NON-ISOTHERMAL FILTRATION AND SEISMIC ACOUSTIC IN POROUS SOILS: THERMO-VISCOELASTIC AND LAMÉ EQUATIONS

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ABSTRACT. The paper considers the linear system of differential equations describing the simultaneous motion of an incompressible elastic porous body and an incompressible fluid filling in the pores. The model considered is very complicated, since the basic differential equations contain nondifferentiable rapidly oscillating small and large coefficients under the derivative signs. On the basis of the Nguetseng two-scaled convergence, the author suggests a correct deduction of averaged equations which are either the thermo-viscoelasticity system of equations (connected pore space) or the anisotropic Lamé system of thermoelasticity.

Introduction

In the present paper, we study the problem on the simultaneous motion of a non-isothermal elastic body perforated by a system of pores and channels (rigid *skeleton*) and a non-isothermal viscous fluid filling in the pores (*pore space*). In dimensionless (not marked by primes) variables

$$\boldsymbol{x}' = L\boldsymbol{x}, \quad t' = \tau t, \quad \boldsymbol{w}' = L\boldsymbol{w}, \quad \theta' = \vartheta_* \frac{L}{\tau v_*}\theta,$$

the differential equations for the dimensionless movements $\boldsymbol{w}$ and the dimensionless temperature θ in a domain $\Omega \subset \mathbb{R}^3$ have the form

$$\alpha_\tau \bar{\rho}\frac{\partial^2 \boldsymbol{w}}{\partial t^2} = \operatorname{div}_x\Big(\bar{\chi}\alpha_\mu \mathbb{D}(\boldsymbol{x}, \frac{\partial \boldsymbol{w}}{\partial t}) + (1-\bar{\chi})\alpha_\lambda \mathbb{D}(x, \boldsymbol{w}) - (p + \pi - \bar{\alpha}_\theta \theta)\mathbb{I}\Big) + \bar{\rho}\boldsymbol{F}, \tag{0.1}$$

$$\alpha_\tau \bar{c}_p \frac{\partial \theta}{\partial t} = \operatorname{div}_x(\bar{\alpha}_\varkappa \nabla_x \theta) - \bar{\alpha}_\theta \frac{\partial}{\partial t}\operatorname{div}_x \boldsymbol{w}, \tag{0.2}$$

$$p + \alpha_p \bar{\chi}\operatorname{div}_x \boldsymbol{w} = 0, \tag{0.3}$$

$$\pi + \alpha_\eta (1-\bar{\chi})\operatorname{div}_x \boldsymbol{w} = 0. \tag{0.4}$$

Here and in what follows, we use the notation

$$\mathbb{D}(x, \boldsymbol{u}) = \frac{1}{2}(\nabla_x \boldsymbol{u} + (\nabla_x \boldsymbol{u})^T),$$

$$\bar{\rho} = \bar{\chi}\rho_f + (1-\bar{\chi})\rho_s,$$
$$\bar{c}_p = \bar{\chi}c_{pf} + (1-\bar{\chi})c_{ps},$$
$$\bar{\alpha}_{\varkappa} = \bar{\chi}\alpha_{\varkappa f} + (1-\bar{\chi})\alpha_{\varkappa s},$$
$$\bar{\alpha}_{\theta} = \bar{\chi}\alpha_{\theta f} + (1-\bar{\chi})\alpha_{\theta s}.$$

We assume that the characteristic function $\bar{\chi}(\boldsymbol{x})$ of the pore spaces $\Omega_f \subset \Omega$ is known.

The deduction of Eqs. (0.1)–(0.3) and the description of dimensionless constants (all of them are strictly positive) are given in [8].

The problem is closed by the homogeneous initial and boundary conditions

$$\boldsymbol{w}|_{t=0} = 0, \quad \frac{\partial \boldsymbol{w}}{\partial t}|_{t=0} = 0, \quad \theta|_{t=0} = 0, \quad \boldsymbol{x} \in \Omega; \tag{0.5}$$

$$\boldsymbol{w} = 0, \quad \theta = 0, \quad \boldsymbol{x} \in S = \partial\Omega, \quad t \geq 0. \tag{0.6}$$

In the proposed model, the ratio of the mean size l of pores and the characteristic size L of the domain considered serves as a natural small parameter:

$$\varepsilon = \frac{l}{L}.$$

We restrict ourselves to the case where the pore space is geometrically periodic.

Our primary goal is to find the limit regimes (averaged equations) as the small parameter tends to zero.

Let the following assumption hold.

Assumption 0.1. The domain $\Omega = (0,1)^3$ is a periodic repetition of the elementary cell $Y^\varepsilon = \varepsilon Y$, where $Y = (0,1)^3$. The quantity $1/\varepsilon$ is an integer, so that Ω contains an integral number of elementary cells. Let Y_s be the "rigid phase" of the cell Y, and let the "fluid phase" Y_f be its open complement. Also, we set $\gamma = \partial Y_f \cap \partial Y_s$. The boundary γ is a surface of class C^1, the pore space Ω_f^ε is a periodic repetition of the elementary cell εY_f, the rigid skeleton Ω_s^ε is a periodic repetition of the elementary cell εY_s, and the boundary $\Gamma^\varepsilon = \partial\Omega_s^\varepsilon \cap \partial\Omega_f^\varepsilon$ is a periodic repetition of the boundary $\varepsilon\gamma$ in Ω. In addition, the domain $\boldsymbol{\Omega}_s$ *is connected.*

Under these assumptions,

$$\bar{\chi}(\boldsymbol{x}) = \chi^\varepsilon(\boldsymbol{x}) = \chi\left(\boldsymbol{x}/\varepsilon\right),$$
$$\bar{c}_p = c_p^\varepsilon(\boldsymbol{x}) = \chi^\varepsilon(\boldsymbol{x})c_{pf} + (1-\chi^\varepsilon(\boldsymbol{x}))c_{ps},$$
$$\bar{\rho} = \rho^\varepsilon(\boldsymbol{x}) = \chi^\varepsilon(\boldsymbol{x})\rho_f + (1-\chi^\varepsilon(\boldsymbol{x}))\rho_s,$$
$$\bar{\alpha}_{\varkappa} = \alpha_{\varkappa}^\varepsilon(\boldsymbol{x}) = \chi^\varepsilon(\boldsymbol{x})\alpha_{\varkappa f} + (1-\chi^\varepsilon(\boldsymbol{x}))\alpha_{\varkappa s},$$
$$\bar{\alpha}_{\theta} = \alpha_{\theta}^\varepsilon(\boldsymbol{x}) = \chi^\varepsilon(\boldsymbol{x})\alpha_{\theta f} + (1-\chi^\varepsilon(\boldsymbol{x}))\alpha_{\theta s},$$

where $\chi(\boldsymbol{y})$ is the characteristic function of Y_f in Y.

We say that the *pore space is disconnected* (isolated pores) if $\gamma \cap \partial Y = \emptyset$.

In the present work, we assume that the dimensionless parameters of the model depend on the small parameter ε, and there exist finite or infinite limits

$$\lim_{\varepsilon\searrow 0}\alpha_\mu(\varepsilon) = \mu_0, \quad \lim_{\varepsilon\searrow 0}\alpha_\lambda(\varepsilon) = \lambda_0, \quad \lim_{\varepsilon\searrow 0}\alpha_\tau(\varepsilon) = \tau_0, \quad \lim_{\varepsilon\searrow 0}\alpha_p(\varepsilon) = p_*, \quad \lim_{\varepsilon\searrow 0}\alpha_\eta(\varepsilon) = \eta_0.$$

The condition $p_* = \eta_0 = \infty$ means that the fluid considered and the rigid skeleton are incompressible.

Simpler model for isothermal media were studied in [1–4, 7, 10, 11].

1. Formulation of Main Results

As usual, Eqs. (0.1)–(0.2) are understood in the distribution theory sense. They properly include Eqs. (0.1)–(0.2) in each of the domains Ω_f^ε and Ω_s^ε and the boundary conditions

$$[\vartheta] = 0, \qquad [\boldsymbol{w}] = 0, \quad \boldsymbol{x}_0 \in \Gamma^\varepsilon, \quad t \geq 0, \tag{1.1}$$

$$[\mathbb{P}\cdot\mathbf{n}] = 0, \quad [\alpha_\varkappa^\varepsilon \nabla_x \theta \cdot \mathbf{n}] = 0, \quad \boldsymbol{x}_0 \in \Gamma^\varepsilon, \quad t \geq 0 \tag{1.2}$$

on the boundary Γ^ε, where $\mathbf{n}$ is the unit normal vector to the boundary and

$$[\varphi](\boldsymbol{x}_0) = \varphi_{(s)}(\boldsymbol{x}_0) - \varphi_{(f)}(\boldsymbol{x}_0),$$

$$\varphi_{(s)}(\boldsymbol{x}_0) = \lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}_0\\ \boldsymbol{x}\in\Omega_s^\varepsilon}} \varphi(\boldsymbol{x}), \qquad \varphi_{(f)}(\boldsymbol{x}_0) = \lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}_0\\ \boldsymbol{x}\in\Omega_f^\varepsilon}} \varphi(\boldsymbol{x}).$$

There exist various forms of writing Eqs. (0.1)–(0.2) and the boundary conditions (1.1)–(1.2), which are equivalent in the distribution theory sense. The writing in the form of integral identities is convenient for us.

Definition 1.1. Functions $(\boldsymbol{w}^\varepsilon,\, \theta^\varepsilon,\, p^\varepsilon)$ are called a generalized solution of the problem (0.1)–(0.6), (1.1)–(1.2) if they satisfy the regularity conditions

$$\boldsymbol{w}^\varepsilon,\, \mathbb{D}(x, \boldsymbol{w}^\varepsilon),\, \operatorname{div}_x \boldsymbol{w}^\varepsilon,\, p^\varepsilon,\, \theta^\varepsilon, \nabla_x\theta^\varepsilon \in L^2(\Omega_T) \tag{1.3}$$

in the domain $\Omega_T = \Omega \times (0, T)$, the boundary conditions (0.5), the equations

$$p^\varepsilon + \alpha_p \chi^\varepsilon \operatorname{div}_x \boldsymbol{w}^\varepsilon = 0, \tag{1.4}$$

$$\pi^\varepsilon + \alpha_\eta (1-\chi^\varepsilon) \operatorname{div}_x \boldsymbol{w}^\varepsilon = 0 \tag{1.5}$$

almost everywhere in the domain Ω_T, the integral identity

$$\left.\begin{aligned} &\int\limits_{\Omega_T} \Big(\alpha_\tau \rho^\varepsilon \boldsymbol{w}^\varepsilon \cdot \frac{\partial^2 \boldsymbol{\varphi}}{\partial t^2} - \rho^\varepsilon \boldsymbol{F}\cdot\boldsymbol{\varphi} - \chi^\varepsilon \alpha_\mu \mathbb{D}(\boldsymbol{x}, \boldsymbol{w}^\varepsilon) : \mathbb{D}\Big(x, \frac{\partial \boldsymbol{\varphi}}{\partial t}\Big) \\ &+\{(1-\chi^\varepsilon)\alpha_\lambda \mathbb{D}(x, \boldsymbol{w}^\varepsilon) - (p^\varepsilon + \pi^\varepsilon - \alpha_\theta^\varepsilon \theta^\varepsilon)\mathbb{I}\} : \mathbb{D}(x, \boldsymbol{\varphi})\Big) d\boldsymbol{x}dt = 0 \end{aligned}\right\} \tag{1.6}$$

for all smooth vector-valued functions $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\boldsymbol{x}, t)$ such that

$$\boldsymbol{\varphi}|_{\partial\Omega} = \boldsymbol{\varphi}|_{t=T} = \partial\boldsymbol{\varphi}/\partial t|_{t=T} = 0,$$

and the integral identity

$$\int\limits_{\Omega_T} \Big((\alpha_\tau c_p^\varepsilon \theta^\varepsilon + \alpha_\theta^\varepsilon \operatorname{div}_x \boldsymbol{w}^\varepsilon)\frac{\partial \xi}{\partial t} - \alpha_\varkappa^\varepsilon \nabla_x \theta^\varepsilon \cdot \nabla_x \xi\Big) d\boldsymbol{x}dt = 0 \tag{1.7}$$

for all smooth functions $\xi = \xi(\boldsymbol{x}, t)$ such that $\xi|_{\partial\Omega} = \xi|_{t=T} = 0$.

In (1.6), we denote by $A : B$ the contraction of two second-rank tensors in both indices, i.e.,

$$A : B = \operatorname{tr}(B^* \circ A) = \sum_{i,j=1}^{3} A_{ij} B_{ji}.$$

In addition to the assumptions made in the Introduction, let there exist finite limits

$$\lim_{\varepsilon\searrow 0} \alpha_{\varkappa s}(\varepsilon) = \varkappa_{0s}, \quad \lim_{\varepsilon\searrow 0} \alpha_{\varkappa f} = \varkappa_{0f}, \quad \lim_{\varepsilon\searrow 0} \alpha_{\theta f}(\varepsilon) = \beta_{0f}, \quad \lim_{\varepsilon\searrow 0} \alpha_{\theta s}(\varepsilon) = \beta_{0s}.$$

In what follows, we assume that the following assumption holds.

Assumption 1.1. (1) The dimensionless parameters in the model $(\boldsymbol{NA})^{\varepsilon}$ satisfy the following restrictions:

$$0<\varkappa_{0f},\ \varkappa_{0s},\ \mu_0,\ \lambda_0<\infty;\quad \tau_0,\ \beta_{0f},\ \beta_{0s},\ p_*^{-1},\ \eta_0^{-1}<\infty.$$

(2) The functions $\boldsymbol{F}$, $\partial\boldsymbol{F}/\partial t$, and $\partial^2\boldsymbol{F}/\partial t^2$ are bounded in $L^2(\Omega_T)$.

Everywhere in what follows, the parameters of the model can assume all admissible values. So, for example, if $\tau_0=0$ or $\eta_0^{-1}=0$, then the summands containing these quantities disappear in all the equations.

The following Theorems 1.1–1.2 are the main results of the present paper.

Theorem 1.1. *Under the assumptions made above, for all $\varepsilon>0$, on an arbitrary interval of time $[0,T]$, there exists a unique generalized solution in the model $(\boldsymbol{NA})^{\varepsilon}$, and the following estimates hold:*

$$\max_{0\leq t\leq T}\left\|\left|\frac{\partial\boldsymbol{w}^{\varepsilon}}{\partial t}(t)\right|+\alpha_{\tau}\left|\frac{\partial^2\boldsymbol{w}^{\varepsilon}}{\partial t^2}(t)\right|+\left|\nabla_x\frac{\partial\boldsymbol{w}^{\varepsilon}}{\partial t}(t)\right|\right\|_{2,\Omega}\leq C_0, \tag{1.8}$$

$$\max_{0\leq t\leq T}\left(\left\|\frac{\partial\theta^{\varepsilon}}{\partial t}\right\|_{2,\Omega}+\|\nabla_x\theta^{\varepsilon}\|_{2,\Omega}\right)\leq C_0, \tag{1.9}$$

$$\max_{0\leq t\leq T}(\|\pi^{\varepsilon}(t)\|_{2,\Omega}+\|p^{\varepsilon}(t)\|_{2,\Omega})\leq C_0, \tag{1.10}$$

where the constant C_0 is independent of the small parameter ε.

Theorem 1.2. *The sequences $\{\boldsymbol{w}^{\varepsilon}\}$ and $\{\theta^{\varepsilon}\}$ strongly converge in $L^2(\Omega_T)$ and weakly converge in $L^2((0,T);W_2^1(\Omega))$ to functions $\boldsymbol{w}$ and θ, whereas the sequences $\{p^{\varepsilon}\}$ and $\{\pi^{\varepsilon}\}$ weakly converge in $L^2(\Omega_T)$ to functions p and π. Moreover, the functions $\boldsymbol{w}$, θ, p, and π satisfy the following initial-value problem in Ω_T:*

$$\left.\begin{array}{l}\tau_0\hat{\rho}\dfrac{\partial^2\boldsymbol{w}}{\partial t^2}+\nabla(q+\pi-\hat{\beta}_0\theta)-\hat{\rho}\boldsymbol{F}=\operatorname{div}_x\Big(\mathbb{A}_1:\mathbb{D}\Big(x,\dfrac{\partial\boldsymbol{w}}{\partial t}\Big)+\mathbb{A}_2:\mathbb{D}(x,\boldsymbol{w})\\ [2mm]+\displaystyle\int_0^t\big(\mathbb{A}_3(t-\tau):\mathbb{D}(x,\boldsymbol{w}(\boldsymbol{x},\tau))+B(t-\tau)\operatorname{div}_x\boldsymbol{w}(\boldsymbol{x},\tau)+B^{\theta}(t-\tau)\theta(\boldsymbol{x},\tau)\big)d\tau\Big),\end{array}\right\} \tag{1.11}$$

$$\left.\begin{array}{l}\dfrac{1}{p_*}p+m\operatorname{div}_x\boldsymbol{w}=-\displaystyle\int_0^t\big(C_1(t-\tau):\mathbb{D}(x,\boldsymbol{w}(\boldsymbol{x},\tau))\\ +a_1(t-\tau)\operatorname{div}_x\boldsymbol{w}(\boldsymbol{x},\tau)+a_1^{\theta}(t-\tau)\theta(\boldsymbol{x},\tau)\big)d\tau,\end{array}\right\} \tag{1.12}$$

$$\left.\begin{array}{l}\dfrac{1}{\eta_0}\pi+(1-m)\operatorname{div}_x\boldsymbol{w}=-\displaystyle\int_0^t\big(C_2(t-\tau):\mathbb{D}(x,\boldsymbol{w}(\boldsymbol{x},\tau))\\ +a_2(t-\tau)\operatorname{div}_x\boldsymbol{w}(\boldsymbol{x},\tau)+a_2^{\theta}(t-\tau)\theta(\boldsymbol{x},\tau)+\big)d\tau,\end{array}\right\} \tag{1.13}$$

$$\left.\begin{array}{l}\tau_0\hat{c}_p\dfrac{\partial\theta}{\partial t}-\dfrac{\beta_{0f}}{p_*}\dfrac{\partial p}{\partial t}-\dfrac{\beta_{0s}}{\eta_0}\dfrac{\partial\pi}{\partial t}+(\beta_{0s}-\beta_{0f})a\dfrac{\partial}{\partial t}\langle\theta\rangle_{\Omega}\\ =\operatorname{div}_x\big(B_0^{\theta}\cdot\nabla_x\theta\big)+\Psi.\end{array}\right\} \tag{1.14}$$

The problem is closed by the homogeneous initial and boundary conditions (0.5) *and* (0.6) *for* θ *and* $\boldsymbol{w}$.

In (1.11)–(1.14),

$$\hat{\rho} = \rho_f m + \rho_s(1-m), \quad \hat{\beta}_0 = \beta_{0f} m + \beta_{0s}(1-m), \quad \hat{c_p} = c_{pf} m + c_{ps}(1-m),$$

$\mathbb{A}_1$, $\mathbb{A}_2$ *and* $\mathbb{A}_3(t)$ *are fourth-rank tensors,* $B(t)$, $B^\theta(t)$, B_0^θ, $C_1(t)$, *and* $C_2(t)$ *are matrices, and* $a_1(t)$, $a_2(t)$, $a_1^\theta(t)$, $a_2^\theta(t)$, *and* $a(t)$ *are scalar. The precise expressions for these object are given by formulas* (4.20)–(4.27) *below. The matrix* B_0^θ *is strictly positive-definite.*

In the case where the pore space is connected, the symmetric tensor $\mathbb{A}_1$ *is strictly positive-definite. Otherwise,* $\mathbb{A}_1 = 0$, *and system* (1.11) *degenerates into a nonlocal anisotropic Lamé system with the symmetric strictly positive-definite tensor* $\mathbb{A}_2$.

2. Preliminaries

The proof of Theorem 1.2 is based on the systematic application of the two-scaled convergence method suggested by G. Nguetseng in [9], which finds wide application in averaging theory (see, e.g., the overview [6]).

Definition 2.1. A sequence $\{\varphi^\varepsilon\} \subset L^2(\Omega_T)$ is said to be *two-scaled convergent* to the limit $\varphi \in L^2(\Omega_T \times Y)$ iff the following limit relation holds for any smooth function $\sigma = \sigma(\boldsymbol{x}, t, \boldsymbol{y})$ 1-periodic in $\boldsymbol{y}$:

$$\lim_{\varepsilon \searrow 0} \int\limits_{\Omega_T} \varphi^\varepsilon(\boldsymbol{x}, t)\sigma(\boldsymbol{x}, t, \boldsymbol{x}/\varepsilon) d\boldsymbol{x} dt = \int\limits_{\Omega_T} \int\limits_{Y} \varphi(\boldsymbol{x}, t, \boldsymbol{y})\sigma(\boldsymbol{x}, t, \boldsymbol{y}) d\boldsymbol{y} d\boldsymbol{x} dt. \tag{2.1}$$

The existence and main properties of two-scaled convergent sequences are stated in the following theorem [6, 9].

Theorem 2.1 (Nguetseng theorem). 1. *From any sequence bounded in* $L^2(\Omega_T)$, *we can choose a subsequence two-scaled converging to a certain limit* $\varphi \in L^2(\Omega_T \times Y)$.

2. *Let sequences* $\{\varphi^\varepsilon\}$ *and* $\{\varepsilon \nabla_x \varphi^\varepsilon\}$ *be uniformly bounded in* $L^2(\Omega_T)$. *Then there exist a function* $\varphi = \varphi(\boldsymbol{x}, t, \boldsymbol{y})$ *1-periodic in* $\boldsymbol{y}$ *and a subsequence of* $\{\varphi^\varepsilon\}$ *such that* $\varphi, \nabla_y \varphi \in L^2(\Omega_T \times Y)$ *and* φ^ε *and* $\varepsilon \nabla_x \varphi^\varepsilon$ *two-scaled converge to* φ *and* $\nabla_y \varphi$, *respectively.*

3. *Let sequences* $\{\varphi^\varepsilon\}$ *and* $\{\nabla_x \varphi^\varepsilon\}$ *be uniformly bounded in* $L^2(\Omega_T)$. *Then there exist functions* $\varphi \in L^2(\Omega_T)$ *and* $\psi \in L^2(\Omega_T \times Y)$ *and also a subsequence of* $\{\varphi^\varepsilon\}$ *such that* ψ *is 1-periodic in* $\boldsymbol{y}$, $\nabla_y \psi \in L^2(\Omega_T \times Y)$, *and* φ^ε *and* $\nabla_x \varphi^\varepsilon$ *two-scaled converge to* φ *and* $\nabla_x \varphi(\boldsymbol{x}, t) + \nabla_y \psi(\boldsymbol{x}, t, \boldsymbol{y})$, *respectively.*

Corollary 2.1. *Let* $\sigma \in L^2(Y)$, *and let* $\sigma^\varepsilon(\boldsymbol{x}) := \sigma(\boldsymbol{x}/\varepsilon)$. *Let a sequence* $\{\varphi^\varepsilon\} \subset L^2(\Omega_T)$ *two-scaled converge to a certain limit* $\varphi \in L^2(\Omega_T \times Y)$. *Then the sequence* $\sigma^\varepsilon \varphi^\varepsilon$ *two-scaled converges to* $\sigma\varphi$.

Everywhere in what follows, we use the following notation:

(1)

$$\langle \Phi \rangle_Y = \int\limits_Y \Phi\, dy, \qquad \langle \Phi \rangle_{Y_f} = \int\limits_Y \chi \Phi\, dy, \qquad \langle \Phi \rangle_{Y_s} = \int\limits_Y (1-\chi)\Phi\, dy,$$

$$\langle \varphi \rangle_\Omega = \int\limits_\Omega \varphi\, dx, \qquad \langle \varphi \rangle_{\Omega_T} = \int\limits_{\Omega_T} \varphi\, dx dt;$$

(2) if $\mathbf{a}$ and $\mathbf{b}$ are two vectors, then the matrix $\mathbf{a} \otimes \mathbf{b}$ is defined as

$$(\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c})$$

for an arbitrary vector $\mathbf{c}$;

(3) if B and C are two matrices, then $B\otimes C$ is a fourth-rank tensor such that its contraction with an arbitrary matrix A is given by the formula

$$(B\otimes C):A=B(C:A);$$

(4) denote by $\mathbb{I}^{ij}$ the matrix such that its unique nonzero entry equal to 1 stands at the intersection of the ith row and jth column;

(5) finally,

$$J^{ij}=\frac{1}{2}(\mathbb{I}^{ij}+\mathbb{I}^{ji})=\frac{1}{2}(\boldsymbol{e}_i\otimes\boldsymbol{e}_j+\boldsymbol{e}_j\otimes\boldsymbol{e}_i),$$

where $(\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3)$ is an orthonormal basis.

3. Proof of Theorem 1.1

The following estimate holds for all $\varepsilon>0$:

$$\max_{0<t<T}\left(\sqrt{\alpha_\lambda}\left\|\nabla_x\frac{\partial\boldsymbol{w}^\varepsilon}{\partial t}(t)\right\|_{2,\Omega_s^\varepsilon}+\sqrt{\alpha_\tau}\left\|\frac{\partial^2\boldsymbol{w}^\varepsilon}{\partial t^2}(t)\right\|_{2,\Omega}+\sqrt{\alpha_p}\left\|\operatorname{div}_x\frac{\partial\boldsymbol{w}^\varepsilon}{\partial t}(t)\right\|_{2,\Omega_f^\varepsilon}+\sqrt{\alpha_\tau}\left\|\frac{\partial\theta^\varepsilon}{\partial t}(t)\right\|_{2,\Omega}\right)$$
$$+\sqrt{\alpha_\mu}\left\|\chi^\varepsilon\nabla_x\frac{\partial^2\boldsymbol{w}^\varepsilon}{\partial t^2}\right\|_{2,\Omega_T}+\left\|\sqrt{\alpha_\theta^\varepsilon}\nabla_x\frac{\partial\theta^\varepsilon}{\partial t}\right\|_{2,\Omega_T}\leq C_0, \tag{3.1}$$

where C_0 is independent of ε. It is obtained after the differentiation of the equations for $\boldsymbol{w}^\varepsilon$ and θ^ε in time, the multiplication of the first equation by $\partial^2\boldsymbol{w}^\varepsilon/\partial t^2$ and the second by $\partial\theta^\varepsilon/\partial t$, the integration by parts, and the summation. This estimate guarantees the existence and uniqueness of a generalized solution of the problem (0.1)–(0.6), (1.1)–(1.2).

Estimates (1.8) and (1.9) follows from estimates (3.1) and the Poincaré inequality.

Estimate (1.10) for the pressures follows from the integral identity (1.6) and estimates (1.8)–(1.9) as an estimate of the corresponding functional if we normalize the pressures in such a way that

$$\int_\Omega(p^\varepsilon(\boldsymbol{x},t)+\pi^\varepsilon(\boldsymbol{x},t))\,d\boldsymbol{x}=0.$$

Indeed, it follows from the integral identity (1.6) that

$$\left|\int_\Omega(p^\varepsilon(\boldsymbol{x},t)+\pi^\varepsilon(\boldsymbol{x},t))\operatorname{div}_x\boldsymbol{\psi}\,d\boldsymbol{x}\right|\leq C\|\nabla\boldsymbol{\psi}\|_{2,\Omega}.$$

Now choosing $\boldsymbol{\psi}$ such that $p^\varepsilon+\pi^\varepsilon=\operatorname{div}_x\boldsymbol{\psi}$, we obtain an estimate for the sum $(p^\varepsilon+\pi^\varepsilon)$ of pressures. Such a choice is always possible (see [5]) if we set

$$\boldsymbol{\psi}=\nabla\varphi+\boldsymbol{\psi}_0,\quad\operatorname{div}_x\boldsymbol{\psi}_0=0,\quad\Delta\varphi=p^\varepsilon+\pi^\varepsilon,\quad\varphi|_{\partial\Omega}=0,\quad(\nabla\varphi+\boldsymbol{\psi}_0)|_{\partial\Omega}=0.$$

It remains to note that by the orthogonality of the functions p^ε and π^ε, the estimate of the sum implies estimates for each of the summands.

Also, we note that the renormalization of the pressures changes the equations of continuity (1.4) and (1.5) to the equations

$$\frac{1}{\alpha_p}p^\varepsilon+\chi^\varepsilon\operatorname{div}_x\boldsymbol{w}^\varepsilon=-\frac{1}{m}\gamma^\varepsilon\chi^\varepsilon \tag{3.2}$$

and

$$\frac{1}{\alpha_p}\pi^\varepsilon+(1-\chi^\varepsilon)\operatorname{div}_x\boldsymbol{w}^\varepsilon=\frac{1}{1-m}\gamma^\varepsilon(1-\chi^\varepsilon), \tag{3.3}$$

where

$$\gamma^\varepsilon=\langle(1-\chi^\varepsilon)\operatorname{div}_x\boldsymbol{w}^\varepsilon\rangle_\Omega.$$

4. Proof of Theorem 1.2

4.1. Weak and two-scaled limits of sequences of movements of temperatures and pressures. By Theorem 1.1, the sequences $\{\boldsymbol{w}^\varepsilon\}$, θ^ε, $\{p^\varepsilon\}$, and $\{\pi^\varepsilon\}$ are bounded in $L^2(\Omega_T)$ uniformly in ε. Hence there exist a subsequence from $\{\varepsilon>0\}$ and functions p, π, $\boldsymbol{w}$, and θ such that as $\varepsilon\searrow 0$,

$$p^\varepsilon \to p, \quad \pi^\varepsilon \to \pi \quad \text{weakly in} \quad L^2(\Omega_T), \tag{4.1}$$

$$\boldsymbol{w}^\varepsilon \to \boldsymbol{w}, \quad \theta^\varepsilon \to \theta \quad \text{strongly in} \quad L^2(\Omega_T), \tag{4.2}$$

$$\nabla_x \boldsymbol{w}^\varepsilon \to \nabla_x \boldsymbol{w}, \quad \nabla_x\theta^\varepsilon \to \nabla_x\theta \quad \text{weakly in} \quad L^2(\Omega_T). \tag{4.3}$$

Using the latter relations and the Nguetseng theorem, we conclude that there exist functions $P(\boldsymbol{x},t,\boldsymbol{y})$, $\Pi(\boldsymbol{x},t,\boldsymbol{y})$, $\Theta(\boldsymbol{x},t,\boldsymbol{y})$, and $\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$ 1-periodic in $\boldsymbol{y}$ such that the sequences $\{p^\varepsilon\}$, $\{\pi^\varepsilon\}$, $\{\nabla\theta^\varepsilon\}$, and $\{\nabla\boldsymbol{w}^\varepsilon\}$ two-scaled converge to P, Π, $\nabla_x\theta+\nabla_y\Theta$, and $\nabla_x\boldsymbol{w}+\nabla_y\boldsymbol{W}$, respectively, as $\varepsilon\searrow 0$.

4.2. Micro- and macroscopic equations.

Lemma 4.1. *The two-scaled limits of the sequences $\{p^\varepsilon\}$, $\{\pi^\varepsilon\}$, $\{\nabla\theta^\varepsilon\}$, and $\{\nabla\boldsymbol{w}^\varepsilon\}$ satisfy the following macroscopic relations in $Y_T=Y\times(0,T)$:*

$$\frac{1}{\eta_0}\Pi+(1-\chi)(\operatorname{div}_x\boldsymbol{w}+\operatorname{div}_y\boldsymbol{W})=\frac{\gamma}{(1-m)}(1-\chi); \tag{4.4}$$

$$\frac{1}{p_*}P+\chi(\operatorname{div}_x\boldsymbol{w}+\operatorname{div}_y\boldsymbol{W})=-\frac{\gamma}{m}\chi; \tag{4.5}$$

$$\left.\begin{aligned}\nabla_y(P+\Pi-(\beta_{0f}\chi+\beta_{0s}(1-\chi))\theta)=\operatorname{div}_y\Big(\chi\mu_0\Big(\mathbb{D}\Big(x,\frac{\partial\boldsymbol{w}}{\partial t}\Big)\\ +\mathbb{D}\Big(y,\frac{\partial\boldsymbol{W}}{\partial t}\Big)\Big)+(1-\chi)\lambda_0(\mathbb{D}(x,\boldsymbol{w})+\mathbb{D}(y,\boldsymbol{W}))\Big),\end{aligned}\right\} \tag{4.6}$$

$$\operatorname{div}_y\big(\chi\kappa_{0f}(\nabla_x\theta+\nabla_y\Theta)+(1-\chi)\kappa_{0s}(\nabla_x\theta+\nabla_y\Theta)\big)=0. \tag{4.7}$$

Lemma 4.2. *The weak and strong limits p, π, θ, and $\boldsymbol{w}$ satisfy the following system of macroscopic equations in Ω_T:*

$$\frac{1}{\eta_0}\pi+(1-m)\operatorname{div}_x\boldsymbol{w}+\langle\operatorname{div}_y\boldsymbol{W}\rangle_{Y_s}=\gamma; \tag{4.8}$$

$$\frac{1}{p_*}p+m\operatorname{div}_x\boldsymbol{w}+\langle\operatorname{div}_y\boldsymbol{W}\rangle_{Y_f}=-\gamma; \tag{4.9}$$

$$\left.\begin{aligned}\tau_0\hat{\rho}\frac{\partial^2\boldsymbol{w}}{\partial t^2}+\nabla(q+\pi-\hat{\beta}_0\theta)-\hat{\rho}\boldsymbol{F}=\operatorname{div}_x\Big(\mu_0\Big(m\mathbb{D}\Big(x,\frac{\partial\boldsymbol{w}}{\partial t}\Big)\\ +\Big\langle\mathbb{D}\Big(y,\frac{\partial\boldsymbol{W}}{\partial t}\Big)\Big\rangle_{Y_f}\Big)+\lambda_0((1-m)\mathbb{D}(x,\boldsymbol{w})+\langle\mathbb{D}(y,\boldsymbol{W})\rangle_{Y_f})\Big);\end{aligned}\right\} \tag{4.10}$$

$$\left.\begin{aligned}\tau_0\hat{c}_p\frac{\partial\theta}{\partial t}-\frac{\beta_{0f}}{p_*}\frac{\partial p}{\partial t}-\frac{\beta_{0s}}{\eta_0}\frac{\partial\pi}{\partial t}+(\beta_{0s}-\beta_{0f})\frac{\partial\gamma}{\partial t}-\Psi\\ =\operatorname{div}_x\big(\kappa_{0f}(m\nabla_x\theta+\langle\nabla_y\Theta\rangle_{Y_f})+\kappa_{0s}((1-m)\nabla_x\theta+\langle\nabla_y\Theta\rangle_{Y_s})\big).\end{aligned}\right\} \tag{4.11}$$

In (4.4)–(4.11), we have

$$\gamma=\langle(\langle\operatorname{div}_y\boldsymbol{W}\rangle_{Y_s})\rangle_\Omega,\quad \hat{\rho}=\rho_f m+\rho_s(1-m),\quad \hat{\beta}_0=\beta_{0f}m+\beta_{0s}(1-m),\quad\text{and}\quad \hat{c_p}=c_{pf}m+c_{ps}(1-m).$$

Proof. To prove (4.4) and (4.5), we multiply Eqs. (3.2) and (3.3) by $\psi^\varepsilon = \psi(\boldsymbol{x}, t, \boldsymbol{x}/\varepsilon)$, where $\psi(\boldsymbol{x}, t, \boldsymbol{y})$ is an arbitrary function 1-periodic in $\boldsymbol{y}$, and integrate the result over the domain Ω. Passing to the limit as $\varepsilon \searrow 0$, we obtain the required relations.

Equations (4.6) and (4.7) follow from the integral identities (1.6) and (1.7) if, as test functions, we consider the functions of the form $\boldsymbol{\varphi}^\varepsilon = \varepsilon\boldsymbol{\varphi}(\boldsymbol{x}, t, \boldsymbol{x}/\varepsilon)$ (identity (1.6)) and $\xi^\varepsilon = \varepsilon\xi(\boldsymbol{x}, t, \boldsymbol{x}/\varepsilon)$ (identity (1.7)), where $\boldsymbol{\varphi}(\boldsymbol{x}, t, \boldsymbol{y})$ and $\xi(\boldsymbol{x}, t, \boldsymbol{y})$ are arbitrary functions 1-periodic in $\boldsymbol{y}$, and then pass to the limit as $\varepsilon \searrow 0$.

Equations (4.8) and (4.9) are the result of averaging Eqs. (4.4) and (4.5) over the elementary cell Y, and Eqs. (4.10) and (4.11) follow from the integral identities (1.6) and (1.7) after passing to the limit as $\varepsilon \searrow 0$ with test functions independent of the "fast" variable y. In this case, in identity (1.7), we have used the equations of continuity (3.2) and (3.3). □

4.3. Averaged equations.

Lemma 4.3. *The weak and strong limits p, π, θ, and $\boldsymbol{w}$ satisfy the following system of averaged equations in Ω_T:*

$$\left.\begin{aligned} &\tau_0\hat{\rho}\frac{\partial^2\boldsymbol{w}}{\partial t^2} + \nabla(q + \pi - \hat{\beta}_0\theta) - \hat{\rho}\boldsymbol{F} = \operatorname{div}_x\Bigg(\mathbb{A}_1 : \mathbb{D}\Big(x, \frac{\partial\boldsymbol{w}}{\partial t}\Big) + \mathbb{A}_2 : \mathbb{D}(x, \boldsymbol{w}) \\ &+ \int_0^t \big(\mathbb{A}_3(t-\tau) : \mathbb{D}(x, \boldsymbol{w}(\boldsymbol{x}, \tau)) + B(t-\tau)\operatorname{div}_x\boldsymbol{w}(\boldsymbol{x}, \tau) + B^\theta(t-\tau)\theta(\boldsymbol{x}, \tau)\big)d\tau\Bigg), \end{aligned}\right\} \tag{4.12}$$

$$\left.\begin{aligned} \frac{1}{p_*}p + m\operatorname{div}_x\boldsymbol{w} = -\int_0^t \big(C_1(t-\tau) &: \mathbb{D}(x, \boldsymbol{w}(\boldsymbol{x}, \tau)) \\ + a_1(t-\tau)\operatorname{div}_x\boldsymbol{w}(\boldsymbol{x}, \tau) &+ a_1^\theta(t-\tau)\theta(\boldsymbol{x}, \tau)\big)d\tau, \end{aligned}\right\} \tag{4.13}$$

$$\left.\begin{aligned} \frac{1}{\eta_0}\pi + (1-m)\operatorname{div}_x\boldsymbol{w} = -\int_0^t \big(C_2(t-\tau) &: \mathbb{D}(x, \boldsymbol{w}(\boldsymbol{x}, \tau)) \\ + a_2(t-\tau)\operatorname{div}_x\boldsymbol{w}(\boldsymbol{x}, \tau) &+ a_2^\theta(t-\tau)\theta(\boldsymbol{x}, \tau) + \big)d\tau, \end{aligned}\right\} \tag{4.14}$$

$$\left.\begin{aligned} \tau_0\hat{c}_p\frac{\partial\theta}{\partial t} - \frac{\beta_{0f}}{p_*}\frac{\partial p}{\partial t} - \frac{\beta_{0s}}{\eta_0}\frac{\partial\pi}{\partial t} + (\beta_{0s} - \beta_{0f})a\frac{\partial}{\partial t}\langle\theta\rangle_\Omega \\ = \operatorname{div}_x\big(B_0^\theta \cdot \nabla_x\theta\big) + \Psi. \end{aligned}\right\} \tag{4.15}$$

Here, $\mathbb{A}_1$, $\mathbb{A}_2$, and $\mathbb{A}_3(t)$ are fourth-order tensors, $B(t)$, $B^\theta(t)$, B_0^θ, $C_1(t)$, and $C_2(t)$ are matrices, and $a_1(t)$, $a_2(t)$, $a_1^\theta(t)$, $a_2^\theta(t)$, and $a(t)$ are scalars. The precise expressions of these objects are given by formulas (4.20)–(4.27) *below.*

Proof. We set

$$Z(\boldsymbol{x}, t) = \mu_0\mathbb{D}\Big(x, \frac{\partial\boldsymbol{w}}{\partial t}\Big) - \lambda_0\mathbb{D}(x, \boldsymbol{w}), \quad Z_{ij} = \mathbf{e}_i \cdot (Z \cdot \mathbf{e}_j), \quad z_1(t) = \langle\theta\rangle_\Omega,$$

$$\boldsymbol{z}(\boldsymbol{x}, t) = \sum_{i=1}^3 z_i(\boldsymbol{x}, t)\boldsymbol{e}_i = (\kappa_{0f} - \kappa_{0f})\nabla_x\theta, \quad z_0(\boldsymbol{x}, t) = \operatorname{div}_x\boldsymbol{w}.$$

As usual, we seek solutions of the macroscopic Eqs. (4.4)–(4.7) in the form

$$\left.\begin{aligned} \boldsymbol{W} = \int\limits_0^t \Big[\boldsymbol{W}^0(\boldsymbol{y}, t-\tau) z_0(\boldsymbol{x},\tau) + \sum_{i,j=1}^3 \boldsymbol{W}^{ij}(\boldsymbol{y}, t-\tau) Z_{ij}(\boldsymbol{x},\tau) \\ + \boldsymbol{W}^\theta(\boldsymbol{y}, t-\tau)(\theta(\boldsymbol{x},\tau) - z_1(\tau)) + \boldsymbol{W}_1^\theta(\boldsymbol{y}, t-\tau) z_1(\tau) \Big] d\tau, \end{aligned}\right\} \tag{4.16}$$

$$\left.\begin{aligned} P = \chi \int\limits_0^t \Big[P^0(\boldsymbol{y}, t-\tau) z_0(\boldsymbol{x},\tau) + \sum_{i,j=1}^3 P^{ij}(\boldsymbol{y}, t-\tau) Z_{ij}(\boldsymbol{x},\tau) \\ + P^\theta(\boldsymbol{y}, t-\tau)(\theta(\boldsymbol{x},\tau) - z_1(\tau)) + P_1^\theta(\boldsymbol{y}, t-\tau) z_1(\tau) \Big] d\tau, \end{aligned}\right\} \tag{4.17}$$

$$\left.\begin{aligned} \Pi = (1-\chi) \int\limits_0^t \Big[\Pi^0(\boldsymbol{y}, t-\tau) z_0(\boldsymbol{x},\tau) + \sum_{i,j=1}^3 \Pi^{ij}(\boldsymbol{y}, t-\tau) Z_{ij}(\boldsymbol{x},\tau) \\ + \Pi^\theta(\boldsymbol{y}, t-\tau)(\theta(\boldsymbol{x},\tau) - z_1(\tau)) + \Pi_1^\theta(\boldsymbol{y}, t-\tau) z_1(\tau) \Big] d\tau, \end{aligned}\right\} \tag{4.18}$$

$$\Theta = \sum_{i=1}^3 \Theta^i(\boldsymbol{y}) z_i(\boldsymbol{x}, t), \tag{4.19}$$

where the functions $\boldsymbol{W}^0$, $\boldsymbol{W}^\theta$, $\boldsymbol{W}^{ij}$, P^0, P^θ, P^{ij}, Π^0, Π^θ, Π^{ij}, and Θ^i 1-periodic in $\boldsymbol{y}$ satisfy the following initial-value problems in the elementary cell Y:

Problem (I):

$$\left.\begin{aligned} &\left.\begin{aligned} \operatorname{div}_y \Big(\chi \Big(\mu_0 \mathbb{D}\Big(y, \frac{\partial \boldsymbol{W}^{ij}}{\partial t} \Big) \\ +(1-\chi)(\lambda_0 \mathbb{D}(y, \boldsymbol{W}^{ij}) - ((1-\chi)\Pi^{ij} + \chi P^{ij})\mathbb{I}) \Big) \Big) = 0, \end{aligned}\right\} \\ &\Big(\frac{1}{p_*}\Big) P^{ij} + \chi \operatorname{div}_y \boldsymbol{W}^{ij} = 0, \quad \Big(\frac{1}{\eta_0}\Big) \Pi^{ij} + (1-\chi) \operatorname{div}_y \boldsymbol{W}^{ij} = 0, \\ &\boldsymbol{W}^{ij}(\boldsymbol{y}, 0) = \boldsymbol{W}_0^{ij}(\boldsymbol{y}), \quad \operatorname{div}_y\big(\chi(\mu_0 \mathbb{D}(y, \boldsymbol{W}_0^{ij}) + J^{ij})\big) = 0. \end{aligned}\right\}$$

Problem (II):

$$\left.\begin{aligned} &\left.\begin{aligned} \operatorname{div}_y \Big(\chi \Big(\mu_0 \mathbb{D}\Big(y, \frac{\partial \boldsymbol{W}^{0}}{\partial t} \Big) + (1-\chi)(\lambda_0 \mathbb{D}(y, \boldsymbol{W}^{0}) \\ -((1-\chi)\Pi^{0} + \chi P^{0})\mathbb{I}) \Big) \Big) = 0, \quad \chi \boldsymbol{W}^0(\boldsymbol{y}, 0) = 0, \end{aligned}\right\} \\ &\Big(\frac{1}{p_*}\Big) P^{0} + \chi (\operatorname{div}_y \boldsymbol{W}^{0} + 1) = 0, \\ &\Big(\frac{1}{\eta_0}\Big) \Pi^{0} + (1-\chi) (\operatorname{div}_y \boldsymbol{W}^{0} + 1) = 0. \end{aligned}\right\}$$

Problem (III):

$$\left.\left.\begin{array}{r}\operatorname{div}_y\Bigg(\chi\Bigg(\mu_0\mathbb{D}\Big(y,\dfrac{\partial \boldsymbol{W}^\theta}{\partial t}\Big)+(1-\chi)(\lambda_0\mathbb{D}(y,\boldsymbol{W}^\theta)\\ -((1-\chi)\Pi^\theta+\chi P^\theta-\beta_{0f}\chi-\beta_{0s}(1-\chi))\mathbb{I})\Bigg)\Bigg)=0,\quad \chi\boldsymbol{W}^\theta(\boldsymbol{y},0)=0,\end{array}\right\}\begin{array}{c}\\ \\ \Big(\dfrac{1}{p_*}\Big)P^\theta+\chi\operatorname{div}_y\boldsymbol{W}^\theta=0,\quad \Big(\dfrac{1}{\eta_0}\Big)\Pi^\theta+(1-\chi)\operatorname{div}_y\boldsymbol{W}^\theta=0.\end{array}\right\}$$

Problem (IV):

$$\left.\begin{array}{r}\left.\begin{array}{r}\operatorname{div}_y\Bigg(\chi\Bigg(\mu_0\mathbb{D}\Big(y,\dfrac{\partial \boldsymbol{W}_1^\theta}{\partial t}\Big)+(1-\chi)(\lambda_0\mathbb{D}(y,\boldsymbol{W}_1^\theta)\\ -((1-\chi)\Pi_1^\theta+\chi P_1^\theta-\beta_{0f}\chi-\beta_{0s}(1-\chi))\mathbb{I})\Bigg)\Bigg)=0,\quad \boldsymbol{W}_1^\theta(\boldsymbol{y},0)=0,\end{array}\right\}\\ \Big(\dfrac{1}{p_*}\Big)P_1^\theta+\chi\operatorname{div}_y\boldsymbol{W}_1^\theta=-(\chi/m)\langle\operatorname{div}_y\boldsymbol{W}_1^\theta\rangle_{Y_s},\\ \Big(\dfrac{1}{\eta_0}\Big)\Pi_1^\theta+(1-\chi)\operatorname{div}_y\boldsymbol{W}_1^\theta=\langle\operatorname{div}_y\boldsymbol{W}_1^\theta\rangle_{Y_s}\dfrac{1-\chi}{1-m}.\end{array}\right\}$$

Problem (V):

$$\operatorname{div}_y\big((\chi\kappa_{0f}+(1-\chi)\kappa_{0s})\nabla_y\Theta^i+\chi\boldsymbol{e}_i\big)=0.$$

Furthermore, substituting expressions (4.16)–(4.19) in the macroscopic Eqs. (4.8)–(4.11), we find that

$$\left.\begin{array}{l}\mathbb{A}_1=\mu_0 m\displaystyle\sum_{i,j=1}^{3}J^{ij}\otimes J^{ij}+\mu_0\mathbb{A}_0^f,\\ \mathbb{A}_0^f=\mu_0\displaystyle\sum_{i,j=1}^{3}\langle\mathbb{D}(y,\boldsymbol{W}_0^{ij})\rangle_{Y_f}\otimes J^{ij},\end{array}\right\}\tag{4.20}$$

$$\left.\begin{array}{l}\mathbb{A}_2=\lambda_0(1-m)\displaystyle\sum_{i,j=1}^{3}J^{ij}\otimes J^{ij}-\lambda_0\mathbb{A}_0^f+\mu_0\mathbb{A}_1^f(0),\\ \mathbb{A}_3(t)=\mu_0\dfrac{d}{dt}\mathbb{A}_1^f(t)-\lambda_0\mathbb{A}_1^f(t),\end{array}\right\}\tag{4.21}$$

$$\mathbb{A}_1^f(t)=\sum_{i,j=1}^{3}\left\{\mu_0\left\langle\mathbb{D}\Big(y,\frac{\partial\boldsymbol{W}^{ij}}{\partial t}\Big)\right\rangle_{Y_f}+\lambda_0\langle\mathbb{D}(y,\boldsymbol{W}^{ij})\rangle_{Y_s}\right\}\otimes J^{ij},\tag{4.22}$$

$$B(t)=\mu_0\left\langle\mathbb{D}\Big(y,\frac{\partial\boldsymbol{W}^{0}}{\partial t}\Big)\right\rangle_{Y_f}+\lambda_0\langle\mathbb{D}(y,\boldsymbol{W}^{0})\rangle_{Y_s},\tag{4.23}$$

$$C_1(t) = -C_2(t) = \sum_{i,j=1}^{3} \langle \mathrm{div}_y\, \boldsymbol{W}^{ij} \rangle_{Y_f} J^{ij}, \qquad a(t) = \langle \mathrm{div}_y\, \boldsymbol{W}_1^{\theta} \rangle_{Y_s}, \tag{4.24}$$

$$a_1(t) = -a_2(t) = \langle \mathrm{div}_y\, \boldsymbol{W}^{0} \rangle_{Y_f}, \qquad a_1^{\theta}(t) = -a_2^{\theta}(t) = \langle \mathrm{div}_y\, \boldsymbol{W}^{\theta} \rangle_{Y_f}, \tag{4.25}$$

$$B^{\theta}(t) = \mu_0 \left\langle \mathbb{D}\left(y, \frac{\partial \boldsymbol{W}^{\theta}}{\partial t}\right) \right\rangle_{Y_f} + \lambda_0 \langle \mathbb{D}(y, \boldsymbol{W}^{\theta}) \rangle_{Y_s}, \tag{4.26}$$

$$B_0^{\theta} = \hat{\kappa}_0 \mathbb{I} + \sum_{i=1}^{3} \{ \kappa_{0f} \langle \nabla \Theta^i \rangle_{Y_f} + \kappa_{0s} \langle \nabla \Theta^i \rangle_{Y_s} \} \otimes \boldsymbol{e}_i, \tag{4.27}$$

where $\hat{\kappa}_0 = m\kappa_{0f} + (1-m)\kappa_{0s}$. □

Lemma 4.4. *The tensors* $\mathbb{A}_1$, $\mathbb{A}_2$, *and* $\mathbb{A}_3$, *the matrices* B, B^{θ}, B_0^{θ}, C_1, *and* C_2, *and the scalars* a_1, a_2, a_1^{θ}, a_2^{θ}, *and* a *are correctly defined and are infinitely many times differentiable functions of time.*

If the pore space is connected, then the symmetric tensor $\mathbb{A}_1$ *is strictly positive-definite. Otherwise* (*isolated pores*), $\mathbb{A}_1 = 0$, *and the symmetric tensor* $\mathbb{A}_2$ *is strictly positive-definite. The symmetric matrix* B_0^{θ} *is strictly positive-definite.*

Except for the assertion on the matrix, B_0^{θ}, the main steps of the proof of the lemma can be found in [7]. The properties of the matrix B_0^{θ} are well known (see [11, 12]).

REFERENCES

1. R. Burridge and J. B. Keller, "Poroelasticity equations derived from microstructure," *J. Acoust. Soc. Am.*, **70**, No. 4, 1140–1146 (1981).
2. Th. Clopeau, J. L. Ferrin, R. P. Gilbert, and A. Mikelić, "Homogenizing the acoustic properties of the seabed: Part II," *Math. Comput. Modelling*, **33**, 821–841 (2001).
3. J. L. Ferrin and A. Mikelić, "Homogenizing the acoustic properties of a porous matrix containing an incompressible inviscid fluids," *Math. Meth. Appl. Sci.*, **26**, 831–859 (2003).
4. R. P. Gilbert and A. Mikelić, "Homogenizing the acoustic properties of the seabed: Part I," *Nonlinear Anal.*, **40**, 85–212 (2000).
5. O. A. Ladyzhenskaya, *Mathematical Problems of the Viscous Incompressible Fluid Dynamics* [in Russian], Nauka, Moscow (1970).
6. D. Lukkassen, G. Nguetseng, and P. Wall, "Two-scale convergence," *Int. J. Pure Appl. Math.*, **2**, No. 1, 35–86 (2002).
7. A. M. Meirmanov, "Nguetseng's two-scaled convergence method for filtration and seismic acoustic problems in elastic porous media," www.arXiv.org/math.AP/0611330 v1 11 Nov (2006).
8. A. M. Meirmanov and S. A. Sazhenkov, "Generalized solutions to the linearized equations of thermoelastic solid and viscous thermofluid," www.arXiv.org/math.AP/0611350 v1 12 Nov (2006).
9. G. Nguetseng, "A general convergence result for a functional related to the theory of homogenization," *SIAM J. Math. Anal.*, **20**, 608–623 (1989).
10. G. Nguetseng, "Asymptotic analysis for a stiff variational problem arising in mechanics," *SIAM J. Math. Anal.*, **21**, 1394–1414 (1990).

11. E. Sanches-Palencia, *Inhomogeneous Media and Vibration Theory* [Russian translation], Mir, Moscow (1984).
12. V. V. Zikov, S. M. Kozlov, and O. A. Oleinik, *Averaging of Differential Operators* [in Russian], Nauka, Moscow (1993).

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