REPRESENTATION OF NATURAL NUMBERS BY SUMS OF FOUR SQUARES OF INTEGERS HAVING A SPECIAL FORM

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Abstract. This paper obtains an asymptotic formula for the number of solutions to the equation $l_1^2 + l_2^2 + l_3^2 + l_4^2 = N$ in integers l_1, l_2, l_3, l_4 such that $a < \{\eta l_j\} < b$, where η is a quadratic irrational number, $0 \le a < b \le 1$, $j = 1, 2, 3, 4$.

1. Introduction

In 1770, Lagrange proved that each natural number N can be represented as the sum of four squares of integers:

$$l_1^2 + l_2^2 + l_3^2 + l_4^2 = N. \tag{1.1}$$

Denote by $I(N)$ the number of representations (1.1).The following asymptotic formula is known for $I(N)$ (see [4]):

$$I(N) = \pi^2 N \sum_{1 \le q} \frac{1}{q^4} \sum_{\substack{1 \le a \le q \\ (a,q)=1}} S_{q,a}^4 e^{-2\pi i N a/q} + O(N^{17/18+\varepsilon}),$$

where

$$S_{q,a} = \sum_{j=1}^{q} e^{2\pi i a j^2/q}$$

is the Gaussian sum, and ε is an arbitrary positive number. Let us consider a subset of the set of integers having the form

$$\mathcal{A} = \{l \mid a < \{\eta l\} < b\},$$

where η is a quadratic irrational number, and a and b are arbitrary numbers from the closed interval $[0, 1]$. Denote by $J(N)$ the number of solutions of Eq. (1.1) in integers from the special set $\mathcal{A}$.

Theorem 1.1. *The following formula holds for any positive small* ε:

$$J(N) = (b - a)^4 I(N) + O(N^{0,9+3\varepsilon}).$$

2. Auxiliary Assertions

Lemma 2.1 (see [5]). *For any tuple $a, b, c, d \in \mathbb{Z}$ such that $ad - bc = 1$, ab and cd are even and for any ξ, $\xi^8 = 1$, the following relation holds:*

$$\theta\left(\frac{z}{cr+d}; \frac{ar+b}{cr+d}\right) = \xi(cr+d)^{1/2} \exp\left(\frac{\pi i c z^2}{cr+d}\right) \theta(z, r),$$

where

$$\theta(z, r) = \sum_{n=-\infty}^{\infty} \exp(\pi i n^2 r + 2\pi i n z)$$

is an analytic function and $z, r \in \mathbb{C}, \operatorname{Im} r > 0$.

Lemma 2.2 (see [1, 3]). *The following assertions hold for the Gauss sum*

$$S(q, a, b) = \sum_{1 \le j \le q} e^{2\pi i (aj^2 + bj)/q}, \quad S_{q,a} = S(q, a, 0):$$

(1) *if $(q, a) = d$, then*

$$S(q, a, b) = \begin{cases} dS(q/d, a/d, b/d) & \text{if} \quad d \mid b, \\ 0 & \text{if} \quad d \nmid b; \end{cases}$$

(2) *if $(a, 2) = 1$, then*

$$S_{2^\gamma, a} = \begin{cases} 0 & \text{if} \quad \gamma = 1, \\ 2^{\gamma/2}(1 + i^\gamma) & \text{if} \quad \gamma \text{ is even}, \\ 2^{(\gamma+1)/2} e^{2\pi i \gamma/8} & \text{if} \quad \gamma > 1 \text{ is odd}; \end{cases}$$

(3) *if $(q, 2a) = 1$, then*

$$S(q, a, b) = e^{-2\pi i \overline{4a} b^2/q} \left(\frac{a}{q}\right) S_{q,1}, \quad 4a\overline{4a} \equiv 1 \pmod{q};$$

(4) *if $(q, 2) = 1$, then*

$$S_{q,1} = \begin{cases} \sqrt{q} & \text{if} \quad q \equiv 1 \pmod 4, \\ i\sqrt{q} & \text{if} \quad q \equiv -1 \pmod 4. \end{cases}$$

Lemma 2.3 (see [2, 7]).

$$K(q, a, b) \ll \tau(q) q^{1/2} (a, b, q)^{1/2},$$

where $K(q, a, b) = \sum_{\substack{1 \le j \le q \\ (j,q)=1}} e^{2\pi i (aj + b\bar{j})/q}$ is the Kloosterman sum, $j\bar{j} \equiv 1 \pmod{q}$.

3. Proof of Theorem 1.1

1. Let us extend the function

$$\psi_0(x) = \begin{cases} 1 & \text{if} \quad a < x < b, \\ 0 & \text{if} \quad 0 \le x \le a \quad \text{or} \quad b \le x \le 1 \end{cases}$$

to the whole real line in a periodic way with period 1. Let

$$S_k(\beta) = \sum_{l=-\infty}^{\infty} e^{2\pi i \beta l^2 - l^2/N} \psi_k(\eta l), \quad k = 0, 1, 2;$$

then

$$J(N) = e \int_0^1 S_0^4(\beta) e^{-2\pi i \beta N} d\beta.$$

By the I. M. Vinogradov "small glass" lemma (see [6, p. 22]), we choose $r = [\log N]$, $\Delta = N^{0,1}$. For $\alpha = a + \Delta/2$ and $\beta = b - \Delta/2$, the function ψ from this lemma is denoted by ψ_1. Setting $\alpha = a - \Delta/2$ and $\beta = b + \Delta/2$, denote by ψ_2 the corresponding function ψ. Then the following inequality hold:

$$J_1(N) \le J(N) \le J_2(N),$$

where

$$J_k(N) = e \int_0^1 S_k^4(\beta) e^{-2\pi i \beta N} d\beta, \quad k = 1, 2. \tag{3.1}$$

For $J_1(N)$ and $J_2(N)$, let us deduce asymptotic formulas whose principal terms are identical. Let us substitute the Fourier series expansion

$$\psi_k(\eta l) = \sum_{|m| \le r\Delta^{-1}} c_k(m) e^{2\pi i m \eta l} + O(N^{-\log \pi})$$

of the function $\psi_k(\eta l)$ in (3.1):

$$J_k(N) = e \sum_{|m_1| \le r\Delta^{-1}} c_k(m_1) \sum_{|m_2| \le r\Delta^{-1}} c_k(m_2) \sum_{|m_3| \le r\Delta^{-1}} c_k(m_3) \sum_{|m_4| \le r\Delta^{-1}} c_k(m_4)$$
$$\times \int_0^1 S(\beta, m_1) S(\beta, m_2) S(\beta, m_3) S(\beta, m_4) e^{-2\pi i \beta N} d\beta + O(N^{2-\log \pi}),$$

where

$$S(\beta, m) = \sum_{l=-\infty}^{\infty} e^{2\pi i \beta l^2 - l^2/N + 2\pi i m \eta l}.$$

Note that for $m_1 = m_2 = m_3 = m_4 = 0$. the principal term $J_k(N)$ has the form

$$c_k^4(0) I(N).$$

2. In the sum $S(\beta, m)$, for $0 < |m| \le r/\Delta$, let us set

$$\beta = \frac{d}{q} + y_1, \quad (d, q) = 1, \quad 1 \le q \le \tau, \quad \tau = [\sqrt{N}], \quad |y_1| < \frac{1}{q\tau},$$
$$l = j + qn.$$

Then

$$S(\beta, m) = \sum_{j=1}^{q} \exp\left(2\pi i \left(\frac{dj^2}{q} + m\eta j\right) + \pi i \left(2y_1 + \frac{i}{\pi N}\right) j^2\right)$$
$$\times \sum_{n=-\infty}^{\infty} \exp\left(\pi i \left(2y_1 + \frac{i}{\pi N}\right) q^2 n^2 + 2\pi i \left(m\eta + j\left(2y_1 + \frac{i}{\pi N}\right)\right) qn\right).$$

Let us make the change

$$t = 2y_1 + \frac{i}{\pi N}, \quad y = m\eta + jt;$$

then we obtain

$$S(\beta, m) = \sum_{j=1}^{q} e^{2\pi i dj^2/q} e^{2\pi i m \eta j} e^{\pi i t j^2} \theta(yq, tq^2).$$

From the functional equation for the theta-function (Lemma 2.1), let us express $\theta(yq, tq^2)$:

$$\theta(yq, tq^2) = \xi^{-1}(tq^2)^{-1/2}e^{-\pi iy^2/t}\theta\left(\frac{y}{tq}; -\frac{1}{tq^2}\right).$$

Then

$$S(\beta, m) = \frac{1}{\xi q\sqrt{t}}\sum_{n=-\infty}^{\infty} \exp\left(-\frac{\pi i}{tq^2}(n - m\eta q)^2\right)S(q, d, n),$$

where

$$S(q, d, n) = \sum_{j=1}^{q} \exp\left(2\pi i\frac{dj^2 + nj}{q}\right)$$

is the Gaussian sum.

3. Let d''/q'', d/q, and d'/q' be neighboring Farey fractions such that

$$\frac{d''}{q''} < \frac{d}{q} < \frac{d'}{q'}, \quad \tau < q + q', \quad q + q'' < \tau + q.$$

For $(m_1, m_2, m_3, m_4) \neq (0, 0, 0, 0)$, let us consider

$$\begin{aligned} &I(N, m_1, m_2, m_3, m_4) \\ &= \int_0^1 S(\beta, m_1)S(\beta, m_2)S(\beta, m_3)S(\beta, m_4)e^{-2\pi i\beta N}\,d\beta \\ &= \frac{1}{\xi^4}\sum_{q=1}^{\tau}\frac{1}{q^4}\int_{-1/(q(q+q''))}^{1/(q(q+q'))} e^{-2\pi iyN}\frac{1}{t^2}\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\sum_{n_3=-\infty}^{\infty} \\ &\times \sum_{n_4=-\infty}^{\infty}\exp\left(-\frac{\pi i}{tq^2}\Big((n_1 - m_1\eta q)^2 + (n_2 - m_2\eta q)^2 + (n_3 - m_3\eta q)^2 + (n_4 - m_4\eta q)^2\Big)\right) \\ &\times \sum_{\substack{1\le d\le q\\ (d,q)=1}} S(q, d, n_1)S(q, d, n_2)S(q, d, n_3)S(q, d, n_4)e^{-2\pi idN/q}dy. \end{aligned}$$

4. Knowing the exact values of the Gaussian sums and the A. Weil estimate for the Kloosterman sum, let us estimate

$$\begin{aligned} W = \sum_{\substack{1\le d\le q\\ (d,q)=1}} S(q, d, n_1)S(q, d, n_2)S(q, d, n_3)S(q, d, n_4)e^{-2\pi idN/q} \\ \ll q^{5/2}\tau(q)(N, q)^{1/2}. \end{aligned}$$

For this purpose, let us represent q in the form $2^\gamma q_1$, where $(2, q_1) = 1$. If x_1, x_2 vary in the complete systems of residues modulo 2^γ and q_1, respectively, then the form $q_1x_1 + 2^\gamma x_2$ varies in the complete system of residues modulo q. Then we have

$$S(q, d, n_i) = S(2^\gamma, dq_1, n_i)S(q_1, d2^\gamma, n_i).$$

Furthermore, let $d = q_1a_1 + 2^\gamma a_2$, where a_1 and a_2 vary in the reduced systems of residues modulo 2^γ and q_1, respectively. Then

$$S(q, d, n_i) = S(2^\gamma, a_1, n_i\bar{q}_1)S(q_1, a_2, n_i\bar{2}^\gamma),$$

where $q_1\bar{q}_1 \equiv 1 \pmod{2^\gamma}$, $2\bar{2} \equiv 1 \pmod{q_1}$. We obtain $W = W_1 \times W_2$, where

$$W_1 = \sum_{\substack{1\le a_1\le 2^\gamma\\(a_1,2^\gamma)=1}} S(2^\gamma, a_1, n_1\bar{q}_1)S(2^\gamma, a_1, n_2\bar{q}_1)S(2^\gamma, a_1, n_3\bar{q}_1) \times S(2^\gamma, a_1, n_4\bar{q}_1)\exp\Big(-2\pi i\frac{a_1}{2^\gamma}N\Big),$$

$$W_2 = \sum_{\substack{1\le a_2\le q_1\\(a_2,q_1)=1}} S(q_1, a_2, n_1\bar{2}^\gamma)S(q_1, a_2, n_2\bar{2}^\gamma)S(q_1, a_2, n_3\bar{2}^\gamma) \times S(q_1, a_2, n_4\bar{2}^\gamma)\exp\Big(-2\pi i\frac{a_2}{q_1}N\Big).$$

For odd $n_i\bar{q}_1$, let us partition the sum $S(2^\gamma, a_1, n_i\bar{q}_1)$ into two sums with summands of the form $j = 2x$ and $j = 2x + 1$. Then

$$S(2^\gamma, a_1, n_i\bar{q}_1) = \sum_{x=1}^{2^{\gamma-1}} \exp\left(\frac{2\pi i(2a_1x^2 + n_i\bar{q}_1x)}{2^{\gamma-1}}\right) + \exp\left(\frac{2\pi i(a_1 + n_i\bar{q}_1)}{2^\gamma}\right)\sum_{x=0}^{2^{\gamma-1}-1} \exp\left(\frac{2\pi i(2a_1x^2 + (2a_1 + n_i\bar{q}_1)x)}{2^{\gamma-1}}\right).$$

In this case, by Lemma 2.2 (1), $S(2^\gamma, a_1, n_i\bar{q}_1) = 0$.

For even $n_i\bar{q}_1$,

$$S(2^\gamma, a_1, n_i\bar{q}_1) = \exp\left(-2\pi i\frac{\bar{a}_1(n_i\bar{q}_1/2)^2}{2^\gamma}\right)\sum_{x=1}^{2^\gamma}\exp\left(2\pi i\frac{a_1(x + \bar{a}_1n_i\bar{q}_1/2)^2}{2^\gamma}\right).$$

By Lemma 2.2 (2), we have

$$S(2^\gamma, a_1, n_i\bar{q}_1) = \exp\left(-2\pi i\frac{\bar{a}_1n_i^2\left(\frac{\bar{q}_1}{2}\right)^2}{2^\gamma}\right) \times \begin{cases} 0 & \text{if } \gamma = 1,\\ 2^{\gamma/2}(1 + i^\gamma) & \text{if } \gamma \text{ is even},\\ 2^{(\gamma+1)/2}e^{2\pi i\gamma/8} & \text{if } \gamma > 1 \text{ is odd}.\end{cases}$$

Then

$$W_1 \le 2^{2\gamma+4}\sum_{\substack{1\le a_1\le 2^\gamma\\(a_1,2^\gamma)=1}}\exp\left(2\pi i\left(\frac{-a_1N - \bar{a}_1(n_1^2 + n_2^2 + n_3^2 + n_4^2)\left(\frac{\bar{q}_1}{2}\right)^2}{2^\gamma}\right)\right).$$

Finally, by Lemma 2.3, we have

$$W_1 \ll \tau(2^\gamma)2^{5\gamma/2}(N, n_1^2 + n_2^2 + n_3^2 + n_4^2, 2^\gamma)^{1/2}.$$

Applying Lemma 2.2 (3) and 2.2 (4) to the sums $S(q_1, a_2, n_i\bar{2}^\gamma)$, we obtain

$$W_2 = q_1^2\sum_{\substack{1\le a_2\le q_1\\(a_2,q_1)=1}}\exp\left(2\pi i\left(\frac{-a_2N - \bar{a}_2(n_1^2 + n_2^2 + n_3^2 + n_4^2)\bar{2}^{2+2\gamma}}{q_1}\right)\right).$$

Then, by Lemma 2.3,

$$W_2 \ll \tau(q_1)q_1^{5/2}(N, n_1^2 + n_2^2 + n_3^2 + n_4^2, q_1)^{1/2}.$$

As a result, we obtain the desired estimate for $|W|$.

5. Let us substitute the estimate for $|W|$ in the integral $I(N,m_1,m_2,m_3,m_4)$:

$$I(N,m_1,m_2,m_3,m_4) \ll \sum_{q=1}^{\tau} \frac{1}{q^{3/2}} q^{\varepsilon}(N,q)^{1/2} \int\limits_{-1/(q(q+q''))}^{1/(q(q+q'))} \frac{N^2 dy}{1+(yN)^2}$$

$$\times \sum_{\substack{|n_i|<\infty\\ i=1,2,3,4}} \exp\left(-\frac{\pi^2 N((n_1-m_1\eta q)^2+(n_2-m_2\eta q)^2+(n_3-m_3\eta q)^2+(n_4-m_4\eta q)^2)}{q^2(1+(2\pi yN)^2)}\right).$$

Denote by $S(\tau)$ the obtained sum in q. Let us divide the interval of integration into intervals:

$$\int\limits_{-1/(q(q+q''))}^{1/(q(q+q'))} = \int\limits_{-1/(q(q+q''))}^{-1/(q(q+\tau))} + \int\limits_{-1/(q(q+\tau))}^{1/(q(q+\tau))} + \int\limits_{1/(q(q+\tau))}^{1/(q(q+q'))};$$

we have $S(\tau) = S_1 + S_2 + S_3$. For an arbitrary positive constant k, we have

$$S_3 \ll \sum_{q=1}^{\tau} \frac{1}{q^{3/2}} q^{\varepsilon}(N,q)^{1/2} N \int\limits_{-N/(q(q+q''))}^{N/(q(q+q'))} \frac{dy}{y^2}$$

$$\times \sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} e^{-k((n_1-m_1\eta q)^2+(n_2-m_2\eta q)^2+(n_3-m_3\eta q)^2+(n_4-m_4\eta q)^2)}$$

$$\ll N^{\varepsilon} \sum_{q=1}^{\tau} \frac{1}{q^{3/2}}(N,q)^{1/2} q\tau \ll \tau N^{\varepsilon} \sum_{\delta|N} \sum_{q_1\le\tau/\delta} \frac{1}{{q_1}^{1/2}} \ll N^{3/4+\varepsilon}.$$

In a similar way, we obtain $S_1 \ll N^{3/4+\varepsilon}$.

For S_2, we have

$$S_2 \ll \sum_{1\le q\le N^{0,2-\varepsilon}} \int\limits_{0}^{1/(qN^{0,8+\varepsilon})} + \sum_{1\le q\le N^{0,2-\varepsilon}} \int\limits_{1/(qN^{0,8+\varepsilon})}^{1/(q(q+\tau))} + \sum_{N^{0,2-\varepsilon}<q\le\tau} \int\limits_{0}^{1/(q(q+\tau))}$$

$$= S_{2,1} + S_{2,2} + S_{2,3}.$$

In the sum $S_{2,1}$, let us isolate n_i such that $n_i - m_i\eta q = \|m_i\eta q\|$, $i=1,2,3,4$. Since η is a quadratic irrational number, it follows that it can be represented in the form

$$\eta = \frac{A}{Q} + y_2, \quad (A,Q)=1, \quad |y_2| < \frac{1}{Q^2}.$$

For

$$Q \asymp N^{0,3},$$

we have

$$\|m_i\eta q\| = \left\{\frac{Am_i q}{Q}\right\} + |y_2 m_i q| > \frac{1}{2Q}.$$

Then there exists $k>0$ such that

$$\exp\left(-\frac{\pi^2 N\|m_i\eta q\|^2}{q^2(1+(2\pi yN)^2)}\right) \ll e^{-kN^{2\varepsilon}}.$$

For other values of n_i and for an arbitrary positive k, we have

$$\exp\left(-\frac{\pi^2 N(n_i-m_i\eta q)^2}{q^2(1+(2\pi yN)^2)}\right) \ll \exp(-kN^{0,6+2\varepsilon}(n_i-m_i\eta q)^2).$$

As a result, we obtain

$$S_{2,1} \ll e^{-kN^{2\varepsilon}} \sum_{1 \le q \le N^{0,2-\varepsilon}} \frac{1}{q^{3/2}} q^{\varepsilon}(N,q)^{1/2} N^2 \int\limits_0^{1/(qN^{0,8+\varepsilon})} dy$$

$$\ll e^{-kN^{2\varepsilon}} N^{1,2} \sum_{\delta|N} \delta^{-2} \sum_{q_1 \le N^{0,2-\varepsilon}/\delta} \frac{1}{{q_1}^{5/2}} \ll N^{0,9},$$

$$S_{2,2} \ll \sum_{1 \le q \le N^{0,2-\varepsilon}} \frac{1}{q^{3/2}} q^{\varepsilon}(N,q)^{1/2} N \int\limits_{(N^{0,2-\varepsilon})/q}^{N/(q(q+\tau))} \frac{dy}{y^2}$$

$$\times \sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} e^{-k((n_1-m_1\eta q)^2+(n_2-m_2\eta q)^2+(n_3-m_3\eta q)^2+(n_4-m_4\eta q)^2)}$$

$$\ll N^{0,8+2\varepsilon} \sum_{\delta|N} \sum_{q_1 \le N^{0,2-\varepsilon}/\delta} \frac{1}{{q_1}^{1/2}} \ll N^{0,9+2\varepsilon},$$

$$S_{2,3} \ll \sum_{N^{0,2-\varepsilon} < q \le \tau} \frac{1}{q^{3/2}} q^{\varepsilon}(N,q)^{1/2} N \int\limits_0^{N/(q(q+\tau))} \frac{dy}{1+y^2}$$

$$\times \sum_{n_1,n_2,n_3,n_4=-\infty}^{\infty} e^{-k((n_1-m_1\eta q)^2+(n_2-m_2\eta q)^2+(n_3-m_3\eta q)^2+(n_4-m_4\eta q)^2)}$$

$$\ll N^{1+\varepsilon} N^{-0,1+\varepsilon} \ll N^{0,9+2\varepsilon}.$$

The theorem is proved. □

REFERENCES

1. T. Estermann, “A new application of the Hardy–Littlewood–Kloosterman method,” *Proc. London Math. Soc.*, **12**, 425–444 (1962).
2. T. Estermann, “On Kloosterman’s sum,” *Mathematica*, **8**, 83–86 (1961).
3. L.-K. Hua, *Introduction to Number Theory*, Springer (1982).
4. H. D. Kloosterman, “On the representation of numbers in the form $ax^2 + by^2 + cz^2 + dt^2$,” *Acta Math.*, **49**, 407–464 (1926).
5. D. Mamford, *Lectures on Theta-Functions* [Russian translation], IO NFMI, Novokuznetsk (1998).
6. I. M. Vinogradov, *Special Variants of Trigonometrical Sum Method* [in Russian], Nauka, Moscow (1983).
7. A. Weil “On some exponential sums,” *Proc. Natl. Acad. Sci.*, **34**, 204–207 (1948).

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