MOTION OF A HEATED SPHEROIDAL PARTICLE AT LOW REYNOLDS NUMBERS

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The problem of motion of a uniformly heated spheroidal hydrosol particle is solved. An expression which allows estimation of the resistance force of the spheroidally shaped solid hydrosol particle at arbitrary differences between the temperatures of the particle surface and the region at a distance from it is obtained with account for the dependence of the viscosity on the temperature.

We consider the motion of a uniformly heated spheroidal solid particle (an oblate spheroid) in a viscous fluid in the gravitational field. By a heated particle is meant a particle whose mean surface temperature greatly exceeds in magnitude the temperature of the surrounding medium. Heating of the particle surface can be caused, for example, by the occurrence of a volume chemical reaction, absorption of electromagnetic radiation, etc.

Of all the parameters of liquid transfer, it is only the coefficient of dynamic viscosity that is strongly related to the temperature [1]. To allow for the dependence of the viscosity on the temperature we use expression (1), which makes it possible to describe a viscosity change over a wide range of temperatures with any required accuracy (at $F_n = 0$ this formula can be reduced to the known Reynolds relation [1])

$$\mu_{\text{liq}} = \mu_\infty \left[1 + \sum_{n=1}^{\infty} F_n \left(\frac{T_{\text{liq}}}{T_\infty} - 1 \right)^n \right] \exp \left\{ -A \left(\frac{T_{\text{liq}}}{T_\infty} - 1 \right) \right\} . \tag{1}$$

Let us change to the reference system related to the particle. The problem in essence is reduced to analysis of the plane-parallel flow of a liquid (at a velocity $\mathbf{U}_\infty (\mathbf{U}_\infty \parallel Oz)$) past a motionless spheroid. The flow past the spheroid is described in the spheroidal coordinate system (ε, η, φ) with origin at the center of the hydrosol particle.

At low Reynolds numbers the distributions of the velocity $\mathbf{U}_{\text{liq}}$, the pressure P_{liq}, and the temperature T_{liq} are described by the following system of equations [2, 3]:

$$\nabla P_{\text{liq}} = \mu_{\text{liq}} \Delta \mathbf{U}_{\text{liq}} + 2 (\nabla \mu_{\text{liq}} \nabla) \mathbf{U}_{\text{liq}} + [\nabla \mu_{\text{liq}} \times \text{rot}\, \mathbf{U}_{\text{liq}}] , \quad \text{div}\, \mathbf{U}_{\text{liq}} = 0 , \tag{2}$$

$$\text{div} (\lambda_{\text{liq}} \nabla T_{\text{liq}}) = 0 . \tag{3}$$

In solving this system of equations, the boundary conditions

$$\mathbf{U}_{\text{liq}} = 0 , \quad T_{\text{liq}} = T_{\text{s}} \quad \text{for} \quad \varepsilon = \varepsilon_0 , \tag{4}$$

$$\mathbf{U}_{\text{liq}} \to U_\infty \cos \eta \, \mathbf{e}_\varepsilon - U_\infty \sin \eta \, \mathbf{e}_\eta , \quad T_{\text{liq}} \to T_\infty , \quad P_{\text{liq}} \to P_\infty \quad \text{for} \quad \varepsilon \to \infty . \tag{5}$$

are taken into account.

The force affecting the particle from the side of the flow is determined from the formula

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$$F_z = \int_S \left(-P_{\text{liq}} \cos\eta + \sigma_{\varepsilon\varepsilon} \cos\eta - \frac{\sinh\varepsilon}{\cosh\varepsilon} \sigma_{\varepsilon\eta} \sin\eta \right) dS, \tag{6}$$

where $dS = c^2 \cosh^2\varepsilon \sin\eta d\eta d\varphi$ is the differential surface element and $\sigma_{\varepsilon\varepsilon}$ and $\sigma_{\varepsilon\eta}$ are the components of the stress tensor in the spheroidal coordinate system [2].

In order to find the force affecting the heated spheroidally shaped solid particle we must know the temperature field in its vicinity. Integration of Eq. (3) with the corresponding boundary conditions yields

$$t_{\text{liq}} = 1 + \frac{\gamma a_0}{c} \operatorname{arcctan}\lambda, \tag{7}$$

where $t_{\text{liq}} = T_{\text{liq}}/T_\infty$, $\lambda = \sinh\varepsilon$, and $\gamma = \dfrac{1}{\sqrt{1+\lambda^2}} \dfrac{t_s - 1}{\operatorname{arcctan}\lambda_0}$ is the dimensionless parameter characterizing the heating of the particle surface, $t_s = T_s/T_\infty$ and $\lambda_0 = b_0/c$.

With account for (7) expression (1) takes the form

$$\mu_{\text{liq}} = \mu_\infty \left[1 + \sum_{n=1}^{\infty} F_n \gamma_0^n (\operatorname{arcctan}\lambda)^n \right] \exp\left\{ -\gamma_0 \operatorname{arcctan}\lambda \right\} \quad \left(\gamma_0 = \frac{A\gamma}{c} a_0 \right). \tag{8}$$

Since the viscosity depends only on the radial coordinate λ (formula (8)), we find the solution of the system of equations (2) by the method of separation of variables, expanding the fields of velocity and pressure in Legendre and Gegenbauer polynomials [2]. In particular, for the components of the mass velocity $\mathbf{U}_{\text{liq}}$ we obtained the following expressions, which satisfy the boundary conditions (5):

$$U_\varepsilon(\varepsilon, \eta) = \frac{U_\infty}{cH_\varepsilon} \cos\eta \, [c^2 + A_1 G_1 + A_2 G_2], \quad U_\eta(\varepsilon, \eta) = -\frac{U_\infty}{cH_\varepsilon} \sin\eta \, [c^2 + A_1 G_3 + A_2 G_4], \tag{9}$$

where

$$G_1 = -\frac{1}{\lambda^3} \sum_{n=0}^{\infty} \frac{\theta_n^{(1)}}{(n+3)\lambda^n}, \quad G_2 = -\frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{\theta_n^{(2)}}{(n+1)\lambda^n} - \frac{\beta}{\lambda^3} \sum_{n=0}^{\infty} \frac{\theta_n^{(1)}}{(n+3)\lambda^n} \left[(n+3) \ln\frac{\lambda_0}{\lambda} - 1 \right],$$

$$G_3 = G_1 + \frac{1+\lambda^2}{2\lambda} G_1^I, \quad G_4 = G_2 + \frac{1+\lambda^2}{2\lambda} G_2^I, \quad \lambda_0 = \sinh\varepsilon_0, \quad H_\varepsilon = c\sqrt{\cosh^2\varepsilon - \sin^2\eta}, \quad \theta_2^{(2)} = 1, \quad \Delta_0 = 1,$$

$$\theta_n^{(1)} = -\frac{1}{n(n+5)} \sum_{k=1}^{n} \left[(n+4-k) \left\{ (n+1-k)\,\alpha_k^{(1)} + \alpha_k^{(2)} \right\} + \alpha_k^{(3)} \right] \theta_{n-k}^{(1)} \quad (n \ge 1), \quad \theta_0^{(1)} = -1, \quad \theta_0^{(2)} = -1,$$

$$\theta_n^{(2)} = -\frac{1}{(n-2)(n+3)} \left[\sum_{k=1}^{n} \left\{ (n+2-k) \left[(n+1-k)\,\alpha_k^{(1)} + \alpha_k^{(2)} \right] + \alpha_k^{(3)} \right\} \theta_{n-k}^2 + \right.$$

$$\left. + \beta \sum_{k=0}^{n} \left[(2n-2k+3)\,\alpha_k^{(1)} + \alpha_k^{(1)} \right] \theta_{n-k-2}^{(1)} - 6\alpha_n^{(4)} \right] \quad (n \ge 3), \quad \theta_1^{(2)} = -\frac{1}{4} \left[2\,(\alpha_1^{(1)} + \alpha_1^{(2)}) + \alpha_1^{(3)} + 6\alpha_1^{(4)} \right],$$

$$\beta = -\frac{1}{5} \left[\left\{ 3\,(2\alpha_1^{(1)} + \alpha_1^{(2)}) + \alpha_1^{(3)} \right\} \theta_1^{(2)} - 2\,(\alpha_2^{(1)} + \alpha_2^{(2)}) - \alpha_2^{(3)} - 6\alpha_2^{(4)} \right],$$

Table 1. Dependence of the Coefficient K on the Mean Temperature of the Surface T_s and the Ratio of the Semiaxes a_0/b_0

a_0/b_0	T_s, K						
	273	283	303	333	343	353	363
0.73	0.947	0.705	0.393	0.163	0.121	0.089	0.065
0.9	0.980	0.727	0.397	0.158	0.116	0.086	0.062

$$\alpha_n^{(1)} = C_n + 12 \sum_{k=0}^{[(n-2)/2]} (-1)^k \frac{C_{n-2k-2}}{(2k+1)(2k+3)(2k+5)}, \quad \alpha_n^{(2)} = (n-2)\, C_n - \gamma_0 C_{n-1} +$$

$$+ 12 \sum_{k=0}^{[(n-2)/2]} (-1)^k \frac{(4k+5)\, C_{n-2k-2}}{(2k+1)(2k+3)(2k+5)} - 3 \sum_{k=0}^{[(n-3)/2]} (-1)^k \frac{1}{(2k+3)(2k+5)} \Big[(n-2k-2)\, C_{n-2k-2} -$$

$$- \gamma_0 C_{n-2k-3} + (n-2k-4)\, C_{n-2k-4}\Big] \;\; (n \ge 1), \quad \alpha_n^{(3)} = -2\,(n+2)\, C_n + 2\gamma_0 C_{n-1} - 2\,(n-2)\, C_{n-2} +$$

$$+ 12 \sum_{k=0}^{[(n-2)/2]} (-1)^k \frac{C_{n-2k-2}}{(2k+5)} + 6 \sum_{k=0}^{[(n-3)/2]} (-1)^k \frac{(k+2)(4k+5)}{(2k+3)(2k+5)} \Big[(n-2k-2)\, C_{n-2k-2} - \gamma_0 C_{n-2k-3} +$$

$$+ (n-2k-4)\, C_{n-2k-4}\Big] \;\; (n \ge 1), \quad \alpha_n^{(4)} = \frac{1}{n!}\Big[\gamma_0 \Delta_{n-1} - (n-1)(n-2)\, \Delta_{n-2}\Big] \;\; (n \ge 1),$$

$$C_k = \sum_{l_1+3l_3+5l_5+\ldots+sl_s=k} \frac{l!}{l_1!\, l_3!\, l_5! \ldots l_s!} F_l f_1^{l_1} \cdot f_3^{l_3} \cdot f_5^{l_5} \cdot \ldots f_s^{l_s}, \quad s = k - \frac{1+(-1)^k}{2},$$

$$l = l_1 + l_3 + l_5 + \ldots + l_s, \quad f_{2k-1} = (-1)^{k-1} \frac{\gamma a_0}{c\,(2k-1)} \quad (k \ge 1).$$

The integral part $k/2$ is denoted by $[k/2]$.

The force affecting the spheroid due to viscous stresses is determined by integration of expression (6) over the surface of the spheroid, and with account for (9) it is

$$\mathbf{F}_z = 6\pi\alpha_0\, \mu_\infty K U_\infty\, \mathbf{n}_z, \tag{10}$$

where $K = \dfrac{2}{3} \dfrac{G_1^I}{\sqrt{1+\lambda_0^2}\,[G_2 G_1^I - G_1 G_2^I]} \exp\left\{-\dfrac{A\gamma}{c} a_0 \operatorname{arccot} \lambda_0\right\}$ and G_1^I and G_2^I are the first derivatives of the corresponding functions with respect to λ.

The effect of the temperature of the particle surface on the resistance force is determined by the coefficient K. As an example, Table 1 presents the results of numerical calculations of the dependence of the coefficient K on the mean temperature of the spheroid surface and the ratio of the semiaxes for solid particles suspended in water at $T_\infty = 273$ K, $A = 5.779$, and $F_n = 0$ for $n \ge 1$. The numerical analysis showed that heating of the spheroid surface substantially affects the resistance force.

In the limit when $\gamma \to 0$ (small temperature differences in the vicinity of the spheroid), $G_1 = 1/(3\lambda^3)$, $G_1^I = -1/\lambda^4$, $G_2 = 1/\lambda$, $G_2^I = -1/\lambda^2$, and $a_0 = b_0 = R$; the coefficient is $K = 1$ and formula (10) becomes the Stokes formula for a spherical solid particle with radius R [2].

In order to obtain the expression for the resistance coefficient of an extended spheroid, in (10) we must replace λ by $i\lambda$ and c by $-ic$ (i is the imaginary unit).

Let us consider the motion of a spheroidal particle in the gravitational field. Equating the gravity force, with account for the buoyancy force, to the resistance force (10), we obtain the velocity of its gravitational fall:

$$U_\infty = (\rho_p - \rho_{liq})\, g \sqrt{1+\lambda_0^2}\, \frac{a_0 b_0}{3\mu_\infty} \frac{G_1 G_2^I - G_2 G_1^I}{G_1^I} \exp\left\{\frac{A\gamma}{c} a_0 \arctan \lambda_0\right\}. \tag{11}$$

Thus, formulas (10) and (11) allow estimation of the hydrodynamic force affecting a uniformly heated spheroidal particle and the velocity of its gravitational force with account for the dependence of the viscosity on the temperature, which is represented in the form of an exponential-power series at arbitrary temperature differences in its vicinity.

NOTATION

$\mu_\infty = \mu_{liq}(T_\infty)$; A, F_n – const; $U_\infty = |\mathbf{U}_\infty|$; T_s, mean temperature of the spheroid surface; $\mathbf{e}_\varepsilon$ and $\mathbf{e}_\eta$, unit vectors of the spheroidal coordinate system; a_0 and b_0, semiaxes of the spheroid; λ_{liq}, thermal conductivity of the liquid; P_∞ and T_∞, undisturbed pressure and temperature in the liquid; $\mathbf{n}_z$, unit vector in the direction of the z axis; ρ, density. Subscripts: p, particle; ∞, values of physical quantities taken at a distance from the particle (at infinity); s, spheroid.

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