MONOTONICITY OF THE PROBABILITY OF PERCOLATION FOR BERNOULLI RANDOM FIELDS ON PERIODIC GRAPHS

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ABSTRACT. In this work, the problem of percolation of the Bernoulli random field on periodic graphs Λ of an arbitrary dimension d is studied. A theorem on nondecreasing dependence of the probability of percolation $Q(c_1, \ldots, c_n)$ with respect to each of the parameters c_i, $i = 1 \div n$, – concentration of the Bernoulli field is proved.

1. Introduction

The study of the percolation theory was initiated in [1] as an attempt to model mathematically physical phenomena occurred in randomly inhomogeneous media. Later, this mathematical theory was divided into two directions substantially distinct from each other called, respectively, the discrete and continuous percolation theory. The first of them, in whose framework a lot of deep results of qualitative character was obtained (see surveys and monographs [2, 4, 5, 7–9]), studies the percolation properties of random sets on connected infinite graphs [6], in particular, on so-called periodic graphs [4]. However, up to now, there is no regular method of calculation with any accuracy given in advance of the main quantity, the probability of percolation $\mathsf{Q}[\mathsf{c}]$ for some wide range of infinite graphs in this theory. This probability is induced by the distribution of probabilities of random sets and is a function of its defining parameters collection $\mathsf{c} = \langle c_1, \ldots, c_m \rangle$. An obstacle in the creation of such a method is the complexity of obtaining apriori estimates for approximations of the function $\mathsf{Q}[\mathsf{c}]$ in the so-called critical domain of variation of parameters c. In particular, it is difficult to give a mathematical substantiation of the statement on the common apriori qualitative properties of the dependence $\mathsf{Q}[\mathsf{c}]$ that are expected for physical reasons. One of such properties is the monotonicity of the dependence of the percolation probability for the Bernoulli random field on probabilities of the filling-in of vertices of the graph. It manifests itself in exactly solvable models and is used in heuristic speculations; however, its proof is absent in the literature. In the present paper, we suggest a rather transparent proof of the statement on monotone change of the probability of percolation for the Bernoulli random field of the general form. Moreover, our proof does not use the so-called FGK-inequalities [3], which are usually a tool for establishing such facts in statistical mechanics.

2. The Problem of Discrete Percolation Theory

Let $\Lambda(V, \Phi)$ be an infinite non-directed graph (without loops and multiple edges) with the set of vertices V and the set of edges Φ. The index of each vertex of the graph is assumed to be finite. Further, speaking of an "arbitrary graph," we mean only graphs of the mentioned type. Following to G. Kesten, we will say that such a graph is periodic of dimension $d \in \mathbb{N}$ if there exists an embedding $\mathsf{M} : V \mapsto \mathbb{R}^d$, for which the image $\mathsf{M}V$ consists of isolated points and is invariant with respect to translations of periods $\langle \mathbf{a}_1, \ldots, \mathbf{a}_d \rangle \in \mathbb{R}^d$ together with the image $\mathsf{M}\Phi$ of the set of edges, i.e.,

$$\mathsf{M}V + n_1\mathbf{a}_1 + \cdots + n_d\mathbf{a}_d = V, \qquad \mathsf{M}\Phi + n_1\mathbf{a}_1 + \cdots + n_d\mathbf{a}_d = \Phi$$

for any $\langle n_1, \ldots, n_d \rangle \in \mathbb{Z}^d$. In this case, the set V is divided into a finite set of classes $\mathfrak{K}_j$, $j = 1 \div m$, $V = \bigcup_{j=1}^{m} \mathfrak{K}_j$ of equivalent vertices, i.e., such that they pass to each other under the mentioned translations. The importance of the study of periodic graphs is stipulated by their importance in applications.

We denote the adjacency relation for the pair of vertices $\mathbf{x}$ and $\mathbf{y}$ from V by $\mathbf{x}\varphi\mathbf{y}$, $\varphi \in \Phi$. Any sequence $\gamma = \langle \mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n \rangle$ of vertices of the graph Λ such that $\mathbf{x}_{j-1}\varphi\mathbf{x}_j$, $j = 1 \div n$ is called a path of length n. The distance between two vertices $\mathbf{x}$ and $\mathbf{y}$ on V is the minimum of lengths of all paths such that $\mathbf{x}$ and $\mathbf{y}$ are their endpoints. The path $\gamma = \langle \mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n \rangle$ is called a path without self-intersections if the distance between any two vertices $\mathbf{x}_j$ and $\mathbf{x}_k$, $k - j > 1$, on this path is greater than 1.

Let, on a periodic graph Λ, a Bernoulli random field $\langle \tilde{c}(\mathbf{x}); \mathbf{x} \in V \rangle$ be given, which is defined by a $\mathsf{c} = \langle c_1, \ldots, c_m \rangle$, corresponding to each of the equivalence classes, so that the random variables $\tilde{c}(\mathbf{x}), \mathbf{x} \in V$ are independent in totality and the probability distribution is given by the relations $\Pr\{\tilde{c}(\mathbf{x}) = 1\} = c_j$, $c_j \in [0, 1]$, $j = 1 \div m$, if the vertex $\mathbf{x}$ belongs to the class $\mathfrak{K}_j$, $j = 1 \div m$.

The random sets $\tilde{C} = \{\mathbf{z} \in V : \tilde{c}(\mathbf{z}) = 1\}$ are called random configurations. By definition, a configuration has a percolation if there exists an infinite path γ without self-intersections in it. If this path starts from the vertex $\mathbf{x}$, then we say that there exists a percolation from the vertex $\mathbf{x}$. Denote by $\mathsf{Q}[\mathbf{x}|\mathsf{c}]$ the probability of a percolation from the vertex $\mathbf{x} \in V$, i.e., the probability of existence of an infinite path starting from the vertex $\mathbf{x}$ located on the set of occupied vertices.

3. The Percolation Expansion

The proof of the main theorem of this work is a consequence of a more general statement. To formulate it, we need some preliminary constructions. We will consider any infinite graphs $\Lambda(V, \Phi)$, not necessarily periodic. On the graph Λ, we introduce an inhomogeneous Bernoulli random field $\{\tilde{c}(\mathbf{x}); \mathbf{x} \in V\}$, i.e., the set of random variables $\tilde{c}(\mathbf{x})$ labelled by vertices $\mathbf{x} \in V$ and independent in totality. The probability distribution of the field is defined by relations $\Pr\{\tilde{c}(\mathbf{x}) = 1\} = c(\mathbf{x})$, where $\mathsf{c} : V \mapsto [0, 1]$ are probabilities of filling-in the vertices $\mathbf{x} \in V$.

We introduce into consideration the probability $\mathsf{Q}_n[\mathbf{x}|\mathsf{c}]$ of existence of a path without self-intersections γ starting from the vertex $\mathbf{x}$ and having length n on the random configuration $\tilde{C} = \{\mathbf{x} \in V : c(\mathbf{x}) = 1\} \subset V$ of occupied vertices of the graph Λ. The probability $\mathsf{Q}_n[\mathbf{x}|\mathsf{c}]$ is a functional of the distribution of concentrations c. We use the following relation for the probability of percolation:

$$\mathsf{Q}[\mathbf{x}|\mathsf{c}] = \lim_{n \to \infty} \mathsf{Q}_n[\mathbf{x}|\mathsf{c}]. \tag{1}$$

Denote by $\mathfrak{P}_n(\mathbf{x})$ the class of finite paths without self-intersections γ of length n starting from the vertex $\mathbf{x}$. Then

$$\mathsf{Q}_n[\mathbf{x}|\mathsf{c}] = \Pr\{\tilde{C} \in \Omega : \exists(\gamma \subset \tilde{C})(\gamma \in \mathfrak{P}_n(\mathbf{x}))\}, \qquad \Omega = \{\tilde{C} : \tilde{C} = \{\mathbf{z} \in V : \tilde{c}(\mathbf{z}) = 1\}\}.$$

We also introduce into consideration classes $\mathfrak{W}_n$ of connected finite subsets of V containing the vertex $\mathbf{x}$ and such that the distance from any vertex of each of the sets, belonging to $\mathfrak{W}_n$, to the vertex $\mathbf{x}$

does not exceed n, $n \in \mathbb{N}_+$. Our goal is to construct for any $n \in \mathbb{N}$ a disjoint decomposition

$$\mathfrak{Q}_n = \{\tilde{C} : \exists(\gamma \subset \tilde{C})(\gamma \in \mathfrak{P}_n(\mathbf{x}))\} = \bigcup_{\langle\sigma_1,\dots,\sigma_n\rangle} A_n(\sigma_1,\dots,\sigma_n), \tag{2}$$

where the sets $A_n(\sigma_1,\dots,\sigma_n)$, belonging to the union, are defined by *admissible* expanding sequences $\langle\sigma_1,\dots,\sigma_n\rangle$ of sets $\sigma_j \in \mathfrak{W}_j$, $j = 1 \div n$. Admissible sequences $\langle\sigma_1,\dots,\sigma_n\rangle$, similarly to the sets $A(\sigma_1,\dots,\sigma_n)$ of configurations, are constructed by induction on the length of the path n.

For $n = 1$, we define the set $D_1 = \{\mathbf{z} \in V : \mathbf{z}\varphi\mathbf{x}\}$ and a nonempty subset $\Delta_1 \subset D_1$. Then we assume that $\sigma_1 = \Delta_1 \cup \{\mathbf{x}\}$ is a subset forming an admissible sequence $\langle\sigma_1\rangle$ of length 1.

Assume that the sequences of sets $\langle D_1,\dots,D_n\rangle$ and $\langle\sigma_1,\dots,\sigma_n\rangle$ such that $\sigma_k = \{\mathbf{x}\}\cup\Delta_1\cup\dots\cup\Delta_k$, $k = 1\div n$, where $\Delta_k = \sigma_k\backslash\sigma_{k-1} \subset D_k\backslash(D_0\cup D_1\cup\dots\cup D_{k-1})$, $D_0 = \{\mathbf{x}\}$, $D_k = \{\mathbf{z} : \exists(\mathbf{y} \in \sigma_{k-1})(\mathbf{z}\varphi\mathbf{y})\}$, $k = 1 \div n$, are constructed. The sequences $\langle\sigma_1,\dots,\sigma_n\rangle$ constructed according to such principle are said to be admissible and having the length n.

The transition to the length of the sequence $(n+1)$ is performed by constructing the set $D_{n+1} = \{\mathbf{z} : \exists(\mathbf{y} \in \sigma_n)(\mathbf{z}\varphi\mathbf{y})\}$ and by choosing a nonempty subset $\Delta_{n+1} \subset D_{n+1}\backslash(D_0\cup D_1\cup\dots\cup D_n)$ such that $\Delta_{n+1}\cap\sigma_n = \emptyset$. After that, we define the set $\sigma_{n+1} = \sigma_n\cup\Delta_{n+1}$ and admissible sequence $\langle\sigma_1,\dots,\sigma_{n+1}\rangle$ of length $(n+1)$.

Now we construct decomposition (2). To each configuration $\tilde{C}$ of the Bernoulli field on Λ, in which at least one path without self-intersections of length n starting from the vertex $\mathbf{x}$ can be embedded, for any $k \le n$, we put in correspondence a unique set $\Gamma_k(\tilde{C}) \subset \mathfrak{W}_k$ of vertices that can be achieved by the path without self-intersections $\gamma \subset \tilde{C}$ of length at most k.

For each $n \in \mathbb{N}$ and each pair of sequences $\langle D_1,\dots,D_n\rangle$ and $\langle\sigma_1,\dots,\sigma_n\rangle$, we define the set of configurations

$$A_n(\sigma_1,\dots,\sigma_n) = \{\tilde{C} : \exists(\gamma \subset \tilde{C})(\gamma \in \mathfrak{P}_n(\mathbf{x})), \Gamma_k(\tilde{C}) = \sigma_k, \tilde{C}\cap(D_k\setminus\sigma_k) = \emptyset,\ k = 1 \div n\}.$$

Then, by definition, the pairs of such sets corresponding to distinct tuples $\langle\sigma_1,\dots,\sigma_n\rangle$ do not intersect, and any configuration from $\mathfrak{Q}_n$ belongs to one of them. Thus, the totality of sets $A_n(\sigma_1,\dots,\sigma_n)$ forms a disjoint decomposition (2).

We denote $\mathsf{P}_n(\sigma_1,\dots,\sigma_n) = \Pr\{A_n(\sigma_1,\dots,\sigma_n)\}$. By (2), the following decomposition holds:

$$\mathsf{Q}_n[\mathbf{x}|\mathsf{c}] = \sum_{\langle\sigma_1,\dots,\sigma_n\rangle} \mathsf{P}(\sigma_1,\dots,\sigma_n). \tag{3}$$

Now we connect probabilities $\mathsf{P}_n(\sigma_1,\dots,\sigma_n)$ with the probability distribution of the Bernoulli field $\{\tilde{c}(\mathbf{z}); \mathbf{z} \in V\}$. For each $n \in \mathbb{N}$, introduce the functions Q_n depending on admissible tuples $\langle\sigma_1,\dots,\sigma_n\rangle$. They depend on concentrations defining the Bernoulli random field, as on parameters. The functions Q_n are defined by the relation

$$\mathsf{Q}_n(\sigma_1,\dots,\sigma_n) = \Bigg(\prod_{\mathbf{z}\in\Delta_n} c(\mathbf{z})\Bigg)\Bigg(\prod_{\mathbf{z}\in D_n^*\backslash\Delta_n} (1 - c(\mathbf{z}))\Bigg), \tag{4}$$

where $D_n^* = D_n \setminus \left(\bigcup_{k=0}^{n-1} D_k\right)$.

By the construction of sequences $\langle\sigma_1,\dots,\sigma_n\rangle$, $\langle D_1,\dots,D_n\rangle$, from the probabilistic point of view, functions (3) are conditional probabilities of the appearance of the set $\Gamma_n(\tilde{C}) = \sigma_n$ for the configurations $\tilde{C}$ containing paths γ from $\mathfrak{P}_n(\mathbf{x})$ under the assumption that $\Gamma_k(\tilde{C}) = \sigma_k$, $k = 1 \div n-1$. Therefore, functions P_n, $n \in \mathbb{N}_+$ are defined inductively by the relation

$$\mathsf{P}_n(\sigma_1,\dots,\sigma_{n-1},\sigma_n) = \mathsf{P}_{n-1}(\sigma_1,\dots,\sigma_{n-1})\mathsf{Q}_n(\sigma_1,\dots,\sigma_{n-1},\sigma_n), \quad n \in \mathbb{N}, \tag{5}$$

and by the value of the probability $\mathsf{P}_0(\sigma_0) = \Pr\{\tilde{c}(\mathbf{x}) = 1\} = c(\mathbf{x})$.

Relations (4) and (5) give the expression for probabilities P_n on the basis of concentrations $c(\mathbf{z})$, $\mathbf{z} \in V$, of the field $\{\tilde{c}(\mathbf{z}); \mathbf{z} \in V\}$ and, together with (3), they form the decomposition, which we have called the *percolation* decomposition.

4. The Main Theorem

Now we can state and prove the main theorem.

Theorem 4.1. *For any graph $\Lambda(V, \Phi)$ and any vertex $\mathbf{x} \in V$, the probability $\mathsf{Q}[\mathbf{x}|\mathsf{c}]$ of percolation from $\mathbf{x}$ of the Bernoulli random field $\{\tilde{c}(\mathbf{z}); \mathbf{z} \in V\}$ with the probability distribution $\Pr\{\tilde{c}(\mathbf{z}) = 1\} = c(\mathbf{z})$, $\mathbf{z} \in V$, is a nondecreasing function with respect to each of the concentrations $c(\mathbf{z})$, $\mathbf{z} \in V$.*

Proof. Note that, according to (5), the probability $\mathsf{Q}_n[\mathbf{x}|\mathsf{c}]$ can be represented by the relation

$$\mathsf{Q}_n[\mathbf{x}|\mathsf{c}] = c(\mathbf{x}) \sum_{\langle \sigma_1, \ldots, \sigma_n \rangle} \prod_{k=1}^{n} \mathsf{Q}_k(\sigma_1, \ldots, \sigma_k). \tag{6}$$

We differentiate the probability $\mathsf{P}_n(\sigma_1, \ldots, \sigma_n)$ with respect to the parameter $c(\mathbf{z})$, $\mathbf{z} \in V$:

$$\frac{\partial}{\partial c(\mathbf{z})} \mathsf{P}_n(\sigma_1, \ldots, \sigma_n) = \sum_{j=1}^{n} \left(\frac{\partial}{\partial c(\mathbf{z})} \mathsf{Q}_k(\sigma_1, ..., \sigma_k) \right) \prod_{\substack{j=1 \\ j \neq k}}^{n} \mathsf{Q}_j(\sigma_1, \ldots, \sigma_j)$$
$$+ \delta_{\mathbf{x},\mathbf{z}} \prod_{j=1}^{n} \mathsf{Q}_j(\sigma_1, \ldots, \sigma_j). \tag{7}$$

For any admissible tuple $\langle \sigma_1, \ldots, \sigma_n \rangle$, each of the functions $\mathsf{Q}_k(\sigma_1, \ldots, \sigma_k)$ depends only on the parameters $c(\mathbf{z})$, for which $\mathbf{z} \in D_k^*$, $k = 0 \div n$. Hence, for any fixed tuple $\langle \sigma_1, \ldots, \sigma_n \rangle$, in sum (7), only one summand is not equal to zero and, by (6), we can write

$$\frac{\partial}{\partial c(\mathbf{z})} \sum_{\langle \sigma_1, \ldots, \sigma_n \rangle} \mathsf{P}_n(\sigma_1, \ldots, \sigma_n)$$
$$= c(\mathbf{x}) \sum_{k=1}^{n} \sum_{\langle \sigma_1, \ldots, \sigma_{k-1} \rangle} \left[\frac{\partial}{\partial c(\mathbf{z})} \sum_{\substack{\sigma_k \\ \mathbf{z} \in D_k^*}} \mathsf{Q}_k(\sigma_1, \ldots, \sigma_k) \right] \sum_{\langle \sigma_{k+1}, \ldots, \sigma_n \rangle} \prod_{\substack{j=1 \\ j \neq k}}^{n} \mathsf{Q}_j(\sigma_1, \ldots, \sigma_j)$$
$$+ \delta_{\mathbf{x},\mathbf{z}} \sum_{\langle \sigma_1, \ldots, \sigma_n \rangle} \prod_{j=1}^{n} \mathsf{Q}_k(\sigma_1, \ldots, \sigma_k).$$

The proof is completed by establishing the positivity of the sum in the square bracket. Since in this sum σ_k are such that $\emptyset \neq \Delta_k = \sigma_k \setminus \sigma_{k-1} \subset D_k^*$, we have for this sum, by (4),

$$\sum_{\emptyset \neq \Delta_k \subset D_k^*} \frac{\partial}{\partial c(\mathbf{z})} \left(\prod_{\mathbf{y} \in \Delta_k} c(\mathbf{y}) \right) \left(\prod_{\mathbf{y} \in D_k^* \setminus \Delta_k} (1 - c(\mathbf{y}) \right)$$
$$= \sum_{\substack{\Delta_k \subset D_k^* \\ \mathbf{z} \in \Delta_k}} \left(\prod_{\mathbf{y} \in \Delta_k \setminus \{\mathbf{z}\}} c(\mathbf{y}) \right) \left(\prod_{\mathbf{y} \in D_k^* \setminus \Delta_k} (1 - c(\mathbf{y})) \right)$$
$$- \sum_{\substack{\emptyset \neq \Delta_k \subset D_k^* \\ \mathbf{z} \notin \Delta_k}} \left(\prod_{\mathbf{y} \in \Delta_k} c(\mathbf{y}) \right) \left(\prod_{\mathbf{y} \in D_k^* \setminus (\Delta_k \cup \{\mathbf{z}\})} (1 - c(\mathbf{y})) \right) = \prod_{\mathbf{y} \in D_k^* \setminus \{\mathbf{z}\}} (1 - c(\mathbf{y})) \geq 0.$$

The latter identity is related to the fact that the first sum is identically equal to 1, and the second one differs from 1 by the summand with $\Delta_k = \emptyset$. ∎

Corollary 4.1. *The probability of percolation* $\mathsf{Q}[\mathbf{x}|\mathsf{c}]$ *of the Bernoulli field on the periodic graph* $\Lambda(V, \Phi)$ *is a nondecreasing function with respect to each of the parameters* c_i, $i = 1 \div n$.

Proof. By the definition of the Bernoulli field on the periodical graph, we have

$$\frac{\partial}{\partial c_i}\mathsf{Q}_n[\mathbf{x}|\mathsf{c}] = \sum_{\mathbf{z}\in\mathfrak{K}_i\cap\mathfrak{W}_n}\left(\frac{\partial}{\partial c(\mathbf{z})}\mathsf{Q}_n[\mathbf{x}|\mathsf{c}]\right)_{\mathbf{y}\in\mathfrak{K}_i, c(\mathbf{y})=c_i, i=1\div n} > 0.$$

Therefore, $\mathsf{Q}_n[\mathbf{x}|\mathsf{c}]$ is an increasing function with respect to each of the parameters c_i, $i = 1 \div n$. After that, passing to the limit as $n \to \infty$, by (1), we obtain the required statement. □

5. Conclusion

We have proved the presence of a very important property of the monotone increase with respect to each of concentrations of the field for the probability of percolation $\mathsf{Q}[\mathbf{x}|\mathsf{c}]$ of Bernoulli random fields. However, in our opinion, this property is not more important in itself but the method based on which it was found, namely, the percolation decomposition (3). This decomposition allows one to find lower bounds for the percolation threshold.

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