(y, \frac{\partial W^{ij}}{\partial t}(\boldsymbol{y}, 0)\Big) : \mathbb{D}\big(y, \boldsymbol{W}_0^{kl}\big) \Big\rangle_Y = -\Big\langle \chi \mathbb{D}\Big(y, \frac{\partial \boldsymbol{W}^{ij}}{\partial t}(\boldsymbol{y}, 0)\Big) : J^{kl} \Big\rangle_Y,$$

which proves our claim.

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