On Discrete Boundary Value Problems

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Abstract. We consider discrete pseudo-differential equations and related discrete boundary value problems in appropriate discrete spaces. First we study simplest types of operators acting in canonical domains like a half-space and a cone. We try to describe solvability conditions for such equations and boundary value problems and further to compare the discrete and continue cases. We use a concept of periodic factorization for elliptic symbols to obtain a form of solution for such equations in canonical domains.

INTRODUCTION

The theory of boundary value problems for elliptic equations with pseudo-differential operators [1] on manifolds with a smooth boundary was constructed in series of papers by M.I. Vishik and G.I. Eskin (see, for example [2]). This theory describes Fredholm properties for such boundary value problems in appropriate Sobolev–Slobodetskii spaces H^s, but there are no any recommendations how to find a solution if it exists. In my opinion it is very hard to find solutions in exact form even for special cases like boundary value problems for the Laplacian. For this purpose I suggest to construct a discrete variant of such a theory to have a possibility to find certain discrete solutions (for some cases) and then to compare the discrete solution with a continual one. Some parts of this approach was realized in author's papers [3, 4, 5, 6, 7, 8].

DISCRETE SPACES AND OPERATORS

Spaces

We consider functions $u_d(\tilde{x})$ of a discrete variable $\tilde{x} \in \mathbf{Z}^m$ such that

$$||u_d||_p \equiv \left(\sum_{\tilde{x} \in \mathbf{Z}^m} |u_d(\tilde{x})|^p \right)^{1/p} < +\infty, \quad 1 < p < +\infty. \tag{1}$$

The space of functions u_d with finite norm (1) will be denoted by $L_p(\mathbf{Z}^m)$.

The discrete Fourier transform F_d for a function $u_d \in L_1(\mathbf{Z}^m)$ is defined as the series

$$(F_d u_d)(\xi) \equiv \tilde{u}(\xi) \equiv \sum_{\tilde{x} \in \mathbf{Z}^m} e^{i\tilde{x}\cdot\xi} u_d(\tilde{x}), \quad \xi \in \mathbf{T}^m \equiv [-\pi, \pi]^m.$$

It is well-known that $F_d : L_2(\mathbf{Z}^m) \to L_2(\mathbf{Z}^m)$ is an isomorphism.

Operators

Let $D \subset \mathbf{R}^m$ be a domain in m-dimensional space, $D_d \equiv D \cap \mathbf{Z}^m$. We introduce a discrete pseudo-differential operator A_d by the formula

$$(A_d u_d)(\tilde{x}) = \int_{\mathbf{T}^m} \sum_{\tilde{y} \in D_d} e^{i(\tilde{x}-\tilde{y})\cdot\xi} \tilde{A}_d(\xi) \tilde{u}_d(\xi) d\xi,$$

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and the function $A_d(\xi)$ is called a symbol of the discrete pseudo-differential operator A_d.

Remark 1. *More general case is that when a symbol under consideration depends on a spatial discrete variable $\tilde{x}$, i.e. one considers a symbol $A_d(\tilde{x},\xi)$ and a pseudo-differential operator is defined by the formula*

$$(A_d u_d)(\tilde{x}) = \int\limits_{\mathbf{T}^m} \sum_{\tilde{y}\in D_d} e^{i(\tilde{x}-\tilde{y})\cdot\xi} \tilde{A}_d(\tilde{x},\xi)\tilde{u}_d(\xi)d\xi.$$

We will study this case also in forthcoming publications using a local principle.

Definition 1. *The symbol $A_d(\xi)$ is called an elliptic symbol if*

$$\inf_{\xi\in\mathbf{T}^m} |A_d(\xi)| > 0.$$

Everywhere below we consider discrete pseudo-differential operators A_d with continuous periodic symbols $A_d(\xi), \xi \in \mathbf{T}^m$.

For this case it is easily seen that the following assertions are valid:

1. the operator $A_d : L_2(D_d) \to L_2(D_d)$ is a linear bounded operator;
2. if $D = \mathbf{R}^m$, then the operator A_d is invertible iff its symbol $A_d(\xi)$ is elliptic.

Rephrasing the last sentence, one can say that the equation

$$(A_d u_d)(\tilde{x}) = v_d(\tilde{x}), \quad \tilde{x} \in \mathbf{Z}^m,$$

with an elliptic symbol is uniquely solvable in the space $L_2(\mathbf{Z}^m)$ for an arbitrary right-hand side $v_d \in L_2(\mathbf{Z}^m)$.

PERIODIC FACTORIZATION AND INDEX

Unfortunately the above second assertion does not hold for $D \neq \mathbf{R}^m$. First difficulties appear for $D = \mathbf{R}^m_+ = \{x \in \mathbf{R}^m : x = (x', x_m), x_m > 0\}$.

Half-spaces

Here we consider the case $D = \mathbf{R}^m_+$ and the corresponding equation

$$(A_d u_d)(\tilde{x}) = v_d(\tilde{x}), \quad \tilde{x} \in \mathbf{Z}^m_+, \tag{2}$$

in the space $L_2(\mathbf{Z}^m_+)$. Let us denote $\xi = (\xi', \xi_m)$.

Definition 2. *A periodic factorization for an elliptic symbol $A_d(\xi)$ is called its representation in the form*

$$A_d(\xi', \xi) = A_d^+(\xi', \xi) \cdot A_d^-(\xi', \xi),$$

where $A_d^{\pm}(\xi)$ admit bounded analytic continuation with respect to the variable ξ_m into half-strips $\Pi_\pm = \{z \in \mathbf{C} : z = \xi_m + i\tau, \xi_m \in [-\pi, \pi], \pm\tau > 0\}$ for almost all ξ'.

An index æ of periodic factorization of elliptic symbol $\sigma_d(\xi)$ is called divided by 2π variation of an argument of the function $\sigma_d(\xi)$ under varying ξ_m from $-\pi$ to π.

This is easily seen that the index of factorization does not depend on ξ'.

Using methods [2] and the theory of periodic Riemann boundary value problem [3, 4], one can obtain the following result on a general solution of the equation (2).

Theorem 1. *If $\text{æ} = n \in \mathbf{N}$, then all solutions of the equation (2) are given by the formula*

$$\tilde{u}_d(\xi) = e^{in\xi_m} a_+^{-1}(\xi', \xi_m)(H^{per}_{\xi'}(e^{-in\xi_m} a_-^{-1}\widetilde{lv_d}))(\xi) + S_n(\xi', \xi_m) a_+^{-1}(\xi', \xi_m), \tag{3}$$

where $a_\pm(\xi', \xi_m)$ are factors of periodic factorization for the symbol

$$e^{-in\xi_m} A_d(\xi),$$

lv_d *is an arbitrary continuation of* v_d *onto the whole* $\mathbf{Z}^m$*,* $S_n(\xi', \xi_m))$ *is an arbitrary polynomial* $\sum_{j=1}^{n} c_j(\xi')e^{-ij\xi_m}$ *with functions* $c_j(\xi')$ *from* $L_2(\mathbf{Z}^{m-1})$.

Factors of a periodic factorization are constructed according to classical books of F.D. Gakhov [9] and N.I. Muskhelishvili [10] with the change of the Hilbert transform H by the operator $H^{per}_{\xi'}$. All details for constructing such factorizations are contained in [3, 4]. Key role plays a periodic analogue of the Hilbert transform, i.e. an operator of the form

$$(H^{per}_{\xi'}\tilde{u}_d)(\xi) = \frac{1}{2\pi i} v.p. \int_{-\pi}^{\pi} \cot\frac{\eta_m - \xi_m}{2} \tilde{u}_d(\xi', \eta_m)d\eta_m.$$

Cones

This case $D = C^a_+ = \{x \in \mathbf{R}^m : x = (x', x_m), x_m > a|x'|, a > 0\}$ is more complicated but it might be studied by an analogue technique. We have to introduce a concept of periodic wave factorization, periodic Bochner kernel and use methodology of the author's book [11]. First considerations were done in [8], and they are related to the theory of holomorphic functions in special multidimensional complex domains [12, 13].

DISCRETE BOUNDARY CONDITIONS

Theorem 1 contains an assertion on non-uniqueness of a solution for the equation (2). To extract an unique solution one needs additional conditions. Usually they use boundary conditions. Simplest variants are given by discrete analogues of Dirichlet or Neumann conditions. But here we consider a general case when boundary conditions are given as traces of some discrete pseudo-differential operators on a discrete hyper-plane $\mathbf{Z}^{m-1}$.

Let $B^k_d, k = 1, \cdots, n$, be discrete pseudo-differential operators with continuous symbols $B^k_d(\xi)$, and let

$$B^k_d u_d|_{\tilde{x}_m=0} = g^k_d(\tilde{x}'), \quad k = 1, \cdots, n, \tag{4}$$

be so-called discrete boundary conditions. We consider below the discrete boundary value problem (3), (4) assuming for simplicity $v_d \equiv 0$. Thus the first summand in the formula (3) disappears.

SOLVABILITY OF DISCRETE BOUNDARY VALUE PROBLEMS

We consider discrete boundary conditions (4), where $g^k_d, k = 1, \cdots, n$, are given function of a discrete variable on a discrete hyper-plane Z^{m-1}.

The conditions (4) for Fourier image take the form

$$\int_{-\pi}^{\pi} B^k_d(\xi', \xi_m)\tilde{u}_d(\xi', \xi_m)d\xi_m = \tilde{g}^k_d(\xi'),$$

and according to theorem 1 it leads to the following system of linear integral equations with respect to unknowns $\tilde{c}_j(\xi')$

$$\sum_{j=1}^{n} c_j(\xi') \int_{-\pi}^{\pi} e^{-ij\xi_m} B^k_d(\xi', \xi_m)a^{-1}_+(\xi', \xi_m)d\xi_m = \tilde{g}^k_d(\xi') \quad k = 1, \cdots, n;$$

we use the notations

$$r_{kj}(\xi') = \int_{-\pi}^{\pi} e^{-ij\xi_m} B^k_d(\xi', \xi_m)a^{-1}_+(\xi', \xi_m)d\xi_m,$$

so we have a non-homogeneous system of linear algebraic equations with the matrix (r_{kj}).

Theorem 2. *If* $v_d = 0$, $æ = n \in \mathbf{N}$, *then discrete boundary value problem (2), (4) is uniquely solvable in the space* $L_2(\mathbf{Z}^m_+)$ *for an arbitrary right-hand side* $v_d \in L_2(\mathbf{Z}^m_+)$ *and an arbitrary boundary functions* $g^k_d \in L_2(\mathbf{Z}^{m-1}), k = 1, \cdots n$ *iff* $\det\ (r_{kj}) \neq 0, k, j = 1, \cdots, n$.

CONCLUSION

These considerations are continuation of author's studies for multidimensional singular integral and pseudo-differential equations and may be helpful for constructing the discrete theory of pseudo-differential equations and boundary value problems.

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