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YU. P. VIRCHENKO AND N. N. VITOKHINA

MULTIPLICATIVE DECOMPOSITION AND INFINITE DIVISIBILITY OF THE MANDEL DISTRIBUTION

Infinite divisibility of the Mandel distribution that arises in quantum optics is proved. On the basis of this fact, the multiplicative decomposition of this distribution into the countable convolution of some Poisson distributions is constructed.

1. Introduction

The registration process of low-intensity electromagnetic radiation represents the absorption of some energy portions named photons. The number $\tilde{n}$ of registered photons during the time T is random according to its physical nature[1)]. Therefore, the description of the registration process is based on the probability distribution $P_n \equiv \Pr\{\tilde{n} = n\}$ of this random value. The determination of this distribution is the quantum optics problem. In this section of theoretical physics, it is considered [1], [2] that, under some certain physical conditions, the distribution P_n is the composite Poissonian one,

$$P_n = \frac{1}{n!}\mathsf{E}\left(\tilde{J}_T^n \exp[-\tilde{J}_T]\right), \tag{1}$$

where $\tilde{J}_T$ is the random variable representing the energy of the electromagnetic field absorbed by a photon counter during the registration time T. In quantum optics, it is referred to as the Mandel distribution [3].

In the work, we study the probability distribution (1) from the mathematical point of view in the physically most simple case where the electromagnetic radiation is the so-called one-mode and completely polarized. In addition, we consider that the electromagnetic radiation is completely noisy. Mathematically, this case is described by the random variable $\tilde{J}_T$ which is defined by the formula [1]

$$\tilde{J}_T \equiv J[\tilde{\zeta}] = \int_0^T \left|\tilde{\zeta}(s)\right|^2 ds\,,$$

where $\tilde{\zeta}(s) = \tilde{\xi}(s) + i\tilde{\eta}(s)$, $s \in \mathbb{R}$, are trajectories of the complex process with real and imaginary parts $\tilde{\xi} = \{\tilde{\xi}(t); t \in \mathbb{R}\}$, $\tilde{\eta} = \{\tilde{\eta}(t); t \in \mathbb{R}\}$ corresponding to Ornstein–Uhlenbeck processes which are stochastically equivalent and independent.

The Ornstein–Uhlenbeck processes are Markovian and Gaussian ones, and they are completely characterized by these properties together with the stationarity condition. This class of processes is parametrized by two numbers $\nu > 0$, $\sigma > 0$. In view of Markovian and Gaussian properties, each Ornstein–Uhlenbeck process is completely determined by the conditional probability density $w(x_0, t_0|x, t)$ of the transition from the

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[1]The sign "tilde" points out further that the corresponding mathematical object is random.

point $x_0 \in \mathbb{R}$ at $t_0 \in \mathbb{R}$ to the point $x \in \mathbb{R}$ at $t \in \mathbb{R}$ (see, for example, [4], III, §8, and also [5]). It depends on the parameters ν and σ and has the following form:

$$w(x_0, t_0|x, t) = \left(\frac{\nu}{\pi\sigma\left(1 - e^{-2\nu|t-t_0|}\right)}\right)^{1/2} \exp\left(-\frac{\nu\left[x - x_0 e^{-\nu|t-t_0|}\right]^2}{\sigma\left(1 - e^{-2\nu|t-t_0|}\right)}\right). \tag{2}$$

Then, fixing the parameters ν and σ completely determines the probability distributions of random variables $J_T[\tilde{\xi}]$, $J_T[\tilde{\eta}]$ and, thus, in view of independence and equivalence of the processes $\{\xi(t); t \in \mathbb{R}\}$, $\{\eta(t); t \in \mathbb{R}\}$, it defines the probability distribution density of the random variable $J_T[\tilde{\zeta}]$.

The characteristic function $Q_\xi(-i\lambda) = \mathsf{E}\exp(i\lambda J_T[\tilde{\xi}])$, $\lambda \in \mathbb{R}$, of the random variable $J_T[\tilde{\xi}]$ has been calculated in [6]:

$$Q_{\tilde{\xi}}(\lambda) = \mathsf{E}\exp(-\lambda J_T[\tilde{\xi}]) = \left(\frac{4r\nu\exp(\nu T)}{(r+\nu)^2\exp(rT) - (r-\nu)^2\exp(-rT)}\right)^{1/2}, \tag{3}$$

where $r = \sqrt{\nu^2 + 2\lambda\sigma}$. From this, the characteristic function is determined as a function of complex variable on a two-list Riemann surface. For the complex process $\{\zeta(t); t \in \mathbb{R}\}$, the similar function is determined by the formula

$$Q(\lambda) = Q_{\tilde{\xi}}(\lambda) Q_{\tilde{\eta}}(\lambda) = \left[Q_{\tilde{\xi}}(\lambda)\right]^2 \tag{4}$$

in view of independence and equivalence of the processes $\tilde{\xi}$, $\tilde{\eta}$. Whence it follows that $Q(\lambda)$ is meromorphic (see, Corollary 1 below).

The availability of the explicit formula of the function $Q(\lambda)$ allows us to use some methods of the theory of complex variable functions to study the properties of the rather complicated Mandel distribution P_n corresponding to this characteristic function. The purpose of the present work is to construct the multiplicative decomposition of the Mandel distribution on Poisson distributions and, in particular, to give the proof of its infinite divisibility.

2. Properties of the function $Q(\lambda)$

According to (3) and (4), the function $Q(\lambda)$ has the form

$$Q(\lambda) = 4\nu r\left[F(r)\right]^{-1} e^{\nu T}, \tag{5}$$

$$F(r) = (\nu + r)^2 e^{rT} - (\nu - r)^2 e^{-rT}. \tag{6}$$

Lemma 1. *The function* $[Q(\lambda)]^{-1}$ *is an entire function of* $\lambda \in \mathbb{C}$.

Proof. The function $F(r)$ is obviously an entire function of the variable r. In addition, one can directly verify that it is odd. Hence, it can be represented in the form $F(r) = rG(r^2)$. Then $G(x)$ is also an entire function of $x \in \mathbb{C}$. Really, for the coefficients $\langle a_k; k \in \mathbb{Z}_+\rangle$ of the decomposition

$$F(r) = \sum_{k=0}^{\infty} a_k r^{2k+1}$$

determining the entire function F, we have

$$\limsup_{k\to\infty} |a_k|^{1/(2k+1)} = 0\,.$$

Then the following decomposition of the function $G(x)$ is valid:

$$G(x) = \sum_{k=0}^{\infty} a_k x^k$$

. Here, the coefficients possess the property

$$\limsup_{k\to\infty} |a_k|^{1/k} = \left(\limsup_{k\to\infty} |a_k|^{1/2k+1}\right)^2 = 0\,.$$

Hence, the last power decomposition has the infinite radius of convergence.

Let us notice that $r^2 = \nu^2 + 2\nu\lambda$ is the entire function of λ. Then, according to (5),

$$[Q(\lambda)]^{-1} = \frac{1}{4\nu} G(r^2) e^{-\nu T}\,, \tag{7}$$

i.e. $[Q(\lambda)]^{-1}$ is the superposition of entire functions and, hence, it is the entire function of λ.

Lemma 2. *The zeros $\{\lambda_n; n \in \mathbb{Z}_+\}$ of the function $[Q(\lambda)]^{-1}$ are real and negative. They are defined by the formula*

$$\lambda_n = -\frac{\nu^2 + x_n^2}{2\sigma}\,, \quad n \in \mathbb{Z}_+, \tag{8}$$

where x_n are positive solutions of the equation

$$\mathrm{tg}\, xT = \frac{2\nu x}{x^2 - \nu^2}\,, \quad x \in \mathbb{R}\,. \tag{9}$$

They satisfy the inequalities

$$\frac{\pi}{T}\left(n - \frac{1}{2}\right) < x_n \le \frac{\pi}{T}\left(n + \frac{1}{2}\right), n \in \mathbb{N}, \tag{10}$$

and x_0 satisfies the inequality $\nu \le x_0 \le \pi/2T$ (x_0 exists only if $\nu \le \pi/2T$).

Proof. According to (7), the zeros of the function $[Q(\lambda)]^{-1}$ are solutions of the equation $G(r^2) = 0$. Then the set of zeros coincides with the set of solutions of the equation

$$e^{2rT} = \left(\frac{\nu - r}{\nu + r}\right)^2 \tag{11}$$

except the point $r = 0$ since

$$\lim_{r\to 0} [Q(\lambda)]^{-1} = \frac{e^{-\nu T}}{4\nu} \lim_{r\to 0} \frac{F(r)}{r} = \frac{e^{-\nu T}}{4\nu} F'(0) = \frac{\nu T}{2} e^{\nu T} \ne 0$$

is valid at this point. Equating the modules of both parts of (11), we obtain

$$\frac{\nu^2 + |r|^2 - 2\nu \mathrm{Re}\, r}{\nu^2 + |r|^2 + 2\nu \mathrm{Re}\, r} = e^{2\nu T\, \mathrm{Re}\, r}\,.$$

This equality is impossible at $\mathrm{Re}\, r \ne 0$ since, at $\mathrm{Re}\, r > 0$, the left-hand side is less than 1, and the right-hand side is more than 1, $\nu \ne 0$. At $\mathrm{Re}\, r < 0$, we have opposite inequalities. Thus, for the validity of (11), it is necessary that $\mathrm{Re}\, r = 0$. Putting $r = ix$, $x \in \mathbb{R}$, in (11) yields

$$\pm e^{ixT} = \frac{\nu - ix}{\nu + ix}\,,$$

whence

$$\pm \cos xT = \frac{\nu^2 - x^2}{\nu^2 + x^2}\,, \quad \pm \sin xT = -\frac{2\nu x}{\nu^2 + x^2}\,,$$

which is equivalent to (9). The solution $x = 0$ of this equation, i.e. $r = 0$, may not be taken into account. Further, if x is the solution of Eq. (9), then $(-x)$ is also its solution. But they give the same value $\lambda = -(x^2 + \nu^2)/2\sigma$. Then, to find zeros λ_n, $n \in \mathbb{N}$, of the function $[Q(\lambda)]^{-1}$, it is necessary to choose only one of them. Therefore, we consider further only positive solutions of Eq. (9).

The function $\mathrm{tg}\, xT$ grows on x, and the function $[-2\nu x/(\nu^2 - x^2)]$ decreases since it has the derivative $[-2\nu(\nu^2 + x^2)/(\nu^2 - x^2)^2]$. Since the function $\mathrm{tg}\, xT$ is periodic with period $\pi/2T$ and varies from $-\infty$ to ∞ on the period, there exists only one intersection of the graphs of these functions in each interval $(\pi(2n-1)/2T, \pi(2n+1)/2T]$. Each intersection gives one solution of Eq. (9). It is a simple solution since the derivatives of both the above-mentioned functions are finite and not equal to zero. Hence, their graphs are intersected transversally. Hence, Eq. (9) has the infinite set of simple solutions $x_n, n \in \mathbb{N}$, satisfying inequalities (10). In addition, this equation has the solution x_0 in the interval $(0, \pi/2T]$ satisfying the inequalities $\nu < x_0 \le \pi/2T$ provided $\nu \le \pi/2T$.

From this analysis, it follows that Eq. (8) has the infinite set of simple zeros λ_n, $n \in \mathbb{Z}_+$, $\lambda_n = -(2\sigma)^{-1}(\nu^2 + x_n^2)$, $n \in \mathbb{Z}_+$ and the zero λ_0 is realized only when the condition $\nu \le \pi/2T$ is valid. Thus, $|\lambda_0| > \nu^2/\sigma$.

Corollary 1. *The simple zeros of the function* $[Q(\lambda)]^{-1}$ *are real and negative poles of the function* $Q(\lambda)$. *Since they satisfy the condition* $|\lambda_n| \to \infty$ *as* $n \to \infty$ *in view of (8) and (10), the function* $Q(\lambda)$ *is meromorphic.*

Since

$$x_n > a = \min\{\nu, \pi/2T\}, \quad n \in \mathbb{Z}_+,$$

the function $Q(\lambda)$ *is analytic in a circle centered at* $\lambda = 0$ *with a radius not less than* $(\nu^2 + a^2)/2\sigma$.

Lemma 3. *The growth order of the entire function* $[Q(\lambda)]^{-1}$ *does not exceed 1/2.*

Proof. The growth order θ of the entire function is defined by

$$\theta = \lim_{x\to\infty} \sup \frac{\ln \ln \max\{|\,[Q(\lambda)]^{-1}\,|; |\lambda| = x\}}{\ln x}. \tag{12}$$

Due to (5) and (6), we get

$$|\,[Q(\lambda)]^{-1}\,| \le \frac{e^{-\nu T}}{2\nu|r|}(\nu + |r|)^2 e^{|r|T},$$

whence

$$\lim_{|r|\to\infty} \frac{\ln \ln |\,[Q(\lambda)]^{-1}\,|}{\ln |r|} \le \lim_{|r|\to\infty} \left(1 + \frac{\ln T}{\ln |r|}\right) = 1.$$

Since $|r| \le (\nu^2 + 2\sigma|\lambda|)^{1/2}$ is valid, i.e.

$$\lim_{|\lambda|\to\infty} \frac{\ln |r|}{\ln |\lambda|} \le \frac{1}{2},$$

relation (12) yields

$$\theta = \left[\lim_{|r|\to\infty} \frac{\ln \ln |\,[Q(\lambda)]^{-1}\,|}{\ln |r|}\right] \left[\lim_{|\lambda|\to\infty} \frac{\ln |r|}{\ln |\lambda|}\right] \le \frac{1}{2}.$$

Corollary 2. *Since the growth order of the function* $[Q(\lambda)]^{-1}$ *is less than 1, the Hadamard decomposition*

$$[Q(\lambda)]^{-1} = [Q(0)]^{-1} \prod_n \left(1 - \frac{\lambda}{\lambda_n}\right), \tag{13}$$

where $Q(0) = 1$, *is valid for it (see, for example,* [7]*).*

3. The multiplicative decomposition of the composite Poisson distribution

In this section, we demonstrate that, under certain conditions, the composite Poisson distribution can be represented as the convolution of an infinite sequence of Poisson distributions $p^{(l)}$, $l \in \mathbb{N}$ with step l.

Definition. *The lattice probability distribution* $p^{(l)} = \langle p_n^{(l)}; n \in \mathbb{Z}_+ \rangle$ *with step* $l \in \mathbb{N}$ *is called the Poisson distribution with parameter* $\alpha_l \in \mathbb{R}_+$ *and step* $l \in \mathbb{N}$ *if*

$$p_n^{(l)} = \{\left(\alpha_l^m e^{-\alpha_l}/m!\right), \ if \ n = ml, m \in \mathbb{Z}_+ ; \ \ 0; \ if \ n \neq ml\}. \tag{14}$$

The generator function of the probability distribution $p^{(l)}$ is presented in the following form:

$$H_l(x) = e^{-\alpha_l} \sum_{m=0}^{\infty} \frac{\alpha_l^m}{m!} x^{lm} = \exp\left(\alpha_l(x^l - 1)\right) . \tag{15}$$

Let $\tilde{n}$ be the lattice random variable with values in $\mathbb{Z}_+$. It is distributed according to a composite Poisson distribution, i.e. the probabilities $\Pr\{\tilde{n} = n\}$, $n \in \mathbb{Z}_+$, of its values are determined by the formula

$$p_n = \frac{1}{n!} \mathsf{E} \tilde{J}^n \exp(-\tilde{J}) , \tag{16}$$

where $\tilde{J}$ is a random variable which takes its values on $[0, \infty)$. Let its characteristic function $\mathsf{E}e^{it\tilde{J}}$ be analytic in a circle with the center at $t = 0$. Then the power series of this function converging in this circle is determined as

$$\mathsf{E}e^{it\tilde{J}} = \sum_{l=0}^{\infty} \frac{(it)^l}{l!} M_l , \tag{17}$$

where $M_l = \mathsf{E}\tilde{J}^l$, $l \in \mathbb{Z}_+$, $M_0 = 1$ are the moments of the random variable $\tilde{J}$. In addition, the power series of the logarithm of the characteristic function is defined, i.e.

$$\ln \mathsf{E}e^{it\tilde{J}} = \sum_{l=0}^{\infty} \frac{(it)^l}{l!} K_l . \tag{18}$$

It converges in a circle that has, generally speaking, a smaller radius with the same center. The coefficients K_l, $l \in \mathbb{Z}_+$, $K_0 = 0$ of this series are named *cumulants* of the random variable $\tilde{J}$. All of them are finite. If there are no zeroes of the function $\mathsf{E}e^{it\tilde{J}}$, then the convergence radii of both decompositions (17) and (18) coincide.

Thus, in the common convergence circle of both series, the following formula is valid:

$$\sum_{l=0}^{\infty} \frac{(it)^l}{l!} M_l = \exp\left(\sum_{l=0}^{\infty} \frac{(it)^l}{l!} K_l\right) . \tag{19}$$

Let

$$H(z) = \sum_{n=0}^{\infty} z^n p_n \tag{20}$$

be the generator function of the random variable $\tilde{n}$. It is obviously analytic in the unit circle. According to the transformations

$$H(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \mathsf{E}\tilde{J}^n e^{-\tilde{J}} = \mathsf{E} \exp\left((z-1)\tilde{J}\right) =$$

$$= \mathsf{E} \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \tilde{J}^n = \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} M_n$$

and to the analyticity property of the function $Q(\lambda)$, the generator function $H(z)$ is analytic in a circle with the center at the point $z = 1$. Then there exists a negative real λ such that the series

$$H(e^{-\lambda}) = \sum_{n=0}^{\infty} e^{-\lambda n} p_n \tag{21}$$

converges and, hence, the convergence radius of the power series (20) is greater than unity due to the fact that

$$\left|\sum_{n=0}^{\infty} p_n z^n\right| \leq \sum_{n=0}^{\infty} p_n e^{-\lambda n} < \infty\,, \quad z \leq e^{-\lambda}.$$

In particular, it follows from the above inequality that all the moments $\mathsf{E}\tilde{n}^l$, $l \in \mathbb{N}$, of the random variable $\tilde{n}$ are finite,

$$\mathsf{E}\tilde{n}^l = \sum_{n=0}^{\infty} p_n n^l < \infty\,.$$

Since $H(0) = \mathsf{E}e^{-\tilde{J}} \neq 0$ and $H'(0) = \mathsf{E}\tilde{J}e^{-\tilde{J}} < M_1$, the analytic function $\ln\, H(z)$ is defined in the neighborhood of the point $z = 0$. Therefore, in this neighborhood, the decomposition

$$\ln\, H(z) = \sum_{l=0}^{\infty} \frac{\bar{K}_l}{l!} z^l \tag{22}$$

is valid. We name the coefficients $\bar{K}_l, l \in \mathbb{N}$ as *reduced cumulants*.

Further, let all the reduced cumulants $\bar{K}_l$ with numbers $l \in \mathbb{N}$ be nonnegative. Then, from (22), we have the series

$$\sum_{l=1}^{\infty} \frac{\bar{K}_l}{l!} z^{l-1} = \frac{1}{z}(\ln\, H(z) - \bar{K}_0)$$

having nonnegative coefficients and converging in a circle with a nonzero radius. The modulus maximum of this analytic function is attained on the positive part of the real axis (see, for example, [9]). Then, if $z_* > 0$ is the divergence point of this series, $H(z_*) = \infty$ is valid. Hence, $z_* > e^{-\lambda}$ and, therefore, its convergence radius is not less than the convergence radius of series (20). From here, it follows that series (22) converges at the point $z = 1$, too. Since $H(1) = 1$, the identity

$$\sum_{l=0}^{\infty} \frac{1}{l!} \bar{K}_l = 0 \tag{23}$$

follows from (22) at $z = 1$. This means that

$$-\sum_{l=1}^{\infty} \frac{1}{l!} \bar{K}_l = \bar{K}_0\,.$$

Then formula (22) takes the form

$$\ln\ H(z) = \sum_{l=1}^{\infty} \frac{z^l - 1}{l!} \bar{K}_l\,. \tag{24}$$

Putting $z = e^{it}$ in (22), we obtain

$$\ln \mathsf{E}e^{it\tilde{n}} = \sum_{l=1}^{\infty} \frac{e^{itl} - 1}{l!} \bar{K}_l\,. \tag{25}$$

This decomposition gives us the Kolmogorov representation

$$\ln \mathsf{E}e^{it\tilde{n}} = it\mathsf{E}\tilde{n} + \int_0^{\infty} \left(e^{itx} - 1 - itx\right) \frac{d\mu(x)}{x^2}$$

of the characteristic function of the infinitely divisible distribution corresponding to the nonnegative random variable with the finite second moment. Here, the measure $\mu(x)$ is a monotone nondecreasing function determining the finite measure on $[0, \infty)$ (see, for example, [8]). In our case, this measure is determined by the formula

$$\mu(x) = \sum_{l=1}^{\infty} \frac{l\bar{K}_l}{(l-1)!}\,\chi(x-l)\,,$$

where $\chi(x)$ is the indicator function of the set $[0,\infty)$. The finiteness of the measure μ, i.e. the convergence of the series

$$\mu(\infty) = \sum_{l=1}^{\infty} \frac{l\bar{K}_l}{(l-1)!} < \infty$$

, is guaranteed by the finiteness of the second moment $\mathsf{E}\tilde{n}^2$ of the random variable $\tilde{n}$.

At last, let us consider the arbitrary pair of distributions (14) $p^{(l)}$ and $p^{(m)}$, $l,m \in \mathbb{N}$, with the generator functions (15) which have $\alpha_l = \left(\bar{K}_l/l!\right)$, $\alpha_m = \left(\bar{K}_m/m!\right)$. The lattice distribution which is formed by the operation $\circ$ of convolution applying to this pair lattice distributions has components determined by the formula

$$(p^{(l)} \circ p^{(m)})_n = \sum_{k=0}^{n} p_k^{(l)} p_{n-k}^{(m)}\,, \quad n \in \mathbb{Z}_+\,.$$

Therefore, its characteristic function is the product $H_l(x)H_m(x)$. Then the distribution p corresponds to the generator function

$$H(x) = \exp\left(\sum_{l=1}^{\infty} \frac{x^l - 1}{l!}\bar{K}_l\right) = \prod_{l=1}^{\infty} H_l(x)\,.$$

Thus, this distribution is performed as the infinite convolution

$$p = p^{(1)} \circ p^{(2)} \circ \ldots \circ p^{(l)} \circ \ldots \tag{26}$$

of the Poisson distributions (14), where $p^{(l)}$ has step l and parameter $\alpha_l = \left(\bar{K}_l/l!\right)$.

The infinite convolution (26) converges component-wise, since the Poisson distributions $p^{(l)}$ with step $l > n$ have the first nonzero component $(p^{(l)})_k$ at such $k > 0$ that $k = l$. Then, for each fixed $n \in \mathbb{Z}_+$, the following formula is valid:

$$p_n = \left(p^{(1)} \circ \ldots \circ p^{(n)}\right)_n \left(p_0^{(n+1)} p_0^{(n+2)} \ldots\right)\,.$$

The expression in the first bracket is a finite sum, and the infinite product of null components in the second bracket converges since

$$\prod_{l=n+1}^{\infty} e^{-\alpha_l} = \exp\left(-\sum_{l=n+1}^{\infty} \alpha_l\right) = \exp\left(-\sum_{l=n+1}^{\infty} \frac{\bar{K}_l}{l!}\right)\,.$$

We summarize the above analysis in the following statement.

Theorem 1. *Let the random variable $\tilde{J}$ be concentrated on $\mathbb{R}_+$, let its characteristic function be analytic in a circle with a nonzero radius, and let its reduced cumulants $\bar{K}_l$, $l \in \mathbb{N}$ be nonnegative. Then the composite Poisson distribution p defined by (16) and constructed on the basis of the random variable $\tilde{J}$ is infinitely divisible. It is represented as the component-wise converging infinite convolution (26) constructed on the basis of Poisson distributions $p^{(l)}$ which have steps l and parameters $\alpha_l = \left(\bar{K}_l/l!\right)$, $l \in \mathbb{N}$, respectively.*

4. The multiplicative decomposition of the Mandel distribution

In this section, we demonstrate that Theorem 1 may be applied to the Mandel distribution that will lead us to the basic result of the work.

Theorem 2. *If the parameters ν, σ of the random variable $\tilde{J}_T$ satisfy the condition $\nu^2/2\sigma > 1$, then its reduced cumulants are positive, and they are determined by the formula*

$$\bar{K}_l = \sum_{n=1}^{\infty} \frac{(l-1)!}{(1+|\lambda_n|)^l} > 0\,.$$

Proof. From expression (13) for the function $[Q(\lambda)]^{-1}$, taking into account the negativity of zeros λ_n, $n \in \mathbb{Z}_+$, we obtain

$$\ln \mathsf{E} e^{\lambda \tilde{J}} = \ln Q(-\lambda) = -\sum_{n=1}^{\infty} \ln \left(1 - \frac{\lambda}{|\lambda_n|}\right) = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=1}^{\infty} \left(\frac{\lambda}{|\lambda_n|}\right)^m .$$

Hence, according to (18), the cumulants of the random variable $\tilde{J}$ are

$$K_m = \sum_{n=1}^{\infty} \frac{(m-1)!}{|\lambda_n|^m}\,. \tag{27}$$

Let the condition $\nu^2/2\sigma > 1$ be fulfilled. According to Corollary 1, all the moduli of the zeros λ_n, $n \in \mathbb{Z}_+$ exceed 1, and therefore the function $Q(\lambda)$ is analytic in the circle centered at $\lambda = 0$ and with radius being more than 1. Then the decomposition

$$\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} M_n = H(z)$$

converges at $|z-1| \le 1+\varepsilon$ at an $\varepsilon > 0$ being sufficiently small. Since there are no zeros of the function $Q(\lambda)$, the power series of ln $Q(\lambda)$ has the convergence radius greater than 1. Applying the logarithm operation to both sides of the last equality and using the definition of reduced cumulants, we obtain the equality of decompositions converging as $|z-1| \le 1+\varepsilon$:

$$\sum_{m=0}^{\infty} \frac{(z-1)^m}{m!} K_m = \sum_{l=0}^{\infty} \frac{z^l}{l!} \bar{K}_l\,.$$

The decomposition in the left-hand side of the equality converges in the neighborhood of the point $z = 0$. Therefore, by differentiating both sides of this equality with respect to z l times and by putting $z = 0$, we obtain

$$\bar{K}_l = \left(\frac{d^l}{dz^l} \sum_{m=0}^{\infty} \frac{(z-1)^m}{m!} K_m \right)_{z=0} .$$

Due to the fact that the power series converges uniformly in its convergence region, all differentiations commute with the summation. As a result, we obtain the following formula for reduced cumulants:

$$\bar{K}_l = \sum_{m=0}^{\infty} K_{m+l} \frac{(-1)^m}{m!}\,. \tag{28}$$

The calculation of the reduced cumulants on the basis of formula (28) gives

$$\bar{K}_l = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} K_{m+l} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{n=1}^{\infty} \frac{(m+l-1)!}{|\lambda_n|^{m+l}} =$$

$$= \sum_{n=1}^{\infty} |\lambda_n|^{-l} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{(m+l-1)!}{|\lambda_n|^m} =$$

$$= \sum_{n=1}^{\infty} |\lambda_n|^{-l} \left[\left(\frac{d}{dy}\right)^{l-1} \sum_{m=0}^{\infty} (-1)^m y^{m+l-1} \right]_{y=|\lambda_n|^{-1}} =$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{l-1}}{|\lambda_n|^l} \left[\left(\frac{d}{dy} \right)^{l-1} \sum_{m=0}^{\infty} (-1)^m y^m \right]_{y=|\lambda_n|^{-1}} =$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{l-1}}{|\lambda_n|^l} \left[\left(\frac{d}{dy} \right)^{l-1} (1+y)^{-1} \right]_{y=|\lambda_n|^{-1}} =$$

$$= (l-1)! \sum_{n=1}^{\infty} |\lambda_n|^{-l} \left(1 + |\lambda_n|^{-1}\right)^{-l} = (l-1)! \sum_{n=1}^{\infty} (1+|\lambda_n|)^{-l} > 0 \,.$$

Then it follows that $\bar{K}_l \geq 0$ and

$$\bar{K}_l = \sum_{n=1}^{\infty} \frac{(l-1)!}{(1+|\lambda_n|)^l} \leq K_l \,, \quad l \in \mathbb{N} \,.$$

From the proved statement, on the basis of Theorems 1 and 2, we come to the basic result of the work.

Theorem 3. *If the condition $\nu^2/2\sigma > 1$ for the parameters ν, σ determining the probability distribution of the random variable $\tilde{J}_T$ is fulfilled, then the Mandel distribution is infinitely divisible. Thus, it decomposes into the infinite convolution*

$$P = p^{(1)} \circ p^{(2)} \circ \ldots \circ p^{(n)} \circ \ldots$$

of Poisson distributions $p^{(l)}$, $l \in \mathbb{N}$, with steps l and parameters

$$\alpha_l = \frac{1}{l} \sum_{n=1}^{\infty} \frac{(l-1)!}{(1+|\lambda_n|)^l} \,.$$

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Kharkiv 61000, Single Crystal Institute of NANU; Belgorod 308000, Belgorod State University

E-mail: virch@isc.kharkov.ua