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MARGINAL PROBABILITY DISTRIBUTIONS OF RANDOM SETS IN $\mathbb{R}$ WITH MARKOVIAN REFINEMENTS

Marginal probability distributions describing statistically random sets in $\mathbb{R}$ closed with probability one are introduced. These distributions are calculated in the case of random sets with Markovian refinements.

1. Introduction

The construction of probability distributions in sample spaces $\mathfrak{A} \subset \Omega = 2^{\mathbb{R}^d}$ with elements which are some sets in $\mathbb{R}^d$, $d \in \mathbb{N}$, due to the difficulty of the description of each separate random element in Ω, is usually realized in applications by the following way. It is implied that there exists the natural coordinatization of elements in $\mathfrak{A}$ by means of elements of the other space $\mathfrak{B}$, so that the latter is described using analytical tools by an essentially simpler way, i.e. there is the definite image $\mathfrak{B} \mapsto \mathfrak{A}$. At the same time, it is natural that the space $\mathfrak{A}$ is found essentially poorer in comparison with the space Ω. Further, the probability distribution is defined on $\mathfrak{B}$, i.e. the probability space $\langle\mathfrak{B}$, T, $\mathbf{Q}\rangle$ is constructed. Then the mapping pointed out generates the measurability structure on $\mathfrak{A}$ by the natural way, i.e. $\mathrm{T} \to \Sigma = \{A : B \mapsto A, B \in \mathrm{T}\ \}$, where the mapping of random events $B \mapsto A$ is constructed by the rule $A = \{\tilde{X} : \tilde{Y} \mapsto \tilde{X}, \tilde{Y} \in B\}$. Under such a definition of random events in $\mathfrak{A}$, the probability distribution $\mathbf{P}$ on $\mathfrak{A}$ is defined by means of the equality $\mathbf{P}(A) = \mathbf{Q}(B)$.

In statistical physics [1], [2], the construction of probability distributions is based namely on the described scheme, and such an approach is found sufficient in that case where the description of each typical random element is realized by a rather simple way. At the same time, the physicists are interesting in the constructions of random sets when the typical realizations have no any simple coordinatization. It is connected with the creation of mathematical models of the so-called fractal structures, in particular, of fractally disordered media. The above-described scheme of the construction of such random sets does not longer act. In this case, it should be natural to define the distribution by the "statistical description" of each realization and to introduce the probability distribution of an infinite set of marginal distributions similarly to the case of separable random processes [3]. But, the natural approach [4] to the construction of marginal probability distributions in Ω based on some cylindrical events $\{\tilde{X} \subset \mathbb{R}^d : \chi(x_i|\tilde{X}) = \alpha_i, i = 1, ..., n\}$, $\alpha_i \in \{0, 1\}$, $n \in \mathbb{N}$, where $\chi(x|X) = 1$ for $x \in X$ and $\chi(x|X) = 0$ for $x \notin X$, which is equivalent to the introducing of the random field $\{\tilde{\theta}(x) = \chi(x|\tilde{X}); x \in \mathbb{R}^d\}$, leads only to the so-called separable random sets which have the topological dimension equal with probability one to the dimension d of *the submersion space*. At the same time, random geometric structures used in physics for the description of the fractal property should be ones possessing arbitrary dimensions, in particular, they may have any value of the

Hausdorff dimension. Therefore, the above way of the introducing of the measurability structure is inadequate. Apparently, to embrace some possible cases which may appear in applications, it is necessary to study the construction of probability distributions of arbitrary random sets closed with probability one [4].

It is necessary to note that some specific constructions of "fractal" random sets were proposed in different works [5—8]. They are based on the above-described idea of constructing the probabilistic space by means of some special maps. In these cases, as is mentioned above, each proposed construction of random sets has inevitably a rather particular character.

In connection with the described situation, it is natural to pose the problem about the method of the probabilistic description of random sets by means of some marginal distributions such that it is useful for arbitrary random sets closed with probability one. Such a method has been proposed in the work [9] for the case $d = 1$. In addition, we have introduced the class of random sets which have been named *random sets with Markovian refinements* [10]. They possess the fractal property, i.e., with probability one, each their realization has the continuum cardinality together with zero Lebesgue's measure defined on the submersion space $\mathbb{R}^d$. It has succeeded to prove that the Hausdorff dimension is not random for all typical realizations of such random sets [11]. In this connection, there is the problem concerning the possibility of the probabilistic description of these random sets on the basis of some marginal distributions introduced in [9]. This work is devoted to the solving of this problem for sets in $\mathbb{R}$.

In Sections 2 and 3, we describe briefly the approach to the construction of probability distributions on the basis of marginal distributions. In Section 4, the random sets with Markovian refinements [10] are introduced, and *the reduction formula* (7) for them is proved in Section 5. This formula expresses multi-interval marginal distributions using the one-interval one. In Section 6, the one-interval probability distribution of random sets with Markovian refinements in $[0, 1] \subset \mathbb{R}$ with the parameter $N = 2$ is calculated.

2. Marginal distributions of random sets in $\mathbb{R}$

In this section, we describe briefly the statistical construction of the probability distribution of random sets in $\mathbb{R}$.

Definition 1. The class of sets $\mathbb{F} \subset 2^{\mathbb{R}^d}$ is called a c-system if it contains the empty set and satisfies the following conditions:

(a) if A and B belong to the system $\mathbb{F}$, there exists the finite disjoint collection $\{C_i \in \mathbb{F} \setminus \{\emptyset\}; i = 1, ..., m\}$, $C_i \neq \emptyset$, $C_i \cap C_j = \emptyset$ as $i \neq j$, $i, j = 1, ..., m$, such that the set $A \cap B$ may be represented in the form of the union

$$A \cap B = \bigcup_{i=1}^{m} C_i \, ;$$

(b) for any $A \in \mathbb{F}$, there exists the finite disjoint collection $\{D_j \in \mathbb{F}; j = 1, ..., n\}$, $D_j \neq \emptyset$, $D_i \cap D_j = \emptyset$ *at* $i \neq j$, $i, j = 1, ..., n$, such that the complement $\overline{A}$ may be represented in the form of the union

$$\overline{A} = \bigcup_{j=1}^{n} D_j \, .$$

It was proved in [9] that each finite measure defined on an arbitrary c-system is uniquely extendable to the minimal σ-algebra containing this c-system.

Let $\Omega = \{\tilde{X}\}$ be the sample space of random sets in $[0, 1]$. We define the bundle $\mathbf{C}(\mathbb{S})$ on the space $\mathbb{S}$ of all possible finite tuples $\Delta = \langle \delta_1, \delta_2, ..., \delta_n \rangle$, $\delta_i = [a_i, b_i) \subset [0, 1)$;

$b_i \leq a_{i+1}$, $i = 1, ..., n-1$, $n \in \mathbb{N}$. Each fiber $\mathbf{C}(\Delta)$ at fixed Δ represents the set of tuples $\Theta = \langle \theta_1, \theta_2, ..., \theta_n \rangle$, $\theta_i \in \{0, 1\}$, $i = 1, ..., n$. We regard two pairs $\langle \Delta, \Theta \rangle$ and $\langle \Delta', \Theta' \rangle$ in the fiber space as equivalent ones if

$$\bigcup_{i:\theta_i=0} \delta_i = \bigcup_{i:\theta_i'=0} \delta_i'$$

and the subtuples $\langle \delta_i; \theta_i = 1 \rangle$, $\langle \delta_i'; \theta_i' = 1 \rangle$ coincide with each other.

It is proved in [9] that the class of all possible events

$$\{X : \chi(X|\delta_i) = \theta_i; i = 1, ..., n\} \; where \; \chi(X|\delta) = \{1, \; X \cap \delta \neq \emptyset \, ; \; 0, \; X \cap \delta = \emptyset\}$$

, constructed on the basis of the fiber space $\langle \mathbb{S}, \mathbf{C}(\mathbb{S}) \rangle$ of tuples $\langle \Delta, \Theta \rangle$ is a c-system. We associate the probability

$$P(\Delta, \Theta) = \Pr\{X : \chi(X|\delta_i) = \theta_i; i = 1, ..., n\}$$

to each pair $\langle \Delta, \Theta \rangle$ in the fiber space $\langle \mathbb{S}, \mathbf{C}(\mathbb{S}) \rangle$. It is done by such a way that all equivalent pairs have equal probabilities. We assign the term of *the marginal n-interval* probability distribution to the collection of all mentioned probabilities at a fixed n which is the length of Δ and Θ. All marginal distributions satisfy, firstly, the normalization condition and, secondly, the consistency condition

$$\sum_{\Theta \in \{0,1\}^n} P(\Delta, \Theta) = 1 \,, \quad P(\Delta, \Theta) = \sum_{\Theta' : \pi(\Theta') = \Theta} P(\Delta', \Theta')$$

, where $\Delta = \pi(\Delta')$ in the last equality and the symbol π denotes the projection operation. It is applied to the tuples Δ', Θ' and consists in the deleting of all those δ' from Δ' which are absent in the tuple Δ.

It has been proved in [9] that the collection of all probabilities $P(\Delta, \Theta)$ defines the probabilistic measure on Ω. This probabilistic measure is concentrated on the class of left-closed random sets. Each left-closed realization (and, in particular, the closed one) is uniquely restored by this measure with probability one.

3. Marginal distributions

We introduce a functional notation for the above-introduced marginal distributions. Let Δ be an arbitrary disjoint tuple of semi-intervals $\langle \delta_i = [a_i, b_i) \,, i = 1, ..., n \rangle$, and let Θ be the corresponding tuple of "filling numbers" $\langle \theta_1, \theta_2, ..., \theta_n \rangle$, where $a_i < b_i \leq a_{i+1}$, $\theta_i \in \{0, 1\}$, $i = 1, 2, ..., n$, and $a_{n+1} \equiv 1$, $[a_1, b_n) \subset [0, 1)$. At the fixed number of intervals, the collection of probabilities $P(\Delta, \Theta)$ with the length of Δ (and, respectively, of Θ) equal to n is obviously the vector function of the variables a_i, b_i $(i = 1, ..., n)$ for each Θ. It has the definite domain which is cut out by the above inequalities, and it has 2^n components enumerated by the values of the parameters $\theta_i \in \{0, 1\}$, $i = 1, 2, ..., n$. We shall call this function the *marginal n-interval probability distribution*. The number n is called the order of this distribution. We denote this vector function as

$$F_{\theta_1, \theta_2, \ldots, \theta_n}(a_1, b_1; a_2, b_2; ...; a_n, b_n) = P(\Delta, \Theta) \,.$$

The consistency conditions show that some components of this vector function are expressed by other components. We introduce the special notation for components with $\Theta = 0$,

$$F(a_1, b_1; a_2, b_2; ...; a_n, b_n) = F_{0,0,\ldots,0}(a_1, b_1; a_2, b_2; ...; a_n, b_n) \,.$$

Let us prove now that all functions $F_{\theta_1, \theta_2, \ldots, \theta_n}(a_1, b_1; a_2, b_2; ...; a_n, b_n)$ at any $n \in \mathbb{N}$ and at arbitrary n-component collection $\Theta \in \{0, 1\}^n$ may be expressed using only the collection of functions $F(a_1, b_1; a_2, b_2; ...; a_n, b_n)$, $n \in \mathbb{N}$.

Theorem 1. *For each $n \in \mathbb{N}$ and for each tuple $\langle \theta_1, ..., \theta_n \rangle \in \{0,1\}^n$, the function $F_{\theta_1,\theta_2,...,\theta_n}(a_1, b_1; a_2, b_2; ...; a_n, b_n)$ belongs to the linear manifold constructed on the basis of the function collection $F(a_{i_1}, b_{i_1}; a_{i_2}, b_{i_2}; ...; a_{i_k}, b_{i_k})$ where $\langle i_1, i_2, ..., i_k \rangle$ are all projections of the tuple $\langle 1, 2, ..., n \rangle$ having the length $k = 0, 1, ..., n$ (it is supposed that $F(\emptyset) = 1$).*

Proof. We prove the theorem by induction on n and on $l = 0, 1, ..., n$ which is the number of nonzero components in the tuple $\langle \theta_1, ..., \theta_n \rangle$. The number l named *the filling index* characterizes the property of being filled up. If $l = 0$, then, for any $n \in \mathbb{N}$, each function $F_{\theta_1,\theta_2,...,\theta_n}(\cdot)$ is equal to the corresponding function $F(\cdot)$. If $n = 1$, then we have, at $l = 1$, $F_1(a_1, b_1) = 1 - F(a_1, b_1)$ from the normalization condition. Let the statement be correct for all filling indices and for all functions with orders up to n inclusively. Further, let the assertion be correct for the order $n+1$ at filling index values up to l inclusively, $0 \leq l < n+1$. Consider the function $F_{\theta_1,\theta_2,...,\theta_{n+1}}(a_1, b_1; a_2, b_2; ...; a_{n+1}, b_{n+1})$ with the filling index $l+1$. Then, without loss of generality, we may suppose that $\theta_{n+1} = 1$. The consistency conditions yield

$$F_{\theta_1,\theta_2,...,\theta_n,1}(a_1, b_1; a_2, b_2; ...; a_{n+1}, b_{n+1}) =$$

$$= F_{\theta_1,\theta_2,...,\theta_n}(a_1, b_1; a_2, b_2; ...; a_n, b_n) - F_{\theta_1,\theta_2,...,\theta_n,0}(a_1, b_1; a_2, b_2; ...; a_{n+1}, b_{n+1}) .$$

The first term has order n and the second one has $n+1$, but its filling index is l. Hence, the induction step has been performed.

4. Random sets with Markovian refinements

We consider now the random sets with Markovian refinements in $[0,1]$ introduced in [10]. Each such set is completely defined by the value of the natural subdivision parameter $N \geq 2$ and by the probability distribution $q[\cdot]$ on the set $2^{\mathfrak{K}_1}$, $\mathfrak{K}_1 = \{[i/N, (i+1)/N); i = 0, 1, ..., N-1\}$. Since the empty realization is not under consideration, we must exclude it. For this, the distribution should satisfy the property $q[\emptyset] = 0$.

We introduce the following notation. Let $\mathfrak{K}_m$ be the collection of all semi-intervals being elements of a refinement of order m with the subdivision parameter N, $\mathfrak{K}_m = \{[i/N^m, (i+1)/N^m); i = 0, 1, ..., N^m - 1\}$. We associate the subset

$$\mathbf{D}_m(\Gamma) = \bigcup_{\delta \in \Gamma} \delta \subset [0, 1)$$

to each set $\Gamma \subset \mathfrak{K}_m$. Denote the class of all such subsets by $\mathfrak{K}_m$, i.e.

$$\mathfrak{K}_m = \{\mathbf{D}_m(\Gamma); \emptyset \neq \Gamma \subset \mathfrak{K}_m\} .$$

Finally, we introduce the projection operation $\mathbf{K}_m : 2^{[0,1)} \to \mathfrak{K}_m$. For any subset X from $[0,1)$, we denote

$$\mathbf{K}_m(X) = \bigcup_{\delta \in \mathfrak{K}_m : \delta \cap X \neq \emptyset} \delta \in \mathfrak{K}_m .$$

Under the fixed value of the subdivision parameter N, each random realization $\tilde{X}$ of the random set with Markovian refinements is defined as the limit in the set theory sense, $\tilde{X} = \lim_{m\to\infty} \tilde{X}_m$, where $\{\tilde{X}_m; m = 1, 2, ...\}$, $\tilde{X}_1 \supset \tilde{X}_2 \supset ...$ is the decreasing sequence of random sets, $\tilde{X}_m \in \mathfrak{K}_m$, $m \in \mathbb{N}$. Components $\tilde{X}_m$, $m \in \mathbb{N}$ are constructed in the form of the union

$$\tilde{X}_m = \bigcup_{\delta \in \tilde{\Delta}_m} \delta ,$$

using some random collections $\tilde{\Delta}_m \subset \mathfrak{K}_m$, $m = 1, 2, ...$ such that the embedding condition $\tilde{X}_{m+1} \supset \tilde{X}_m$ is satisfied. At the same time, $\tilde{X}_m = \mathbf{K}_m(\tilde{X})$, $m = 1, 2, ...$, is fulfilled.

The probability distribution of a random set with Markovian refinements is defined by means of the introduction of probability distributions of each sequence component $\tilde{X}_m$, $m \in \mathbb{N}$. These distributions $P_m(Z) = \mathsf{P}r\{\tilde{X}_m = Z\}$, $Z \in \mathfrak{K}_m$, are united in the uniform Markovian chain. Namely, for any pair $X \in \mathfrak{K}_{m+1}$, $Y \in \mathfrak{K}_m$ such that $X \subset Y$, the equality

$$P_{m+1}(X) = \mathsf{P}r\{\tilde{X}_{m+1} = X|\tilde{X}_m = Y\}P_m(Y) \tag{1}$$

takes place where the conditional probability is defined by the Markovian "branching condition"

$$\mathsf{P}r\{\tilde{X}_{m+1} = X|\tilde{X}_m = Y\} = \prod_{\delta:\delta\in\mathbf{S}_m(Y)} q\left(\mathsf{T}_m(X\cap\delta)\right) . \tag{2}$$

The operation T_m shifts the set so that the initial point of the semi-interval $\delta \in \mathcal{K}_m$ becomes zero and extends the shifted semi-interval by N^m times so that it becomes $[0, 1)$. In addition, in (2), the semi-interval tuple belonging to $\mathcal{K}_m$ and forming the set Y is denoted by $\mathbf{S}_m(Y)$, $\mathbf{S}_m(Y) = \{\delta \in \mathcal{K}_m : \delta\cap Y \neq \emptyset\}$, and $q(\cdot)$ is the probability distribution on $\mathfrak{K}_1$ defined by the formula $q(Z) = q[\mathbf{S}_1(Z)]$, $Z \in \mathfrak{K}_1$, where $q(\emptyset) = 0$.

On the basis of probabilities $P_m(Z)$, $Z \in \mathfrak{K}_m$, $m \in \mathbb{N}$, the probability distribution $P(\Delta,\Theta)$ of random sets with Markovian refinements for arbitrary pairs $\langle\Delta,\Theta\rangle$, $\Theta \equiv \langle\theta(\delta); \delta \in \Delta\rangle$ is calculated as follows:

$$P(\Delta,\Theta) = \lim_{m\to\infty} \sum_{Z\in\mathfrak{K}_m:\,\chi(Z|\delta)=\theta(\delta),\,\delta\in\Delta} P_m(Z) . \tag{3}$$

To study probabilities $P(\Delta,\Theta)$, we prove a remarkable formula.

Theorem 2. *For any pair $X \in \mathfrak{K}_{m+l}$, $Y \in \mathfrak{K}_m$ with $m, l \in \mathbb{N}$ such that $\mathbf{K}_m(X) = Y$, the probabilities $P_m(Y)$, $P_{m+l}(X)$ corresponding to random sets with Markovian refinements satisfy the identity*

$$P_{m+l}(X) = \left[\prod_{\delta\in\mathbf{S}_m(Y)} P_l\left(\mathsf{T}_m(X\cap\delta)\right)\right] P_m(Y) . \tag{4}$$

Proof. We use the induction on $l = 1, 2, \ldots$. For $l = 1$ and $m \in \mathbb{N}$, formula (4) coincides with (1) since, by definition, $P_1(Z) = q(Z)$ for any $Z \in \mathfrak{K}_1$. Supposing that formula (4) is valid for any m and for the given l, we perform the induction step to the value $l + 1$. For this, using the induction assumption, we represent the function value $P_{m+l+1}(X)$ at any $X \in \mathfrak{K}_{m+l+1}$ in the form

$$P_{m+l+1}(X) = \left[\prod_{\delta\in\mathbf{S}_{m+1}(Z)} P_l\left(\mathsf{T}_{m+1}(X\cap\delta)\right)\right] P_{m+1}(Z) \tag{5}$$

where $\mathbf{K}_{m+1}(X) = Z$. Further, using formulas (1), (2), we represent each probability $P_{m+1}(Z)$ in the form

$$P_{m+1}(Z) = \left[\prod_{\sigma\in\mathbf{S}_m(Y)} q\left(\mathsf{T}_m(Z\cap\sigma)\right)\right] P_m(Y)$$

, where $\mathbf{K}_m(Z) = Y$. Substituting this representation to (5), we transform the product of two expressions $\left[\prod_{\delta\in\mathbf{S}_{m+1}(Z)} (\cdot)\right] \left[\prod_{\sigma\in\mathbf{S}_m(Y)} (\cdot)\right]$ to the iterated product

$$P_{m+l+1}(X) = \left[\prod_{\sigma\in\mathbf{S}_m(Y)} R(\sigma)\right] P_m(Y) , \tag{6}$$

$$R(\sigma) \equiv q\left(\mathsf{T}_m(Z \cap \sigma)\right) \prod_{\delta \in \mathbf{S}_{m+1}(Z):\, \delta \subset \sigma} P_l\left(\mathsf{T}_{m+1}(X \cap \delta)\right)$$

, where we used the fact that $\mathbf{K}_m(Z) = Y$ when each semi-interval $\delta \in \mathbf{S}_{m+1}(Z)$ is contained in some semi-interval $\sigma \in \mathbf{S}_m(Y)$. We associate the set $\gamma = \mathsf{T}_m(\delta)$ to each $\delta \in \mathbf{S}_{m+1}(Z)$, $\delta \subset \sigma$ at fixed $\sigma \in \mathbf{S}_m(Y)$. In this case, $\gamma \in \mathcal{K}_1$ and, in addition, $\gamma \in \mathbf{S}_1(\mathsf{T}_m(Z \cap \sigma))$ where $\mathsf{T}_m(Z \cap \sigma) \in \mathfrak{K}_1$ since $Z \in \mathfrak{K}_{m+1}$ and $\sigma \in \mathfrak{K}_m$. Conversely, each definite $\delta \in \mathbf{S}_{m+1}(Z)$, $\delta \subset \sigma$, corresponds uniquely to each $\gamma \in \mathbf{S}_1(\mathsf{T}_m(Z \cap \sigma))$. Therefore, it is possible to enumerate the multipliers in the product with all possible γ:

$$R(\sigma) \equiv q\left(\mathsf{T}_m(Z \cap \sigma)\right) \prod_{\gamma \in \mathbf{S}_1(\mathsf{T}_m(Z \cap \sigma))} P_l\left(\mathsf{T}_{m+1}(X \cap \delta)\right) .$$

We denote $\mathsf{T}_m(Z \cap \sigma) = V$. Since $\sigma \in \mathfrak{K}_m$ and $Z \in \mathfrak{K}_{m+1}$, $Z \cap \sigma \in \mathfrak{K}_{m+1}$ and, hence, $V \in \mathfrak{K}_1$. Herein, $q(V) = P_1(V)$ is fulfilled. So

$$R(\sigma) \equiv P_1\left(V\right) \prod_{\gamma \in \mathbf{S}_1(V)} P_l\left(\mathsf{T}_{m+1}(X \cap \delta)\right) .$$

We notice that it is possible to represent the extension of a set by N^{m+1} times in any interval $\delta \in \mathcal{K}_{m+1}$ as two extensions, firstly, by N^m times and, secondly, by N times. That is, for any δ having a nonempty intersection with $\sigma \in \mathcal{K}_m$, it may be possible to write the variable in the product in the form

$$\mathsf{T}_1\left(\mathsf{T}_m(X \cap \delta)\right) = \mathsf{T}_1\left([\mathsf{T}_m(X \cap \sigma)] \cap \delta\right) =$$
$$= \mathsf{T}_1\left([\mathsf{T}_m(X \cap \sigma)] \cap [\mathsf{T}_m(\delta)]\right) = \mathsf{T}_1\left([\mathsf{T}_m(X \cap \sigma)] \cap \gamma\right)$$

, where $\mathsf{T}_m(X \cap \sigma) \in \mathfrak{K}_{l+1}$. Then $R(\sigma)$ is represented in the form

$$R(\sigma) \equiv P_1\left(V\right) \prod_{\gamma \in \mathbf{S}_1(V)} P_l\left(\mathsf{T}_1\left([\mathsf{T}_m(X \cap \sigma)] \cap \gamma\right)\right) = P_{l+1}(\mathsf{T}_m(X \cap \sigma))$$

on the basis of the induction assumption. The substitution of this expression in (6) completes the induction step.

5. Reduction of multi-interval distributions

We now consider the problem of calculating the probabilities $P(\Delta, \Theta)$ for random sets with Markovian refinements. On the basis of Theorem 2, we shall obtain the general formula which expresses all multi-interval marginal distributions via the one-interval distribution.

A tuple Δ of semi-intervals is called *the subordinated tuple* to the subdivision of order m (subordinated to $\mathcal{K}_m$) if, for each $\delta \in \Delta$, there exists a semi-interval $\sigma \in \mathcal{K}_m$ such that $\delta \subset \sigma$. If a tuple Δ is subordinated to $\mathcal{K}_m$ in a pair $\langle \Delta, \Theta \rangle$ where $\Theta = \langle \theta(\delta); \delta \in \Delta \rangle$, then, for each $\sigma \in \mathcal{K}_m$ containing, at least, one component of Δ, it is possible to define the pair $\langle \Delta_\sigma, \Theta_\sigma \rangle$ by the rule $\Delta_\sigma = \langle \delta \in \Delta; \delta \subset \sigma \rangle$, $\Theta_\sigma = \langle \theta(\delta); \delta \in \Delta_\sigma \rangle$.

For each fixed $m \in \mathbb{N}$ and for each pair $\langle \Delta, \Theta \rangle$ with Δ subordinated to $\mathcal{K}_m$, we define two sets to formulate below the reduction theorem. At first, we put

$$\mathfrak{J}_{\Delta,m} = \mathbf{K}_m \left(\bigcup_{\delta \in \Delta; \theta(\delta)=1} \delta \right) .$$

The second set $\mathcal{N}_{\Delta,m}$ plays an important role in the degenerate case where semi-intervals $\delta \in \Delta$ are some elements of the subdivision with parameter N. Consider the class $\mathcal{N}_\Delta = \{\delta \in \Delta : \theta(\delta) = 0\}$. Define $\mathcal{N}_\Delta(\Gamma) = \bigcup_{\delta \in \Gamma} \delta$ for each $\Gamma \subset \mathcal{N}_\Delta$. For fixed m and each semi-interval $\sigma \in \mathcal{K}_m$, we denote, by Γ_σ, the subset of $\mathcal{N}_\Delta$ such that $\mathcal{N}_\Delta(\Gamma_\sigma) = \sigma$.

It is clear that the set Γ_σ is unique. But if such a set is absent for a given σ, then we put $\Gamma_\sigma = \emptyset$. Further, we define

$$\mathfrak{N}_{\Delta,m} = \bigcup_{\sigma \in \mathfrak{K}_m} \Gamma_\sigma \,.$$

The following assertion gives the main analytic tool, i.e. *the reduction formula* (7).

Theorem 3. *For each pair $\langle \Delta, \Theta \rangle$ and for any $m \in \mathbb{N}$ such that the tuple Δ is subordinated to $\mathfrak{K}_m$, the expansion*

$$P(\Delta, \Theta) = \sum_{Y \in \mathfrak{K}_m :\, [0,1) \setminus \mathfrak{N}_{\Delta,m} \supset Y \supset \mathfrak{J}_{\Delta,m}} P_m(Y) \left[\prod_{\sigma \in \mathbf{S}_m(Y)} P(\mathsf{T}_m(\Delta_\sigma), \Theta_\sigma) \right] \tag{7}$$

takes place with semi-interval tuples $\mathsf{T}_m(\Delta_\sigma)$ consisting of semi-intervals $\mathsf{T}_m(\delta)$, $\bigcup_{\delta \in \Delta_\sigma} \delta \subset \sigma$.

Proof. Since the semi-interval tuple Δ is subordinated to $\mathfrak{K}_m$, the pair $\langle \Delta_\sigma, \Theta_\sigma \rangle$ is defined, and two sets $\mathfrak{J}_{\Delta,m}$, $\mathfrak{N}_{\Delta,m}$ are determined at each $\sigma \in \mathfrak{K}_m$ and for each pair $\langle \Delta, \Theta \rangle$.

According to (3), the probability $P(\Delta, \Theta)$ is presented in the form

$$P(\Delta, \Theta) = \lim_{l \to \infty} \sum_{Z \in \mathfrak{K}_{m+l} : \chi(Z|\delta) = \theta(\delta), \delta \in \Delta} P_{m+l}(Z) \,. \tag{8}$$

For each $\sigma \in \mathfrak{K}_m$ and $Z \in \mathfrak{K}_{m+l}$, we define the set $\sigma \cap Z$. This set is empty in a trivial way if $\sigma \notin \mathbf{S}_m(\mathbf{K}(Z))$. It also can be empty for the other reason. If, for the given σ, there is $\delta \subset \mathfrak{N}_{\Delta,m}$, $\delta \in \Delta$, then, due to the definition $\mathfrak{N}_{\Delta,m}$, it follows that there exists such $\Gamma_\sigma \subset \mathfrak{K}_\Delta$ that the equality $\mathfrak{N}_\Delta(\Gamma_\sigma) = \sigma$ holds. Then $\theta(\delta') = 0$ takes place for all semi-intervals $\delta' \in \Gamma_\sigma$ and, therefore, $\chi(Z|\sigma) = 0$, $Z \cap \sigma = \emptyset$. The converse statement is valid, too. If the set Z corresponds to the term in (8), then $\sigma \cap \mathfrak{N}_{\Delta,m} = \emptyset$. In this connection, the property restricting the set of configurations $\{Z \in \mathfrak{K}_{m+l} : Z \subset [0,1) \setminus \mathfrak{N}_{\Delta,m}\}$ included in the sum in (8) can be written as follows:

$$[\forall \sigma \in \mathbf{S}_m(\mathbf{K}_m(Z))] \, [Z \cap \sigma \neq \emptyset, [\chi(Z \cap \sigma | \delta) = \theta(\delta), \delta \in \Delta_\sigma]] \,. \tag{9}$$

According to (4) and this requirement, all terms $P_{m+l}(Z)$ in (8) are represented in the form

$$P_{m+l}(Z) = P_m(\mathbf{K}_m(Z)) \left[\prod_{\sigma \in \mathbf{S}_m(\mathbf{K}_m(Z))} P_l(\mathsf{T}_m(Z \cap \sigma)) \right] . \tag{10}$$

In this case, $\mathbf{K}_m(Z) \subset [0,1) \setminus \mathfrak{N}_{\Delta,m}$. In addition, if there exists $\delta \in \Delta_\sigma$ with $\theta(\delta) = 1$ for anyone $\sigma \in \mathfrak{K}_m$, then necessarily $\sigma \subset \mathbf{K}_m(Z)$. Therefore, $\mathbf{K}_m(Z) \supset \mathfrak{J}_{\Delta,m}$.

Taking into account formula (10) and requirement (9) for configurations $Z \in \mathfrak{K}_{m+l}$, we represent the sum in (8) in the form of iterated sums

$$\sum_{Z \in \mathfrak{K}_{m+l} : \chi(Z|\delta) = \theta(\delta), \delta \in \Delta} P_{m+l}(Z) =$$

$$= \sum_{\substack{Y \in \mathfrak{K}_m : \\ \mathfrak{J}_{\Delta,m} \subset Y \subset [0,1) \setminus \mathfrak{N}_{\Delta,m}}} P_m(Y) \left(\prod_{\sigma \in \mathbf{S}(Y)} \sum_{\substack{V \in \mathfrak{K}_{m+l} : V \subset \sigma \\ \chi(V|\delta) = \theta(\delta), \delta \in \Delta, \delta \subset \sigma}} P_l(\mathsf{T}_m(V)) \right)$$

, where changes of the summation variables $Y = \mathbf{K}_m(Z)$ and $V = Z \cap \sigma$ have been done. Let us consider the multiplied sums on the right-hand side of this formula. Producing the replacement of the summation variables in each of them $U = \mathsf{T}_m(V) \in \mathfrak{K}_l$ and taking into account that T_m transforms $V \cap \delta = (V \cap \sigma) \cap \delta$ to $U \cap \mathsf{T}_m(\delta)$, we obtain

$$\sum_{\substack{V \in \mathfrak{K}_{m+l}: V \subset \sigma \\ \chi(V|\delta)=\theta(\delta), \delta \in \Delta, \delta \in \Delta_\sigma}} P_l(\mathsf{T}_m(V)) = \sum_{\substack{U \in \mathfrak{K}_l: \\ \chi(U|\mathsf{T}_m(\delta))=\theta(\delta), \delta \in \Delta_\sigma}} P_l(U) =$$

$$= \sum_{\substack{U \in \mathfrak{K}_l: \\ \chi(U|\delta')=\theta(\delta'), \delta' \in \mathsf{T}_m(\Delta_\sigma)}} P_l(U) = \mathsf{Pr}\{\tilde{X} : \chi(\mathsf{K}_l(\tilde{X})|\delta) = \theta(\delta), \delta \in \mathsf{T}_m(\Delta_\sigma)\}\,.$$

Here, it is taken into account that the condition $\delta \in \Delta$, $\delta \subset \sigma$ in the initial sum can be changed by $\delta \in \Delta_\sigma$ and, in this case, $\mathsf{T}_m(\delta) \in \mathsf{T}_m(\Delta_\sigma)$. Passing to the limit $l \to \infty$ in the last expression, we obtain (7) on the basis of definition (3) since

$$\lim_{l\to\infty} \mathsf{Pr}\{\tilde{X} : \chi(\mathsf{K}_l(\tilde{X})|\delta) = \theta(\delta), \delta \in \mathsf{T}_m(\Delta_\sigma)\} =$$

$$= \mathsf{Pr}\{\tilde{X} : \chi(\tilde{X}|\delta) = \theta(\delta), \delta \in \mathsf{T}_m(\Delta_\sigma)\} = P(\mathsf{T}_m(\Delta_\sigma), \Theta_\sigma)\,.$$

Remark 1. For factors corresponding to those σ when $\Delta_\sigma = \emptyset$ on the right-hand side of formula (7), it is necessary to put $P(\emptyset, \Theta) = 1$.

Remark 2. The tuple of semi-intervals may be such that it is not subordinated to $\mathfrak{K}_m$ for all $m \in \mathbb{N}$. But, in some cases, this difficulty can be passed over, namely, it may be if it is possible to change the pair $\langle \Delta, \Theta \rangle$ by the equivalent one obtained by a subdivision of semi-intervals $\delta \in \Delta$ having $\theta(\delta) = 0$. For the equivalent pair consisting of a semi-interval collection obtained by the subdivision, it may be already found such $m \in \mathbb{N}$ that it is subordinated to $\mathfrak{K}_m$. In particular, if $\Theta = 0$, then all $m \in \mathbb{N}$ will be such that there is an equivalent pair $\langle \Sigma, 0 \rangle$ for the pair $\langle \Delta, 0 \rangle$, where Σ is subordinated to $\mathfrak{K}_m$.

The following theorem allows us to express all multi-interval marginal distributions by means of the one-interval one.

Theorem 4. *For random sets with Markovian refinements, all multi-interval marginal distributions $F(a_1, b_1; ...; a_n, b_n)$, $n \in \mathbb{N}$, are expressed by the one-interval distribution due to the following formula*

$$F(a_1, b_1; ...; a_n, b_n) = \sum_{Y \in \mathfrak{K}_m : [0,1) \setminus \mathbf{N}_{\Delta,m} \supset Y} P_m(Y) \prod_{\sigma \in \mathbf{S}_m(Y)} F^{(\sigma)}(a'_i, b'_i), \tag{11}$$

where m is the smallest order of the subdivision of the interval $[0, 1)$, when there exists the tuple of semi-intervals $\Delta' = \langle [a'_i, b'_i); i = 1..., n' \rangle$ subordinated to $\mathfrak{K}_m$ and such that the pair $\langle \Delta', 0 \rangle$ being equivalent to the pair $\langle \Delta, 0 \rangle$ possesses the following property. In each semi-interval $\sigma \in \mathfrak{K}_m$, there exists no more than one semi-interval of Δ'. In this case, $F^{(\sigma)}(a'_i, b'_i) = 1$ if $\sigma \cap [a'_i, b'_i) = \emptyset$ and $F^{(\sigma)}(a'_i, b'_i) = P(\mathsf{T}_m([a'_i, b'_i)), 0)$ otherwise.

Proof. In view of the above notice, there is a tuple pointed out in the theorem. After that, we apply formula (7).

6. One-interval distribution

Let us calculate the one-interval distribution in the elementary special case where the subdivision parameter N is equal to 2 and the subdivision is spatially uniform $q[\delta_1] = q[\delta_2]$, where $\mathfrak{K}_1 = \{\delta_1, \delta_2\}$, $\delta_1 = [0, 1/2)$, $\delta_2 = [1/2, 1)$. Herewith, the random set is defined completely by two probabilities $q_2 = \mathsf{Pr}\{\tilde{X}_1 = [0, 1)\}$, $q_1 = \mathsf{Pr}\{\tilde{X}_1 = \delta\}$, $\delta \in \mathfrak{K}_1$, $2q_1 + q_2 = 1$. Hereinafter, we use a binary presentation of numbers in $[0, 1)$ in the calculations. For any number $\xi \in [0, 1)$ with the binary decomposition $\xi = 0.\xi_1\xi_2...$, $\xi_i \in \{0, 1\}$, $i \in \mathbb{N}$, we introduce the notations of n-component restrictions of the fraction ξ, namely, $\xi|_n = 0.\xi_1\xi_2...\xi_n$ with the lack and $\xi|^n = \xi|_n + 2^{-n}$ with the excess, correspondingly. At the same time, $\xi|_0 = 0$, $\xi|^0 = 1$ if $\xi \in (0, 1)$. Firstly, let us calculate the probability $F(0,b) = \mathsf{P}r\{\tilde{X} : \tilde{X} \cap [0,b) = \emptyset\}$. To this end, we consider the probability $F_1(0,b)$ and represent it as the limit

$$F_1(0,b) = \lim_{m\to\infty} F_1(0,b|_m) = \lim_{m\to\infty} \mathsf{P}r\{\tilde{X} : \mathbf{K}_m(\tilde{X}) \cap [0,b|_m) \neq \emptyset\}\,. \tag{12}$$

Thus, it is necessary to calculate the probability $F_1(0,b|_m)$, $m \in \mathbb{N}$. Considering the function $F_1(0,b|_{m+1})$ as the probability of the sum of two disjoint events, we represent it in the form

$$F_1(0,b|_{m+1}) = F_1(0,b|_m) + F_{01}(0,b|_m; b|_m, b|_{m+1})\,. \tag{13}$$

The following relation is established similarly:

$$f_m \equiv F_{01}(0,b|_m; b|_m, b|^m) = F_{01}(0,b|_{m+1}; b|_{m+1}, b|^m) + F_{01}(0,b|_m; b|_m, b|_{m+1})\,.$$

At $b_{m+1} = 0$, we have $b|_m = b|_{m+1}$ and, therefore, the last summand is equal to zero. If $b_{m+1} = 1$, then $b|^{m+1} = b|^m$. Hence, in both cases,

$$F_{01}(0,b|_m; b|_m, b|_{m+1}) = b_{m+1}(f_m - f_{m+1})\,. \tag{14}$$

Thus, it is sufficient to calculate the sequence f_m, $m \in \mathbb{N}$. We construct the recurrent relation for f_m. For this, we represent f_{m+1} as the value $P(\Delta, \Theta)$ at $\Delta = \langle \delta_i^{(m+1)}; i = 1, ..., 2^{m+1} b|_{m+1} + 1\rangle$ and $\Theta = \langle 0, ..., 0, 1\rangle$, where $\delta_i^{(m+1)} = [(i-1)/2^{m+1}, i/2^{m+1})$. Then, on the basis of (7), we have

$$f_{m+1} = \sum_{\substack{Y \in \mathfrak{K}_m: \\ [0,1)\setminus \mathcal{N}_{\Delta,m} \supset Y \supset [b|_{m+1}, b|^{m+1})}} P_m(Y) \left[\prod_{\sigma \in \mathbf{S}_m(Y)} P(\mathbf{T}_m(\Delta_\sigma), \Theta_\sigma) \right]. \tag{15}$$

Notice that $\mathcal{N}_{\Delta,m} = [0, b|_m)$ and only one semi-interval $\sigma = [b|_m, b|^m)$ contained in $\mathbf{S}_m(Y)$ can have a nonempty intersection with semi-intervals of Δ. In this case, the tuple Δ_σ consists of semi-intervals $[b|_m, b|_{m+1})$, $[b|_{m+1}, b|^{m+1})$, respectively (the first is empty as $b_{m+1} = 0$), the tuple $\mathbf{T}_m(\Delta_\sigma)$ consists of $[0, 0.1)$ as $b_{m+1} = 0$ and of the pair $\langle [0, 0.1), [0.1, 1)\rangle$ as $b_{m+1} = 1$. Respectively, Θ_σ is either $\langle 1\rangle$ or $\langle 0, 1\rangle$. Therefore, in the first case, $P(\mathbf{T}_m(\Delta_\sigma), \Theta_\sigma)$ equals

$$F_{10}(0, 0.1; 0.1, 1) + F_{11}(0, 0.1; 0.1, 1) = q_1 + q_2$$

and, in the second one, it equals $F_{01}(0, 0.1; 0.1, 1) = q_1$. Consequently, we rewrite (15) in the form

$$f_{m+1} = (q_1 + (1 - b_{m+1})q_2) \sum_{\substack{Y \in \mathfrak{K}_m: \\ [b|_m, 1) \supset Y \supset [b|_m, b|^m)}} P_m(Y) = (q_1 + (1 - b_{m+1})q_2) f_m.$$

At $f_0 = \mathsf{P}r\{\tilde{X} \cap [0,0) = \emptyset, \tilde{X} \cap [0,1) \neq \emptyset\} = 1$, the obtained difference equation gives us the formula for f_m,

$$f_m = \prod_{j=1}^{m} (q_1 + (1 - b_j)q_2) = q_1^{r_m}(q_1 + q_2)^{m - r_m}$$

, where the sequence $\langle r_m; m = 1, 2...\rangle$ determines the number of units in the binary decomposition $b|_m$. Substituting this expression to (14) and using (12), (13), we find

$$F_1(0,b) = (q_1 + q_2) \sum_{m=0}^{\infty} b_{m+1} f_m\,, \quad F(0,b) = 1 - F_1(0,b)\,. \tag{16}$$

The probability $F(b,1)$ is calculated in the same way, i.e.

$$F_1(b,1) = (q_1 + q_2) \sum_{m=0}^{\infty} (1 - b_{m+1}) g_m\,, \quad F(b,1) = 1 - F_1(b,1)\,,$$

$$g_m = \prod_{j=1}^{m}(q_1 + b_j q_2) = q_1^{m-r_m}(q_1+q_2)^{r_m}\,.$$

At last, in the general case, the probability $F(a,b)$ is expressed via the probabilities $F_- = F(2^m(a|^m - a), 1)$, $F_+ = F(0, 2^m(b - b|_m))$, $b|_m = a|^m$ on the basis of the reduction formula. Namely, we find the minimal subdivision order m, when there exists the unique point $i/2^m \in [a,b]$. Then, we apply the reduction formula to $F(a,b) = P(\Delta, 0)$, where the equivalent pair $\langle \Delta, 0\rangle$ with $\Delta = \langle \delta_-, \delta_+\rangle = \langle [a, i/2^m), [i/2^m, b)\rangle$ is used. Semi-intervals of this pair may have a nonempty intersection with no more than two semi-intervals $\sigma \in \mathbf{S}_m(Y)$. Then it follows from (7) that

$$F(a,b) = \sum_{Y \in \mathfrak{K}_m : Y \subset [0,1)} P_m(Y) P(\mathsf{T}_m(\delta_-), 0) P(\mathsf{T}_m(\delta_+), 0)\,. \tag{17}$$

Here, it is taken into account that $\mathbf{N}_{\Delta,m} = \emptyset$ under the minimality condition of the subdivision order m. The sum in (17) is split into four parts in accordance with the following summand groups $\{Y : Y \cap (\delta_+ \cup \delta_-) = \emptyset\}$, $\{Y : Y \cap \delta_+ = \emptyset, Y \cap \delta_- \neq \emptyset\}$, $\{Y : Y \cap \delta_+ \neq \emptyset, Y \cap \delta_- = \emptyset\}$, $\{Y : Y \cap \delta_+ \neq \emptyset, Y \cap \delta_- \neq \emptyset\}$. Accordingly, in the first group, both factors $P(\mathsf{T}_m(\delta_\pm), 0)$ are equal identically to unity (the first group is absent as $m = 1$), in the second and in the third group, one of the summands differs from unity and, in last, both of them possess this property. If these factors differ from zero, then they are equal to $P(\mathsf{T}_m(\delta_\pm), 0) = F_\pm$. Therefore, (17) is represented in the form

$$\begin{aligned} F(a,b) = {} & \mathsf{P}r\{\mathbf{K}_m(\tilde{X}) \cap \mathbf{K}_m(\delta_+ \cup \delta_-) = \emptyset\} + \\ & + \mathsf{P}r\{\mathbf{K}_m(\delta_-) \subset \mathbf{K}_m(\tilde{X}) \subset [0,1) \setminus \mathbf{K}_m(\delta_+)\} F_+ + \\ & + \mathsf{P}r\{\mathbf{K}_m(\delta_+) \subset \mathbf{K}_m(\tilde{X}) \subset [0,1) \setminus \mathbf{K}_m(\delta_-)\} F_- + \\ & + \mathsf{P}r\{\mathbf{K}_m(\delta_+ \cup \delta_-) \subset \mathbf{K}_m(\tilde{X})\} F_- F_+ = \\ & = 1 - (q_1+q_2)^{m-1}\left[1 - q_1(F_- + F_+) - q_2 F_- F_+\right]\,,\ m \in \mathbb{N}\,. \end{aligned} \tag{18}$$

The simple calculation of each summand in (18) is based on formulas (1) and (2).

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