A Minimal Polynomial for Finding the Switching Instants and the Support Vector of the Controllability Domain

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One of the directions in the development of linear time optimization theory is based on its relationship with the classical moment problem. The investigation of the time optimization problem for the canonical system

$$\begin{aligned} &\dot{x}_1 = u, \qquad |u| \le 1, \qquad \dot{x}_i = x_{i-1}, \qquad i = 2, \ldots, n, \\ &x(0) = x, \qquad x(\Theta) = 0, \qquad \Theta \to \min \end{aligned} \tag{1}$$

is important in this connection.

By [1], this problem is equivalent to the power moment problem on the minimal possible interval. This approach has permitted obtaining [1, 2] an analytic solution of problem (1) for a system of arbitrary order n, including methods for finding the control $u(t)$, the optimal time Θ, and the switching instants $T_1, T_2, \ldots, T_{n-1}$ [points of discontinuity of the function $u(t)$]. The explicit form of the polynomials $\Delta_n(\Theta, x)$ and $\tilde{\Delta}_n(\Theta, x)$ of degree p^2 for $n = 2p - 1$ and degree $p(p+1)$ for $n = 2p$ such that the maximum real root Θ_0 of the equation $\Delta_n(\Theta, x)\tilde{\Delta}_n(\Theta, x) = 0$ is the optimal time from x to 0 was given in [1]. After finding the optimal time, the next step is to find the switching instants for the control $u(t)$. In the method suggested in [1], one successively seeks the switching instants as the minimal roots of the corresponding polynomials, whose total degree is equal to $p(p-1)$ for $n = 2p - 1$ and p^2 for $n = 2p$. The procedure is rather cumbersome in that the sequence of polynomials determining the next switching instant is evaluated only after the previous instant has been found.

In the present paper, we show that finding switching instants can be implemented in a simpler way in the framework of the theory constructed in [1]. More precisely, we explicitly write out a polynomial of degree $n-1$ such that all switching instants $T_1, T_2, \ldots, T_{n-1}$ are its roots. Furthermore, by multiplying the coefficients of the polynomial by some known factors, one obtains the components of the support vector at the point x of the domain of controllability in time Θ_0. This is another advantage of this polynomial, since the support vector is used in the solution of optimal control problems [3].

Let $u(t)$ be a time optimal control from x to 0 for system (1). This is a piecewise constant function that has at most $n-1$ points of discontinuity [4] and assumes the values ± 1. Since $u(t)$ is piecewise continuous, it follows from the relation

$$x = -\int_0^{\Theta} e^{-A\tau} b u(\tau) d\tau, \tag{2}$$

where A is an $n \times n$ matrix with unit entries on the first subdiagonal and remaining entries equal to zero and $b = \mathrm{col}(1, 0, \ldots, 0)$ is an n-dimensional vector, that

$$(-1)^k x_k = (-1)^{n+1} \frac{2\tilde{u}}{k!} \left[\sum_{j=1}^{n-1} (-1)^{j+1} T_j^k + (-1)^{n+1} \frac{T_n^k}{2} \right], \qquad k = 1, \ldots, n. \tag{3}$$

Here and in the following, $T_n = \Theta$ and $\tilde{u}$ is the control on the last interval $[T_{n-1}, T_n]$, i.e., either $\tilde{u} = +1$ or $\tilde{u} = -1$. Just as in [2], here we have two systems of relations, one for initial points x brought into the origin by a control of the first kind ($\tilde{u} = -1$) and the other for the case in which $\tilde{u} = +1$ (a control of the second kind). Since $\tilde{u} = \pm 1$ (i.e., $\tilde{u} = 1/\tilde{u}$), it follows that relation (3) can be rewritten in the form

$$G_k = (-1)^n \sum_{j=1}^{n-1} (-1)^{j+1} T_j^k, \qquad k = 1, \ldots, n; \tag{4}$$

here we have used the notation $G_k = \left[T_n^k + (-1)^{k+1} \tilde{u} k! x_k\right]/2$, $k = 1, \ldots, n$.

Following [1, 2], to find the optimal time Θ, the time optimal control $\tilde{u}$, and the switching instants $T_1, \ldots, T_{n-1}$, we introduce the sequence of polynomials $\{\gamma_k(x, \Theta, \tilde{u})\}$ given by the recursion formula

$$\gamma_1 = \frac{\Theta + \tilde{u} x_1}{2}, \qquad \gamma_k = \frac{1}{k}\left(G_k - \sum_{i=1}^{k-1} \gamma_i G_{k-i}\right), \qquad k = 2, \ldots, n \tag{5}$$

(we also set $\gamma_0 = -1$).

As was mentioned above, the optimal time Θ_0 is the maximum real root of the equation

$$\Delta_n(\gamma_1(x, \Theta, -1), \ldots, \gamma_n(x, \Theta, -1))\, \Delta_n(\gamma_1(x, \Theta, 1), \ldots, \gamma_n(x, \Theta, 1)) = 0.$$

Here Δ_n (with $n = 2p$ or $n = 2p - 1$) is given by the relations

$$\Delta_{2p-q} = \det\left(\gamma_{i+j-q}\right)_{i,j=1}^{p}, \qquad q \in \{0, 1\}, \tag{6}$$

where i is the row index and j the column index of the corresponding matrix. Moreover, if $\tilde{u} = -1$, then Θ_0 is the maximum real root of the equation $\Delta_n(\gamma_1(x, \Theta, -1), \ldots, \gamma_n(x, \Theta, -1)) = 0$; if $\tilde{u} = +1$, then Θ_0 is the maximum real root of the equation $\Delta_n(\gamma_1(x, \Theta, +1), \ldots, \gamma_n(x, \Theta, +1)) = 0$, i.e., Θ_0 is the maximum real root of the equation

$$\Delta_n(\gamma_1(x, \Theta, \tilde{u}), \ldots, \gamma_n(x, \Theta, \tilde{u})) = 0. \tag{7}$$

Theorem 1. *The switching instants $T_1, \ldots, T_{n-1}$ are found from the equation*

$$\sum_{l=1}^{n} t^{l-1} \sum_{k=l}^{n} \frac{\partial \Delta_n(\gamma_1(x, \Theta_0, \tilde{u}), \ldots, \gamma_n(x, \Theta_0, \tilde{u}))}{\partial \gamma_k(x, \Theta_0, \tilde{u})} \gamma_{k-l}(x, \Theta_0, \tilde{u}) = 0, \tag{8}$$

where x is the initial point, Θ_0 is the optimal time from x into 0, and $\tilde{u}$ is the control on the last interval $[T_{n-1}; \Theta_0]$.

The proof is based on the following assertion.

Lemma. *One has*

$$\partial \gamma_k / \partial G_k = 1/k, \qquad \partial \gamma_k / \partial G_{k-j} = -\gamma_j/(k-j), \qquad j = 1, \ldots, k-1. \tag{9}$$

Proof of the theorem. It follows from (4)–(6) that, for given x and the above-defined Θ_0 and $\tilde{u}$, the determinants Δ_n are functions of $T_1, \ldots, T_{n-1}$ [$\Delta_n = \Delta_n(\gamma_1, \ldots, \gamma_n)$; $\gamma_k = \gamma_k(G_1, \ldots, G_k)$, $k = 1, \ldots, n$; and $G_l = G_l(T_1, \ldots, T_{n-1})$, $l = 1, \ldots, n$].

For these Θ_0 and $\tilde{u}$, Eq. (7) becomes an identity. Differentiating it with respect to T_j, we obtain the system

$$\partial \Delta_n / \partial T_j = 0, \qquad j = 1, \ldots, n-1,$$

or

$$\frac{\partial \Delta_n}{\partial G_1} \frac{\partial G_1}{\partial T_j} + \frac{\partial \Delta_n}{\partial G_2} \frac{\partial G_2}{\partial T_j} + \cdots + \frac{\partial \Delta_n}{\partial G_n} \frac{\partial G_n}{\partial T_j} = 0, \qquad j = 1, \ldots, n-1. \tag{10}$$

It follows from (4) that

$$\partial G_k/\partial T_j = (-1)^{n+j+1} k T_j^{k-1}. \tag{11}$$

Substituting (11) into (10), we obtain

$$(-1)^{n+j+1}\left(\frac{\partial \Delta_n}{\partial G_1} + 2\frac{\partial \Delta_n}{\partial G_2}T_j + 3\frac{\partial \Delta_n}{\partial G_3}T_j^2 + \cdots + n\frac{\partial \Delta_n}{\partial G_n}T_j^{n-1}\right) = 0, \qquad j = 1, \ldots, n-1.$$

This means that the switching instants $T_1, \ldots, T_{n-1}$ are the only roots of the equation

$$\partial \Delta_n/\partial G_1 + 2\left(\partial \Delta_n/\partial G_2\right)t + 3\left(\partial \Delta_n/\partial G_3\right)t^2 + \cdots + n\left(\partial \Delta_n/\partial G_n\right)t^{n-1} = 0,$$

or

$$\sum_{l=1}^{n} l\frac{\partial \Delta_n}{\partial G_l}t^{l-1} = 0. \tag{12}$$

For a given initial point x and for these Θ_0 and $\tilde{u}$, the left-hand side of Eq. (12) is a polynomial of degree $n-1$, which has $n-1$ real roots.

Taking account of the relation $\partial \Delta_n/\partial G_l = \sum_{k=l}^{n}\left(\partial \Delta_n/\partial \gamma_k\right)\left(\partial \gamma_k/\partial G_l\right)$, $l = 1, \ldots, n$ (here $\partial \gamma_k/\partial G_l = 0$ for $k < l$) and using relations (9) with $l = k - j$, we obtain

$$\frac{\partial \Delta_n}{\partial G_l} = -\frac{1}{l}\sum_{k=l}^{n}\frac{\partial \Delta_n}{\partial \gamma_k}\gamma_{k-l}.$$

Substituting the last expression into (12), we arrive at (8), which completes the proof.

Let us now find the derivatives $\partial \Delta_n/\partial \gamma_k$. From the form of the determinants (6), we find that the derivative $\partial \Delta_n/\partial \gamma_k$ is the sum of the cofactors of γ_k. By $\Delta_n^{(i,j)}$ we denote the determinant obtained from the determinant Δ_n by deleting the ith row and the jth column. Then

$$\frac{\partial \Delta_{2p-1}}{\partial \gamma_k} = (-1)^{k+1}\sum_{\substack{i+j=k+1\\ 1\le i,j\le p}}\Delta_{2p-1}^{(i,j)}, \qquad k = 1, \ldots, 2p-1;$$

$$\frac{\partial \Delta_{2p}}{\partial \gamma_k} = (-1)^{k}\sum_{\substack{i+j=k\\ 1\le i,j\le p}}\Delta_{2p}^{(i,j)}, \qquad k = 1, \ldots, 2p.$$

Substituting the resulting expressions into (8), we obtain the following equations for all switching instants:

$$\sum_{l=1}^{2p-1} t^{l-1}\sum_{k=l}^{2p-1}(-1)^{k+1}\gamma_{k-l}\sum_{\substack{i+j=k+1\\ 1\le i,j\le p}}\Delta_{2p-1}^{(i,j)} = 0$$

for $n = 2p-1$ and

$$\sum_{l=1}^{2p} t^{l-1}\sum_{k=l}^{2p}(-1)^{k}\gamma_{k-l}\sum_{\substack{i+j=k\\ 1\le i,j\le p}}\Delta_{2p}^{(i,j)} = 0$$

for $n = 2p$. The proof of the theorem is complete.

Consider the time optimization problem for system (1). We rewrite it in the vector form

$$\dot{x} = Ax + bu, \qquad |u| \le 1, \qquad x(0) = x_0, \qquad x(\Theta) = 0, \tag{13}$$

where the $n \times n$ matrix A and the n-dimensional vector b are the same as in (2).

By $S(\Theta)$ we denote the zero controllability domain for system (13) in time Θ, i.e.,

$$S(\Theta) = \left\{x:\ x = -\int_0^{\Theta} e^{-A\tau}bu(\tau)d\tau\right\},$$

where $u(\tau)$ is an arbitrary measurable function such that $|u(\tau)| \le 1$. Here x_0 is a boundary point of the set $S(\Theta)$.

Theorem 2. *The vector*

$$g = \left(\sum_{k=1}^{n} \frac{\partial \Delta_n}{\partial \gamma_k} \gamma_{k-1}; -\sum_{k=2}^{n} \frac{\partial \Delta_n}{\partial \gamma_k} \gamma_{k-2}; \ldots; (-1)^{n-2}(n-2)! \sum_{k=n-1}^{n} \frac{\partial \Delta_n}{\partial \gamma_k} \gamma_{k-n+1}; (-1)^n (n-1)! \frac{\partial \Delta_n}{\partial \gamma_n} \right) \tag{14}$$

is a support vector of the controllability domain $S(\Theta)$ *of system* (1) *at the point* $x(0)$.

Proof. Let A^* be the transpose of the matrix A of system (13). We consider the adjoint system of (13) in reverse time for the auxiliary variables $\psi_1, \psi_2, \ldots, \psi_n$:

$$\dot{\psi} = -A^* \psi. \tag{15}$$

As the initial vector $\psi(0) = \psi^0 = (\psi_1^0, \psi_2^0, \ldots, \psi_n^0)$ we take the support vector of the controllability domain $S(\Theta)$ at the point x_0 [5]. System (15) has the nontrivial solution $\psi(t) = \psi^0 e^{-A^* t}$.

Let $u^*(t)$ be the time optimal control for system (1) from x_0 to 0; by [5], it has the form

$$u^*(t) = \operatorname{sgn}(-\psi(t), b) = \operatorname{sgn}\left(-\psi^0 e^{-A^* t}, b\right) = -\operatorname{sgn}\left(\psi^0, e^{-At} b\right). \tag{16}$$

The inner product $(\psi^0, e^{-At} b)$ is the polynomial

$$\left(\psi^0, e^{-At} b\right) = \sum_{l=0}^{n-1} (-1)^l \psi_{l+1}^0 t^l / l!, \tag{17}$$

whose roots, by (16), are the switching instants. Consequently, neglecting a constant factor C, the polynomial (17) coincides with the left-hand side of Eq. (8); i.e., we can write

$$\sum_{l=0}^{n-1} \frac{(-1)^l \psi_{l+1}^0 t^l}{l!} = C \sum_{l=0}^{n-1} t^l \sum_{k=l+1}^{n} \frac{\partial \Delta_n}{\partial \gamma_k} \gamma_{k-l-1}.$$

Hence, $(-1)^l \psi_{l+1}^0 / l! = C \sum_{k=l+1}^{n} \left(\partial \Delta_n / \partial \gamma_k\right) \gamma_{k-l-1}$, or

$$\psi_{l+1}^0 = (-1)^l C l! \sum_{k=l+1}^{n} \left(\partial \Delta_n / \gamma_k\right) \gamma_{k-l-1}, \qquad l = 0, \ldots, n-1.$$

By setting $C = 1$, we find that the vector

$$g = \left(\sum_{k=1}^{n} \frac{\partial \Delta_n}{\partial \gamma_k} \gamma_{k-1}; -\sum_{k=2}^{n} \frac{\partial \Delta_n}{\partial \gamma_k} \gamma_{k-2}; \ldots; (-1)^{n-1}(n-1)! \frac{\partial \Delta_n}{\partial \gamma_n} \gamma_0 \right)$$

is the support vector of the controllability domain $S(\Theta)$ of system (1) at the point x_0. Since $\gamma_0 = -1$, we obtain the vector (14).

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