

Discrete Pseudo-Differential Operators and Boundary Value Problems in a Half-Space and a Cone

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Abstract—We consider a certain class of discrete pseudo-differential operators and related equations in a sharp convex cone and describe their invertibility conditions in L_2 spaces. For this purpose we introduce a concept of periodic wave factorization for elliptic symbol and show its applicability for the studying. For a half-space case we consider the Laplace equation and describe a solution of the discrete Dirichlet problem.

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1. INTRODUCTION

A classical pseudo-differential operator in Euclidean space \mathbb{R}^m is defined by the formula [1–4]

$$(Au)(x) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \tilde{A}(x, \xi) e^{i(\xi \cdot x - y)} \tilde{u}(\xi) d\xi dy,$$

where the sign \sim over a function denotes its discrete Fourier transform

$$\tilde{u}(\xi) = \int_{\mathbb{R}^m} u(x) e^{ix \cdot \xi} dx.$$

Given function u_d of a discrete variable $\tilde{x} \in \mathbb{Z}^m$ we define its discrete Fourier transform by the series

$$(F_d u_d)(\xi) \equiv \tilde{u}_d(\xi) = \sum_{\tilde{x} \in \mathbb{Z}^m} e^{i\tilde{x} \cdot \xi} u(\tilde{x}), \quad \xi \in \mathbb{T}^m \equiv [-\pi, \pi]^m,$$

where partial sums are taken over cubes $Q_N = \{\tilde{x} \in \mathbb{Z}^m : \tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_m), \max_{1 \leq k \leq m} |\tilde{x}_k| \leq N\}$.

Let $D \subset \mathbb{R}^m$ be a sharp convex cone, $D_d \equiv D \cap \mathbb{Z}^m$, and $L_2(D_d)$ be a space of functions of discrete variable defined on D_d , and $A(\tilde{x})$ be a given function of a discrete variable $\tilde{x} \in \mathbb{Z}^m$. We introduce the function $\tilde{A}_d(\xi) = \sum_{\tilde{x} \in \mathbb{Z}^m} e^{i\tilde{x} \cdot \xi} A(\tilde{x})$, $\xi \in \mathbb{T}^m$, and consider the following types of operators

$$(A_d u_d)(\tilde{x}) = \int_{\mathbb{T}^m} \sum_{\tilde{y} \in D_d} e^{i(\tilde{y} - \tilde{x}) \cdot \xi} \tilde{A}(\xi) \tilde{u}_d(\xi) d\xi, \quad \tilde{x} \in D_d, \tag{1}$$

according to a standard definition.

Definition 1. *The function $\tilde{A}_d(\xi)$ is called a symbol of the operator A_d , and this symbol is called an elliptic symbol if $\tilde{A}_d(\xi) \neq 0, \forall \xi \in \mathbb{T}^m$.*

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Our main goal is describing a periodic variant of wave factorization for an elliptic symbol [10] and showing its usability for studying invertibility for the operator A_d .

Let us denote P_{D_d} projection operator on D_d , $P_{D_d} : L_2(\mathbb{Z}^m) \rightarrow L_2(D_d)$ so that for arbitrary function $u_d \in L_2(\mathbb{Z}^m)$

$$(P_{D_d}u_d)(\tilde{x}) = \begin{cases} u_d(\tilde{x}), & \text{if } \tilde{x} \in D_d; \\ 0, & \text{otherwise.} \end{cases}$$

If we consider a half-space case then the Fourier image of the operator P_{D_d} is evaluated [12, 13] and we'll demonstrate it in the following

Example 1. If $D = \mathbb{R}_+^m$ then

$$(F_d P_{D_d} u_d)(\xi', \xi_m) = \frac{1}{4\pi i} \lim_{\tau \rightarrow 0+} \int_{-\pi}^{\pi} u_d(\xi', \eta_m) \cot \frac{\xi_m - \eta_m + i\tau}{2} d\eta_m.$$

If D is a sharp convex cone $C_+^a = \{\tilde{x} \in \mathbb{Z}^m : \tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_m), \tilde{x}_m > a|\tilde{x}'|, \tilde{x}' = (\tilde{x}_1, \dots, \tilde{x}_{m-1}), a > 0\}$ then we introduce the function $B_d(z) = \sum_{\tilde{x} \in D_d} e^{i\tilde{x} \cdot z}, z = \xi + i\tau, \xi \in \mathbb{T}^m, \tau \in C_+^a$, and define the operator

$$(B_d u)(\xi) = \lim_{\tau \rightarrow 0} \int_{\mathbb{T}^m} B_d(z - \eta) u_d(\eta) d\eta.$$

Lemma 1. For arbitrary $u_d \in L_2(\mathbb{Z}^m)$ the property $F_d P_{D_d} u_d = B_d F_d u_d$ holds.

Proof. Let $\chi_+(\tilde{x})$ be an indicator of the set D_d . Thus $(P_{D_d} u_d)(\tilde{x}) = \chi_+(\tilde{x}) \cdot u_d(\tilde{x})$. Further since the function $\chi_+(\tilde{x})$ is not summarizable we can't apply directly a convolution property of the Fourier transform. We choose the function $e^{i\tilde{x} \cdot \tau}$ so the product $\chi_+(\tilde{x}) e^{i\tilde{x} \cdot \tau}$ will be summarizable for some admissible τ . Taking into account a forthcoming passing to a limit under $\tau \rightarrow 0+$ we have $F_d(\chi_+(\tilde{x}) e^{i\tilde{x} \cdot \tau}) = B_d(z)$. Thus we can use the Fourier transform obtaining convolution of functions $B_d(z)$ and $\tilde{u}_d(\xi)$. It is left passing to a limit. \square

2. MULTIDIMENSIONAL PERIODIC RIEMANN BOUNDARY VALUE PROBLEM

For $D = \mathbb{R}_+^m$ we'll remind some author's constructions [12, 13] for discrete equations in a half-space. We have $B_d(z) = \cot \frac{z}{2}, z = (\xi', \xi_m + i\tau), \xi' = (\xi_1, \dots, \xi_{m-1}), \tau > 0$. Thus (see Example 1) we use a periodic one-dimensional Riemann problem with a parameter $\xi' \in \mathbb{T}^{m-1}$ which is the following. Finding a pair of functions $\Phi^\pm(\xi', \xi_m)$ which are boundary values of holomorphic in half-strips $\Pi_\pm = \{z \in \mathbb{C} : z = \xi_m \pm i\tau, \tau > 0\}$ such that these are satisfied a linear relation

$$\Phi^+(\xi)(\xi', \xi_m) = G(\xi', \xi_m) \Phi^-(\xi)(\xi', \xi_m) + g(\xi), \quad \xi \in \mathbb{T}^m,$$

for almost all $\xi' \in \mathbb{T}^{m-1}$, where $G(\xi), g(\xi)$ are given periodic functions.

Now we will consider an essential multidimensional case. Let $\overset{*}{D}$ be a conjugate cone for D i.e. $\overset{*}{D} = \{x \in \mathbb{R}^m : x \cdot y > 0, y \in D\}$, and $T(\overset{*}{D}) \subset \mathbb{C}^m$ be a set of the type $\mathbb{T}^m + i \overset{*}{D}$. For $\mathbb{T}^m \equiv \mathbb{R}^m$ such a domain of multidimensional complex space is called a radial tube domain over the cone $\overset{*}{D}$ [7, 8, 10].

Let us define the subspace $A(\mathbb{T}^m) \subset L_2(\mathbb{T}^m)$ consisting of functions which admit a holomorphic continuation into $T(\overset{*}{D})$ and satisfy the following condition

$$\sup_{\tau \in \overset{*}{D} \mathbb{T}^m} \int |\tilde{u}_d(\xi + i\tau)|^2 d\xi < +\infty. \tag{2}$$

In other words the space $A(\mathbb{T}^m) \subset L_2(\mathbb{T}^m)$ consists of boundary values of holomorphic in $T(\overset{*}{D})$ functions.

Let us denote $B(\mathbb{T}^m) = L_2(\mathbb{T}^m) \ominus A(\mathbb{T}^m)$, so that $B(\mathbb{T}^m)$ is a direct complement of $A(\mathbb{T}^m)$ in $L_2(\mathbb{T}^m)$. A simplest variant of this problem is so called a jump problem. We formulate the problem by the following way: finding a pair of functions $\Phi^\pm, \Phi^+ \in A(\mathbb{T}^m), \Phi^- \in B(\mathbb{T}^m)$, such that

$$\Phi^+(\xi) - \Phi^-(\xi) = g(\xi), \quad \xi \in \mathbb{T}^m, \tag{3}$$

where $g(\xi) \in L_2(\mathbb{T}^m)$ is given.

Lemma 2. *The operator $B_d : L_2(\mathbb{T}^m) \rightarrow A(\mathbb{T}^m)$ is a bounded projector. A function $u_d \in L_2(D_d)$ iff its Fourier transform $\tilde{u}_d \in A(\mathbb{T}^m)$.*

Proof. According to standard properties of the discrete Fourier transform F_d we have

$$F_d(\chi_+(\tilde{x})u_d(\tilde{x})) = \lim_{\tau \rightarrow 0} \int_{\mathbb{T}^m} B_d(z - \eta)\tilde{u}_d(\eta)d\eta,$$

where $\chi_+(\tilde{x})$ is an indicator of the set D_d . It implies a boundedness of the operator B_d . The second assertion follows from holomorphic properties of the kernel $B_d(z)$. In other words for arbitrary function $v \in A(\mathbb{T}^m)$ we have

$$v(z) = \int_{\mathbb{T}^m} B_d(z - \eta)v(\eta)d\eta, \quad z \in T(\overset{*}{D}).$$

It is an analogue of the Cauchy integral formula. □

Theorem 1. *The jump problem has unique solution for arbitrary right-hand side from $L_2(\mathbb{T}^m)$.*

Proof. Indeed it is equivalent to one-to-one representation of the space $L_2(D_d)$ as a direct sum of two subspaces. If we'll denote $\chi_+(x), \chi_-(x)$ indicators of discrete sets $D_d, \mathbb{Z}^m \setminus D_d$ respectively then the following representation $u_d(\tilde{x}) = \chi_+(\tilde{x})u_d(\tilde{x}) + \chi_-(\tilde{x})u_d(\tilde{x})$ is unique and holds for arbitrary function $u_d \in L_2(\mathbb{Z}^m)$. After applying the discrete Fourier transform we have $F_d u_d = F_d(\chi_+ u_d) + F_d(\chi_- u_d)$, where $F_d(\chi_+ u_d) \in A(\mathbb{T}^m)$ according to Lemma 2, and thus $F_d(\chi_- u_d) = F_d u_d - F_d(\chi_+ u_d) \in B(\mathbb{T}^m)$ because $F_d u_d \in L_2(\mathbb{T}^m)$. □

Example 2. *If $m = 2$ and C_+^2 is the first quadrant in a plane then a solution of a jump problem is given by formulas*

$$\begin{aligned} \Phi^+(\xi) &= \frac{1}{(4\pi i)^2} \lim_{\tau \rightarrow 0} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cot \frac{\xi_1 + i\tau_1 - t_1}{2} \cot \frac{\xi_2 + i\tau_2 - t_2}{2} g(t_1, t_2) dt_1 dt_2, \\ \Phi^-(\xi) &= \Phi^+(\xi) - g(\xi), \quad \tau = (\tau_1, \tau_2) \in C_+^2. \end{aligned}$$

Now we can formulate a general statement for the multidimensional periodic Riemann boundary value problem. It looks as follows. Finding a pair of functions $\Phi^\pm, \Phi^+ \in A(\mathbb{T}^m), \Phi^- \in B(\mathbb{T}^m)$, such that

$$\Phi^+(\xi) = G(\xi)\Phi^-(\xi) + g(\xi), \quad \xi \in \mathbb{T}^m, \tag{4}$$

where $G(\xi), g(\xi)$ are given periodic functions. If $G(\xi) \equiv 1$ we have the jump problem (3).

Like classical studies [5, 6] we want to use a special representation for an elliptic symbol to solve the problem (4). We can easily obtain so-called characteristic singular integral equation associated with multidimensional periodic Riemann boundary value problem (4) like [5, 6] (see also [12, 13]).

Let us denote $Q_{D_d} = I - P_{D_d}$ and consider so-called paired operator composed by two operators $A_d^{(1)}, A_d^{(2)}$ of the following type

$$A_d^{(1)} P_{D_d} + A_d^{(2)} Q_{D_d} : L_2(\mathbb{Z}^m) \rightarrow L_2(\mathbb{Z}^m). \tag{5}$$

One can easily obtain the following

Lemma 3. *The invertibility of the operator (1) in the space $L_2(D_d)$ is equivalent to invertibility of the operator (5) in the space $L_2(\mathbb{Z}^m)$ with $A_d^{(1)} = A_d, A_d^{(2)} = I$.*

The Fourier image for the operator (5) is the following operator

$$\tilde{u}_d(\xi) \longmapsto ((A_d^{(1)}(\xi)B_d + A_d^{(2)}(\xi)(I - B_d)\tilde{u}_d)(\xi). \quad (6)$$

If $D = \mathbb{R}_+^m$ then the operator (6) is a one-dimensional singular integral operator with periodic Cauchy kernel and a parameter ξ' [12, 13]

3. PERIODIC WAVE FACTORIZATION

Definition 2. *Periodic wave factorization for elliptic symbol $\tilde{A}(\xi)$ is called its representation in the form $\tilde{A}_d(\xi) = \tilde{A}_{\neq}(\xi)\tilde{A}_{=}(\xi)$, where the factors $A_{\neq}^{\pm 1}(\xi)$, $A_{=}^{\pm 1}(\xi)$ admit bounded holomorphic continuation into domains $T(\pm \overset{*}{D})$.*

3.1. Sufficient Conditions

We'll give here certain sufficient conditions for an existence of the periodic wave factorization for an elliptic symbol.

Theorem 2. *Let an elliptic symbol $\tilde{A}_d(\xi) \in C(\mathbb{T}^m)$ be a such that*

$$\text{supp } F_d^{-1}(\ln \tilde{A}_d(\xi)) \subset D_d \cup (-D_d), \quad (7)$$

$$\int_{-\pi}^{\pi} d \arg \tilde{A}_d(\dots, \xi_k, \dots) = 0, \quad k = 1, \dots, m. \quad (8)$$

Then the symbol $\tilde{A}_d(\xi)$ admits the wave factorization.

Proof. If we start from equality $\tilde{A}_d(\xi) = \tilde{A}_{\neq}(\xi)\tilde{A}_{=}(\xi)$ then by logarithm we obtain $\ln \tilde{A}_d(\xi) = \ln \tilde{A}_{\neq}(\xi) + \ln \tilde{A}_{=}(\xi)$ and we have a special kind of a jump problem.

Namely if we'll denote by $A_1(\mathbb{T}^m)$ a subspace of the space $L_2(\mathbb{T}^m)$ consisting of functions which admit a holomorphic continuation into $T(-\overset{*}{D})$ and satisfy the condition (2) for $\tau \in -\overset{*}{D}$. So evidently we speak on a possibility of decomposition of the function $\ln \tilde{A}_d(\xi)$ into two summands one of which belongs to the space $A(\mathbb{T}^m)$ and the second one belongs to the space $A_1(\mathbb{T}^m)$. Let us denote $F^{-1}(\ln \tilde{A}_d(\xi)) \equiv v(x)$. If $\text{supp } v \subset D_d \cup (-D_d)$ then we have the unique representation $v = \chi_+ v + \chi_- v$, where χ_{\pm} is an indicator of the discrete set $\pm D_d$. Further passing to the Fourier transform and potentiating we obtain the required factorization. \square

Remark 1. *The condition (7) is not necessary but we have no an algorithm for constructing a periodic wave factorization. For $D = \mathbb{R}_+^m$ a such algorithm exists always (see [13]).*

3.2. Factorization and Index

There is one point in previous considerations from proof of the Theorem 2 for which one needs an explanation. Indeed the function $\ln \tilde{A}(\xi)$ is defined correctly because the condition (8) provides an absence of bifurcation points. That's why one can call this factorization with vanishing index.

4. INVERTIBILITY OF DISCRETE OPERATORS

Lemma 4. *If $f \in B(\mathbb{T}^m), g \in A_1(\mathbb{T}^m)$ then $f \cdot g \in B(\mathbb{T}^m)$.*

Proof. According to properties of discrete Fourier transform F_d we have

$$(F_d^{-1}(f \cdot g))(\tilde{x}) - ((F_d^{-1}f) * (F_d^{-1}g))(\tilde{x}) \equiv \sum_{\tilde{y} \in \mathbb{Z}^m} f_1(\tilde{x} - \tilde{y})g_1(\tilde{y}) = \sum_{\tilde{y} \in -D_d} f_1(\tilde{x} - \tilde{y})g_1(\tilde{y}),$$

where $f_1 = F_d^{-1}f, g_1 = F_d^{-1}g$ and according to Lemma 2 $supp g_1 \subset -D_d$. Further since we have $supp f_1 \subset \mathbb{Z}^m \setminus (-D_d)$ then for $\tilde{x} \in D_d, \tilde{y} \in -D_d$ we have $\tilde{x} - \tilde{y} \in D_d$ so that $f_1(\tilde{x} - \tilde{y}) = 0$ for such \tilde{x}, \tilde{y} . Thus $supp f_1 * g_1 \subset \mathbb{Z}^m \setminus D_d$. \square

Theorem 3. *If the elliptic symbol $\tilde{A}_d(\xi) \in C(\mathbb{T}^m)$ admits periodic wave factorization then the operator A_d is invertible in the space $L_2(D_d)$.*

Proof. We'll remind that according to the Lemma 3 an invertibility of the operator A_d in the space $L_2(D_d)$ is equivalent to an invertibility of the operator $A_d P_{D_d} + I Q_{D_d}$ in the space $L_2(\mathbb{Z}^m)$. It is easily concluding the last invertibility is equivalent to solving the Riemann problem (4) for arbitrary right-hand side $g(\xi) \in L_2(\mathbb{Z}^m)$ with $G(\xi) \equiv \tilde{A}_d^{-1}(\xi)$. If we have the periodic wave factorization for the symbol $\tilde{A}_d(\xi)$ then

$$\tilde{A}_{\neq}(\xi)\Phi^+(\xi) = \tilde{A}_{=}^{-1}(\xi)\Phi^-(\xi) + \tilde{A}_{\neq}(\xi)g(\xi), \quad \xi \in \mathbb{T}^m, \tag{9}$$

and we have a jump problem. The first summand $\tilde{A}_{\neq}(\xi)\Phi^+(\xi) \in A(\mathbb{T}^m)$ according to a holomorphic property, and the second one $\tilde{A}_{=}^{-1}(\xi)\Phi^-(\xi) \in B(\mathbb{T}^m)$ according to the lemma 3. Taking into account the theorem 2 we conclude that the Riemann problem (9) has a unique solution for arbitrary $g(\xi) \in L_2(\mathbb{T}^m)$. \square

5. DISCRETE BOUNDARY VALUE PROBLEMS IN A HALF-SPACE

We will consider here certain pseudo-differential operators using general concepts of the theory of special operator equations.

5.1. Discrete Laplacian

We define the discrete Laplacian by divided difference of second order instead of partial derivatives

$$\begin{aligned} \frac{\partial^2 u}{\partial x_k^2} &\sim u_d(x_1, \dots, \tilde{x}_k + 2, \dots, x_m) - 2u_d(x_1, \dots, \tilde{x}_k + 1, \dots, x_m) \\ &+ u_d(x_1, \dots, \tilde{x}_k, \dots, x_m) \equiv \Delta_{x_k}^2 u_d, \end{aligned}$$

so that

$$\sum_{k=1}^m \frac{\partial^2 u}{\partial x_k^2} \equiv \Delta u \sim \sum_{k=1}^m \Delta_{x_k}^2 u_d \equiv \Delta_d u_d.$$

Remark 2. *This is not a convolution operator but a difference one in a discrete space.*

Applying the discrete Fourier transform to the last expression one can easily obtain the formula

$$(F_d(\Delta_d u_d))(\xi) = \left(\sum_{k=1}^m (e^{i\xi_k} - 1)^2 \right) \tilde{u}_d(\xi).$$

It is natural the function $\Delta_d(\xi) \equiv \sum_{k=1}^m (e^{i\xi_k} - 1)^2$ should be called **a symbol** of the discrete Laplacian.

Let A_d be a linear bounded operator acting in the discrete space $L_2(\mathbb{Z}^m)$. A rule this operator is considered with the following operator equation

$$A_d u_d = v_d, \quad v_d \in L_2(\mathbb{Z}^m), \tag{10}$$

and if the operator is such that one can apply the discrete Fourier transform to the equation (10) reducing it to the multiplier $\sigma_d(\xi)\tilde{u}_d(\xi) = \tilde{v}_d(\xi), \xi \in \mathbb{R}^m$, then the function $\sigma_d(\xi)$ is called **a symbol** of operator

A_d . Of course A_d means not an arbitrary operator but such that above multiplier representation is valid; it may be a convolution operator or a difference one. Let us note in the last equation all functions are periodic, thus we can consider these functions on \mathbb{T}^m .

Factorization technique for a symbol [3, 9] is used if the equation (10) is considered not on the whole lattice \mathbb{Z}^m but on its part, in our case on $\mathbb{Z}_+^m = \{\tilde{x} \in \mathbb{Z}^m : \tilde{x}_m \geq 0\}$. In other words instead of the equation (10) one considers the equation

$$P_+ A_d u_d^+ = v_d^+, \quad v_d^+ \in L_2(\mathbb{Z}_+^m), \tag{11}$$

where $P_+ : L_2(\mathbb{Z}^m) \rightarrow L_2(\mathbb{Z}_+^m)$ is a projector, and a solution u_d^+ belongs to $L_2(\mathbb{Z}_+^m)$.

Everywhere below we consider discrete operators with continuous periodic symbols.

Definition 3. The symbol $\sigma_d(\xi)$ is called an elliptic symbol if $\inf_{\xi \in \mathbb{T}^m} |\sigma_d(\xi)| > 0$.

Remark 3. For our case it is equivalent that the symbol is not vanishing everywhere.

6. PERIODIC FACTORIZATION

Let us denote $\xi = (\xi', \xi_m)$.

Definition 4. A periodic factorization for elliptic symbol $\sigma_d(\xi)$ is called its representation in the form $\sigma_d(\xi', \xi) = \sigma_d^+(\xi', \xi) \cdot \sigma_d^-(\xi', \xi)$, where $\sigma_d^\pm(\xi)$ admit bounded analytic continuation with respect to the variable ξ_m into half-strips $\Pi_\pm = \{z \in \mathbb{C} : z = \xi_m + i\tau, \xi_m \in [-\pi, \pi], \pm\tau > 0\}$ for almost all ξ' .

Definition 5. An index of periodic factorization of elliptic symbol $\sigma_d(\xi)$ is called divided by 2π variation of an argument of the function $\sigma_d(\xi)$ under varying ξ_m from $-\pi$ to π .

Remark 4. This is easily seen that the index of factorization does not depend on ξ' .

Since the discrete Laplacian with the symbol $\Delta_d(\xi)$ is not an elliptic symbol according the definition 3 we will consider here the discrete Helmholtz operator $H_d \equiv \Delta_d - k^2 I$, where I is an identity operator with some fixed $k \in \mathbb{R}$ to satisfy the definition 1.

Lemma 5. If index of factorization for discrete operator A_d is zero then its symbol admits a periodic factorization.

All details for constructing such factorizations are contained in [12, 13]. Key role plays a periodic analogue of the Hilbert transform, i.e. an operator of the form

$$(H_{\xi'}^{per} \tilde{u}_d)(\xi) = \frac{1}{2\pi i} v.p. \int_{-\pi}^{\pi} \cot \frac{\eta_m - \xi_m}{2} \tilde{u}_d(\xi', \eta_m) d\eta_m.$$

This is a simple corollary of the theory of periodic Riemann boundary value problem which permits in future considering non-vanishing indices.

Lemma 6. Index of factorization for the discrete operator H_d is equal to 1.

This fact is verified immediately. Indeed, under fixed ξ' the argument varying is determined by the term with $e^{i\xi_m}$ for arbitrary k .

7. A GENERAL SOLUTION

Lemmas 5 and 6 permit describing a structure of a general solution for the equation (11) for the case $A_d \equiv H_d$ in the space $L_2(\mathbb{Z}_+^m)$ using methods [3, 9] and the theory of periodic Riemann boundary value problem [12, 13].

Theorem 4. For $A_d = H_d$ all solutions of the equation (3) are given by the formula

$$\tilde{u}_d(\xi) = e^{-i\xi_m} h_+^{-1}(\xi', \xi_m) (H_{\xi'}^{per} (h_-^{-1} \widetilde{lv}_d^+))(\xi) + \tilde{c}(\xi') e^{-i\xi_m} h_+^{-1}(\xi', \xi_m),$$

where $h_\pm(\xi', \xi_m)$ are factors of periodic factorization for the symbol $e^{-i\xi_m} (\Delta_d(\xi) + k^2)$, lv_d^+ is an arbitrary continuation of v_d^+ onto the whole \mathbb{Z}^m , $c(x')$ is an arbitrary function from $L_2(\mathbb{Z}^{m-1})$.

Factors of a periodic factorization are constructed according to classical books of F.D. Gakhov [5] and N.I. Muskhelishvili [6] with the change of the Hilbert transform H by the operator $H_{\xi'}^{per}$.

7.1. Boundary Value Problems

The Theorem 4 contains an assertion on non-uniqueness of a solution for the equation (11). To extract an unique solution one needs additional conditions. Usually they use boundary conditions. Simplest variants are given by discrete analogues of Dirichlet or Neumann conditions. We will consider here discrete Dirichlet conditions. We will require that a solution of the equation (11) satisfies the following discrete Dirichlet condition

$$u_d|_{\tilde{x}_m=0} = g_d(\tilde{x}'), \tag{12}$$

where g_d is given function of a discrete variable on a discrete hyper-plane Z^{m-1} . The condition (12) for Fourier image takes the form

$$\int_{-\pi}^{\pi} \tilde{u}_d(\xi', \xi_m) d\xi_m = \tilde{g}_d(\xi'),$$

and according to the Theorem 4 it leads to the following integral equation with respect to $\tilde{c}(\xi')$

$$\int_{-\pi}^{\pi} c(\xi') e^{-i\xi_m} h_+^{-1}(\xi', \xi_m) d\xi_m = \tilde{f}_d(\xi');$$

we use the notations

$$\tilde{f}_d(\xi') = \tilde{g}_d(\xi') - \int_{-\pi}^{\pi} e^{-i\xi_m} h_+^{-1}(\xi', \xi_m) 1/2(I + H_{\xi'}^{per})(h_-^{-1} \widetilde{lv}_d^+)(\xi) d\xi_m.$$

Further, denoting

$$\int_{-\pi}^{\pi} e^{-i\xi_m} h_+^{-1}(\xi', \xi_m) d\xi_m \equiv b(\xi'),$$

and supposing $b(\xi') \neq 0$ one can easily find $\tilde{c}(\xi') = b^{-1}(\xi') \tilde{f}_d(\xi')$. Then a solution of the problem (11), (12) for Fourier image takes the form

$$\tilde{u}_d(\xi) = e^{-i\xi_m} h_+^{-1}(\xi', \xi_m) (1/2(I + H_{\xi'}^{per})(h_-^{-1} \widetilde{lv}_d^+))(\xi) + b^{-1}(\xi') \tilde{f}_d(\xi') e^{-i\xi_m} h_+^{-1}(\xi', \xi_m),$$

Remark 5. One can verify that really $b(\xi')$ is a non-vanishing complex constant.

Theorem 5. Discrete boundary value problem (11), (12) is uniquely solvable in the space $L_2(\mathbb{Z}_+^m)$ for an arbitrary right-hand side $v_d \in L_2(\mathbb{Z}_+^m)$ and an arbitrary boundary function $g_d \in L_2(\mathbb{Z}^{m-1})$.

7.2. Discrete Poisson Formula

If the right-hand side of the equation is a zero, $v_d^+ \equiv 0$, then formula for a solution of the discrete boundary value problem (11), (12) will be very simplified $\tilde{u}_d(\xi) = b^{-1}(\xi') \tilde{g}_d(\xi') e^{-i\xi_m} h_+^{-1}(\xi', \xi_m)$, and after applying the inverse discrete Fourier transform it will be like this expression

$$u_d(\tilde{x}', \tilde{x}_m) = \sum_{\tilde{y}' \in \mathbb{Z}^{m-1}} P_d(\tilde{x}' - \tilde{y}', \tilde{x}_m) g_d(\tilde{y}'), \tag{13}$$

where the function of a discrete variable $P_d(\tilde{x})$ is defined by inverse discrete Fourier transform of the function $b^{-1}(\xi') e^{-i\xi_m} h_+^{-1}(\xi', \xi_m)$. The formula (13) is a discrete analogue of the Poisson formula for solving the Dirichlet problem in a half-space.

CONCLUSION

Taking into account these results one can wait some interesting interrelations between discrete and continue theory which will permit to justify obtaining discrete solutions with computer calculations for a lot of distinct boundary value problems and wide classes of equations. Moreover such technique are also useful for studying difference equations in multidimensional spaces [10, 11, 14–16]. Last these “discrete” considerations can be transferred on more general situations and operators. It will be a subject of forthcoming papers of the author.

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