On Some Equations on Non-Smooth Manifolds: Canonical Domains and Model Operators

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Abstract We describe different aspects of the theory of pseudo-differential equations on manifolds with non-smooth boundaries. Using a concept of special factorization for an elliptic symbol we consider distinct variants of this approach including asymptotic and discrete situations.

1 Introduction

From 90s the author develops the theory of boundary value problems based on two principles [9]. These are a local principle and factorizability principle at a boundary point like [2, 15, 3]. The first principle was known earlier and it also was known as a freezing coefficients principle. Usually the second name corresponds to partial differential equations theory but the first name was introduced for multidimensional singular integral equations and more general for pseudo-differential equations. Main difference between differential and pseudo-differential operator is the following. A differential operator $\mathscr{D}$ has a local property i.e. if one takes two smooth functions φ, ψ with non-intersecting supports and compose the operator $\psi \cdot \mathscr{D} \cdot \varphi$ then it leads to a zero operator. For a pseudo-differential operator $\mathscr{P}$ this property does not hold and we obtain for $\psi \cdot \mathscr{P} \cdot \varphi$ a compact operator only. This case permits to obtain rough properties for pseudo-differential equations and related boundary value problems namely Fredholm properties only in comparison with differential operators and boundary value problems where one has as a rule results on existence and uniqueness.

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There are a lot of approaches to construct such a theory (see for example papers [5, 7, 8]). I have written many times [11, 17] what is difference between this consideration and others, it is choice of distinct key principles. In any case one needs to declare an invertibility of so-called local representatives of an initial pseudo-differential operator to describe its Fredholm properties.

Local principle and factorizability was first introduced in papers of I.B. Simonenko [15] (for multidimensional singular integral operators in Lebesgue L_p-spaces) and M.I. Vishik – G.I. Eskin [3] (for pseudo-differential operators in Sobolev – Slobodetskii H^s-spaces). For manifolds with a smooth boundary one uses an idea of "rectification of a boundary", and the problems reduces to a half-space case, for which a factorizability principle holds immediately because under localization at a boundary point and applying the Fourier transform we obtain well known one-dimensional classical Riemann boundary value problem for upper and lower complex half-planes with a multidimensional parameter. This approach does not work if a boundary has at least one singular point like a conical point. One needs here other considerations and approaches.

2 Domains and operators

Our main goal is to describe possible solvability conditions for the pseudo-differential equation

$$(Au)(x) = f(x), \ x \in D,$$

where D is manifold with a boundary, A is pseudo-differential operator with the symbol $A(x, \xi)$.

Such operators are defined locally by the formula

$$u(x) \longmapsto \int\limits_{\mathbf{R}^m} \int\limits_{\mathbf{R}^m} A(x, \xi) \tilde{u}(y) e^{-i(x-y)\cdot\xi} \, d\xi \, dy, \tag{1}$$

if D is a smooth compact manifold, because can use "freezing coefficients principle", or in other words "local principle". For manifold with a smooth boundary we need new local formula for defining the operator A: more precisely in inner points of D we use the formula 1, but in boundary points we need another formula

$$u(x) \longmapsto \int\limits_{\mathbf{R}^m_+} \int\limits_{\mathbf{R}^m} A(x, \xi) u(y) e^{-i(x-y)\cdot\xi} \, d\xi \, dy. \tag{2}$$

For invertibility of such operator (2) with symbol $A(\cdot, \xi)$ non-depending on spatial variable x one can apply the theory of classical Riemann boundary problem for upper and lower complex half-planes with a parameter ξ'. This step was systematically studied in the book [3]. But if the boundary ∂D has at least one conical point, this approach is not effective.

The conical point at the boundary is a such point, for which its neighborhood is diffeomorphic to the cone

$$C_+^a = \{x \in \mathbf{R}^m : x_m > a|x'|, \ x' = (x_1, ..., x_{m-1}), \ a > 0\},$$

hence the local definition for pseudo-differential operator near the conical point is the following

$$u(x) \longmapsto \int\limits_{C_+^a} \int\limits_{\mathbf{R}^m} A(x, \xi) u(y) e^{-i(x-y)\cdot\xi} d\xi dy. \tag{3}$$

We consider the operator 1 in the Sobolev – Slobodetskii space $H^s(\mathbf{R}^m)$ with norm

$$||u||_s^2 = \int\limits_{\mathbf{R}^m} |\tilde{u}(\xi)|^2 (1+|\xi|)^{2s} d\xi,$$

where $\tilde{u}(\xi)$ denotes the Fourier transform for u, and introduce the following class of symbols non-depending on spatial variable x: $\exists c_1, c_2 > 0$, such that

$$c_1 \leq |A(\xi)(1+|\xi|)^{-\alpha}| \leq c_2, \ \xi \in \mathbf{R}^m. \tag{4}$$

The number $\alpha \in \mathbf{R}$ we call the order of pseudo-differential operator A.

It is well-known that pseudo-differential operator with symbol $A(\xi)$ satisfying 3, is linear bounded operator acting from $H^s(\mathbf{R}^m)$ into $H^{s-\alpha}(\mathbf{R}^m)$ [3].

We are interested in studying invertibility operator 3 in corresponding Sobolev – Slobodetskii spaces. By definition, $H^s(C_+^a)$ consists of distributions from $H^s(\mathbf{R}^m)$ with support in C_+^a. The norm in the space $H^s(C_+^a)$ is induced by the norm $H^s(\mathbf{R}^m)$. We associate such operator with corresponding equation

$$(Au_+)(x) = f(x), \ x \in C_+^a, \tag{5}$$

where right-hand side f is chosen from the space $H_0^{s-\alpha}(C_+^a)$.

$H_0^s(C_+^a)$ is the space of distributions $S'(C_+^a)$, which admit continuation on $H^s(\mathbf{R}^m)$. The norm in $H_0^s(C_+^a)$ is defined by

$$||f||_s^+ = \inf ||lf||_s,$$

where *infimum* is chosen for all possible continuations l.

3 Complex variables and wave factorization

Below we will consider the symbols $A(\xi)$ satisfying the condition 4.

Definition 1. Wave factorization of symbol $A(\xi)$ is called its representation in the form

$$A(\xi) = A_{\neq}(\xi) A_{=}(\xi),$$

where the factors $A_{\neq}(\xi), A_{=}(\xi)$ satisfy the following conditions:

1) $A_{\neq}(\xi), A_{=}(\xi)$ are defined everywhere without may be the points $\{\xi \in \mathbf{R}^m : |\xi'|^2 = a^2 \xi_m^2\}$;

2) $A_{\neq}(\xi), A_{=}(\xi)$ admit an analytical continuation into radial tube domains $T(\overset{*}{C^a_+})$, $T(\overset{*}{C^a_-})$ respectively, which satisfy the estimates

$$|A_{\neq}^{\pm 1}(\xi + i\tau)| \leq c_1(1 + |\xi| + |\tau|)^{\pm æ},$$

$$|A_{=}^{\pm 1}(\xi - i\tau)| \leq c_2(1 + |\xi| + |\tau|)^{\pm(\alpha - æ)}, \ \forall \tau \in \overset{*}{C^a_+} .$$

The number $æ$ is called *index of wave factorization.*

Here $\overset{*}{C^a_+}$ is conjugate cone to C^a_+, and $\overset{*}{C^a_-} = - \overset{*}{C^a_+}$.

Example 1. Let

$$A = -\frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_m^2} + k^2, \quad k \in \mathbf{R} \setminus \{\mathbf{0}\},$$

and then according to some properties of the Fourier transform the symbol of this operator has the form

$$A(\xi) = \xi_1^2 + \xi_2^2 + \cdots + \xi_m^2 + k^2.$$

The following equality is the wave factorization of the Helmholtz operator. We will write it as

$$\xi_m^2 + |\xi'|^2 + k^2 =$$

$$\left(\sqrt{a^2 + 1} \, \xi_m + \sqrt{a^2 \xi_m^2 - |\xi'|^2 - k^2} \right) \left(\sqrt{a^2 + 1} \, \xi_m - \sqrt{a^2 \xi_m^2 - |\xi'|^2 - k^2} \right)$$

meaning for $\sqrt{a^2 \xi_m^2 - |\xi'|^2 - k^2}$ the boundary value

$$\sqrt{a^2(\xi_m + i0)^2 - |\xi'|^2 - k^2}.$$

4 Pseudo-differential equations and solvability

To describe a solvability picture for a model elliptic pseudo differential equation (5) in 2-dimensional cone $C^a_+ = \{x \in \mathbf{R}^2 : x_2 > a|x_1|, a > 0\}$ the author earlier considered a special singular integral operator [9]

$$(K_a u)(x) = \frac{a}{2\pi^2} \lim_{\tau \to 0+} \int_{\mathbf{R}^2} \frac{u(y)dy}{(x_1 - y_1)^2 - a^2(x_2 - y_2 + i\tau)^2}. \tag{6}$$

This operator served a conical singularity in the general theory of boundary value problems for elliptic pseudo differential equations on manifolds with a non-smooth boundary. The operator K_a is a convolution operator, and the parameter a is a size of an angle, $x_2 > a|x_1|, a = \cot \alpha$.

One of author's main result [9] is the following (we formulate it for $m = 2$ for simplicity)

Theorem 1. *If elliptic symbol $A(\xi)$ admits wave factorization with respect to the cone C_+^a and $|æ - s| < 1/2$ then the equation (5) has a unique solution*

$$\widetilde{u}(\xi) = A_{\neq}^{-1}(\xi)(K_a \widetilde{lv})(\xi),$$

where lv is an arbitrary continuation of v on the whole $H^s(\mathbf{R}^2)$.

A priori estimate holds

$$||u_+||_s \leq c||f||_{s-\alpha}^+.$$

Below we will mention other possible situations.

4.1 Boundary value problems

If $æ - s = n + \delta, n \in \mathbf{N}, |\delta| < 1/2$, then one has the formula for a general solution of the equation (5), and this formula contains a certain number of arbitrary functions from corresponding Sobolev–Slobodetskii spaces [9]. To obtain the uniqueness theorem one needs to add some complementary conditions as a rule these are boundary conditions.

Some classical variants are considered in [9], some new constructions are described in [17].

4.2 Equations with potentials

It is possible that $æ - s = n + \delta, -n \in \mathbf{N}, |\delta| < 1/2$, then the equation (5) is overdetermined so that one needs to add some unknowns. According to the special representation for a solution of the equation (5) these unknowns should have a potential like form [9].

5 Asymptotical variants

For $|æ - s| < 1/2$ one has the existence and uniqueness theorem [9]

$$\widetilde{u}(\xi) = A_{\neq}^{-1}(\xi)(K_a \widetilde{lv})(\xi),$$

where lv is an arbitrary continuation of v on the whole $H^s(\mathbf{R}^2)$.

5.1 Preliminaries

The formula (6) can be treated as a convolution of the distribution

$$K_a(\xi) = \frac{a}{2\pi^2} \frac{1}{\xi_1^2 - a^2\xi_2^2}$$

with a basic function $u(\xi)$. If so it is interesting to study behavior of the operator (6) for limit cases ($a = 0, a = +\infty$) from convolution point of view.

Let $S(\mathbf{R}^2)$ be the Schwartz space of infinitely differentiable rapidly decreasing at infinity functions then $S'(\mathbf{R}^2)$ is a corresponding space of distributions over $S(\mathbf{R}^2)$.

When $a \to +\infty$ one obtains [10] the following limit distribution

$$\lim_{a \to \infty} \frac{a}{2\pi^2} \frac{1}{\xi_1^2 - a^2\xi_2^2} = \frac{i}{2\pi} \mathscr{P} \frac{1}{\xi_1} \otimes \delta(\xi_2), \tag{7}$$

where the notation for distribution $\mathscr{P}$ is taken from V.S. Vladimirov's books [18, 19], and $\otimes$ denotes the direct product of distributions. Here δ denotes one-dimensional Dirac mass-function which acts on $\varphi \in S(\mathbf{R})$ in the following way

$$(\delta, \varphi) = \varphi(0),$$

and the distribution $\mathscr{P} \frac{1}{x}$ is defined by the formula

$$\left(\mathscr{P}\frac{1}{x}, \varphi\right) = v.p. \int_{-\infty}^{+\infty} \frac{\varphi(x)dx}{x} \equiv \lim_{\varepsilon \to 0+} \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{+\infty} \right) \frac{\varphi(x)dx}{x}.$$

Our main goal in this paper is obtaining an asymptotical expansion for the two-dimensional distribution

$$K_a(\xi_1, \xi_2) \equiv \frac{a}{2\pi^2} \frac{1}{\xi_1^2 - a^2\xi_2^2}$$

with respect to small a^{-1}. It is defined by the corresponding formula $\forall \varphi \in S(\mathbf{R}^2)$

$$(K_a, \varphi) = \frac{a}{2\pi^2} \int_{\mathbf{R}^2} \frac{\varphi(\xi_1, \xi_2)d\xi}{\xi_1^2 - a^2\xi_2^2}. \tag{8}$$

5.2 Asymptotical representation for a solution

Below we denote $lv \equiv V$.

Theorem 2. *If the symbol $A(\xi)$ admits a wave factorization with respect to the cone C_+^a and $|\text{æ} - s| < 1/2$ then the equation 1 has a unique solution in the space $H^s(C_+^a)$, and for a large a it can be represented in the form*

$$\widetilde{u}(\xi) = \frac{i}{2\pi} A_{\neq}^{-1}(\xi) v.p. \int\limits_{-\infty}^{+\infty} \frac{(A_{\underline{=}}^{-1}\widetilde{V})(\eta_1, \xi_2) d\eta_1}{\xi_1 - \eta_1} +$$

$$A_{\neq}^{-1}(\xi) \sum_{m,n} c_{m,n}(a) \int\limits_{-\infty}^{+\infty} (\xi_1 - \eta_1)^m (A_{\underline{=}}^{-1}\widetilde{V})_{\xi_2}^{(n)}(\eta_1, \xi_2) d\eta_1 \qquad (9)$$

assuming $\widetilde{V} \in S(\mathbf{R}^2), A_{\underline{=}}^{-1}\widetilde{V}$ means the function $A_{\underline{=}}^{-1}(\xi)\widetilde{V}(\xi)$.

Proof. Let $\varphi \in S(\mathbf{R}^2)$.

A formal using the Maclaurin formula for the first integral in 4 will lead to the following result

$$(K_a, \varphi) = \frac{1}{2\pi^2} \sum_{k=0}^{\infty} \frac{b^k}{k!} \int\limits_{-N}^{+N} \varphi_{\xi_2}^{(k)}(\xi_1, 0) \left(\int\limits_{-N}^{+N} \frac{t^k dt}{\xi_1^2 - t^2} \right) d\xi_1, \qquad (10)$$

and we need to give a certain sense for the expression in brackets.

Let us denote

$$T_{k,N}(\xi_1) \equiv \int\limits_{-N}^{+N} \frac{t^k dt}{\xi_1^2 - t^2}$$

and reproduce some calculations.

First $T_{k,N}(\xi_1) \equiv 0, \forall k = 2n-1, n \in \mathbf{N}$. So the non-trivial case is $k = 2n, n \in \mathbf{N}$. Let us remind $T_{0,\infty}(\xi_1) = \pi i 2^{-1} \xi_1^{-1}$ [10, 11]. For other cases we can calculate this integral. we have the following

$k = 2$,

$$T_{2,N}(\xi_1) = -2N - 2^{-1}\xi_1^{-1} \ln \frac{N - \xi_1}{N + \xi_1} + \pi i 2^{-1} \xi_1^{-1};$$

$k = 4$,

$$T_{4,N}(\xi_1) = -2/3N^3 - 2\xi_1^2 N - 2^{-1}\xi_1^3 \ln \frac{N - \xi_1}{N + \xi_1} + \pi i 2^{-1} \xi_1^3;$$

$k = 6$,

$$T_{6,N}(\xi_1) = -2/5N^5 - 2/3\xi_1^2 N^3 - 2\xi_1^5 N - 2^{-1}\xi_1^5 \ln \frac{N - \xi_1}{N + \xi_1} + \pi i 2^{-1}\xi_1^5,$$

and so on. One can easily write all expressions for arbitrary $T_{2n,N}(\xi_1)$.

In general one can write

$$T_{2n,N}(\xi_1) = P_{2n-1}(N, \xi_1) - 2^{-1}\xi_1^{2n-1} \ln \frac{N - \xi_1}{N + \xi_1} + \pi i 2^{-1}\xi_1^{2n-1}$$

where $P_{2n-1}(N, \xi_1)$ is a certain polynomial of order $2n - 1$ on variables N, ξ_1.

Therefore instead of the formula (10) we can write

$$(K_a, \varphi) = \frac{i}{2\pi}(\mathscr{P}\frac{1}{\xi_1} \otimes \delta(\xi_2), \varphi)+ \tag{11}$$

$$\frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{b^{2n}}{(2n)!} \int_{-N}^{+N} \varphi_{\xi_2}^{(2n)}(\xi_1, 0) \left(P_{2n-1}(N, \xi_1) - 2^{-1}\xi_1^{2n-1} \ln \frac{N - \xi_1}{N + \xi_1} + \pi i 2^{-1}\xi_1^{2n-1} \right) d\xi_1.$$

Let us describe the polynomial $P_{2n-1}(N, \xi_1)$ more precisely. Obviously

$$P_{2n-1}(N, \xi_1) = c_{2n-1}N^{2n-1} + c_{2n-3}N^{2n-3}\xi_1^2 + \cdots + c_1 N\xi_1^{2n-1}.$$

Further we rewrite the equality (11) in the following form

$$(K_a, \varphi) = \frac{i}{2\pi}(\mathscr{P}\frac{1}{\xi_1} \otimes \delta(\xi_2), \varphi)+$$

$$\frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{b^{2n}}{(2n)!} \sum_{k=1}^{n} c_{2k-1}N^{2k-1} \int_{-N}^{+N} \varphi_{\xi_2}^{(2n)}(\xi_1, 0)\xi_1^{2k-1}d\xi_1-$$

$$\frac{1}{4\pi^2} \sum_{n=1}^{\infty} \frac{b^{2n}}{(2n)!} \int_{-N}^{+N} \varphi_{\xi_2}^{(2n)}(\xi_1, 0)\xi_1^{2n-1} \ln \frac{N - \xi_1}{N + \xi_1}d\xi_1 + \frac{i}{4\pi} \sum_{n=1}^{\infty} \frac{b^{2n}}{(2n)!} \int_{-N}^{+N} \varphi_{\xi_2}^{(2n)}(\xi_1, 0)\xi_1^{2n-1}d\xi_1$$

We will start from two last summands. The second summand does not play any role because

$$\lim_{N\to+\infty} \ln \frac{N - \xi_1}{N + \xi_1} = 0.$$

The third summand we will represent according to lemma 1 (see below) taking into account that we can pass to the limit under $N \to +\infty$

$$\frac{i}{4\pi} \sum_{n=1}^{\infty} \frac{b^{2n}}{(2n)!}(\widetilde{\delta^{(2n-1)}}(\xi_1) \otimes \delta^{(2n)}(\xi_2), \varphi)$$

For the first summand we consider separately the case $Nb \sim 1(N \to \infty, b \to 0)$. in other words we consider a special limit to justify the decomposition. Then

$$\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{b^{2n}}{(2n)!}\sum_{k=1}^{n}c_{2k-1}N^{2k-1}\int\limits_{-N}^{+N}\varphi_{\xi_2}^{(2n)}(\xi_1,0)\xi_1^{2k-1}d\xi_1\sim$$

$$\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n)!}\sum_{k=1}^{n}c_{2k-1}b^{2n-2k+1}\int\limits_{-\infty}^{+\infty}\varphi_{\xi_2}^{(2n)}(\xi_1,0)\xi_1^{2k-1}d\xi_1.$$

Therefore

$$\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n)!}\sum_{k=1}^{n}c_{2k-1}b^{2n-2k+1}\int\limits_{-\infty}^{+\infty}\varphi_{\xi_2}^{(2n)}(\xi_1,0)\xi_1^{2k-1}d\xi_1=$$

$$\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n)!}\sum_{k=1}^{n}c_{2k-1}b^{2n-2k+1}(\widetilde{\delta^{(2k-1)}}(\xi_1)\otimes\delta^{(2n)}(\xi_2),\varphi)$$

One can note if desirable

$$c_{2k-1}=-2(1+\frac{1}{3}+\cdots+\frac{1}{2k-1}).$$

Further details one can find in [13]. $\triangle$

6 Discrete variants

6.1 Discrete functions and operators: preliminaries and examples

Given function u_d of a discrete variable $\tilde{x}\in \mathbf{Z}^m$ we define its discrete Fourier transform by the series

$$(F_d u_d)(\xi)\equiv\widetilde{u}_d(\xi)=\sum_{\tilde{x}\in\mathbf{Z}^m}e^{i\tilde{x}\cdot\xi}u_d(\tilde{x}),\quad \xi\in\mathbf{T}^m,$$

where $\mathbf{T}^m=[-\pi,\pi]^m$ and partial sums are taken over cubes

$$Q_N=\{\tilde{x}\in\mathbf{Z}^m:\tilde{x}=(\tilde{x}_1,\cdots,\tilde{x}_m),\max_{1\leq k\leq m}|\tilde{x}_k|\leq N\}.$$

One can define some discrete operators for such functions u_d.

Example 2. If $K(x),x\in\mathbf{R}^m\setminus\{0\}$, is a Calderon–Zygmund kernel, then the corresponding operator is defined by the formula

$$(K_d u_d)(\tilde{x})=\sum_{\tilde{y}\in\mathbf{Z}^m,\tilde{y}\neq\tilde{x}}K(\tilde{x}-\tilde{y})u_d(\tilde{y}),\quad \tilde{x}\in\mathbf{Z}^m.$$

Example 3. If a first order finite difference of a discrete variable $\tilde{x}_k$ is defined by

$$\delta_k u_d(\tilde{x}) = u_d(\tilde{x}_k + 1) - u_d(\tilde{x}_k),$$

then the discrete Laplacian is

$$(\Delta_d u_d)(\tilde{x}) = \sum_{k=1}^{m} \left(u_d(\tilde{x}_k + 2) - 2u_d(\tilde{x}_k + 1) + u_d(\tilde{x}_k) \right),$$

and its discrete Fourier transform is the function

$$(F_d \Delta_d u_d)(\xi) = \sum_{k=1}^{m} (e^{i\xi_k} - 1)^2.$$

Let $D \subset \mathbf{R}^m$ be a sharp convex cone, $D_d \equiv D \cap \mathbf{Z}^m$, and $L_2(D_d)$ be a space of functions of discrete variable defined on D_d, and $A(\tilde{x})$ be a given function of a discrete variable $\tilde{x} \in \mathbf{Z}^m$. We consider the following types of operators

$$(A_d u_d)(\tilde{x}) = \int_{\mathbf{T}^m} \sum_{\tilde{y} \in D_d} e^{i(\tilde{y} - \tilde{x}) \cdot \xi} \widetilde{A}_d(\xi) \tilde{u}_d(\xi) d\xi, \quad \tilde{x} \in D_d, \tag{12}$$

and introduce the function

$$\widetilde{A}_d(\xi) = \sum_{\tilde{x} \in \mathbf{Z}^m} e^{i\tilde{x} \cdot \xi} A(\tilde{x}), \quad \xi \in \mathbf{T}^m.$$

Definition 2. The function $\widetilde{A}_d(\xi)$ is called a symbol of the operator A_d, and this symbol is called an elliptic symbol if $\tilde{A}_d(\xi) \neq 0, \forall \xi \in \mathbf{T}^m$.

Remark 1. If $D = \mathbf{R}^m$ then an ellipticity is necessary and sufficient condition for the operator A_d to be invertible in the space $L_2(\mathbf{Z}^m)$.

Remark 2. One can define a general pseudo-differential operator with symbol $\tilde{A}(\tilde{x}, \xi)$ depending on a spatial discrete variable $\tilde{x}$ by the similar formula

$$(A_d u_d)(\tilde{x}) = \int_{\mathbf{T}^m} \sum_{\tilde{y} \in D_d} e^{i(\tilde{y} - \tilde{x}) \cdot \xi} \widetilde{A}(\tilde{x}, \xi) \tilde{u}_d(\xi) d\xi, \quad \tilde{x} \in D_d,$$

but taking into account a local principle [2] the main aim in this situation is describing invertibility conditions for model operators like (12) in canonical domains D_d.

Below we will refine the lattice $\mathbf{Z}^m$ and introduce more convenient space scale.

6.2 Discrete Sobolev–Slobodetskii spaces

We consider here refined lattice $h\mathbf{Z}^m, h > 0$, and define corresponding discrete Fourier transform. If a function of a discrete variable is defined on a lattice $h\mathbf{Z}^m$ then its discrete Fourier transform can be introduced by the formula

$$(\tilde{u}_d)(\xi) = \sum_{\tilde{x} \in h\mathbf{Z}^m} u_d(\tilde{x}) e^{i\tilde{x}\cdot\xi} h^m, \quad \xi \in \hbar\mathbf{T}^m,$$

where $\hbar = h^{-1}$.

Let $H^s(h\mathbf{Z}^m)$ denotes the space of functions of a discrete variable for which

$$||u_d||_s^2 \equiv \int\limits_{\hbar\mathbf{T}^m} |\tilde{u}_d(\xi)|^2 (1 + |\sigma_{\Delta_d}(h)(\xi)|)^s d\xi < +\infty,$$

where

$$\sigma_{\Delta_d}(h)(\xi) = h^{-2} \sum_{k=1}^m (e^{ih\xi_k} - 1)^2, \quad \xi \in \hbar\mathbf{T}^m.$$

6.3 Solvability for discrete equations

6.3.1 Conical case and periodic Bochner kernel

Let D be a sharp convex cone, and $\overset{*}{D}$ be a conjugate cone for D, i.e.,

$$\overset{*}{D} = \{x \in \mathbf{R}^m : x \cdot y > 0, \ y \in D\}.$$

Let $T(\overset{*}{D}) \subset \mathbf{C}^m$ be a set of the type $\mathbf{T}^m + i\overset{*}{D}$. For $\mathbf{T}^m \equiv \mathbf{R}^m$ such a domain of multidimensional complex space is called a radial tube domain over the cone $\overset{*}{D}$ ([1, 18, 19]). We introduce the function

$$B_d(z) = \sum_{\tilde{x} \in D_d} e^{i\tilde{x}\cdot z}, \quad z = \xi + i\tau, \quad \xi \in \mathbf{T}^m, \quad \tau \in \overset{*}{D},$$

and define the operator

$$(B_d u)(\xi) = \lim_{\tau \to 0} \int\limits_{\mathbf{T}^m} B_d(z - \eta) u_d(\eta) d\eta.$$

Lemma 1. *For arbitrary $u_d \in L_2(\mathbf{Z}^m)$, the following property*

$$F_d P_{D_d} u_d = B_d F_d u_d$$

holds.

Let us define the subspace $A(\mathbf{T}^m) \subset L_2(\mathbf{T}^m)$ consisting of functions which admit a holomorphic continuation into $T(\overset{*}{D})$ and satisfy the condition

$$\sup_{\tau \in \overset{*}{D}} \int\limits_{\mathbf{T}^m} |\tilde{u}_d(\xi + i\tau)|^2 d\xi < +\infty.$$

In other words, the space $A(\mathbf{T}^m) \subset L_2(\mathbf{T}^m)$ consists of boundary values of holomorphic in $T(\overset{*}{D})$ functions.

Let us denote

$$B(\mathbf{T}^m) = L_2(\mathbf{T}^m) \ominus A(\mathbf{T}^m),$$

so that $B(\mathbf{T}^m)$ is a direct complement of $A(\mathbf{T}^m)$ in $L_2(\mathbf{T}^m)$.

6.3.2 A jump problem

We formulate the problem in the following way: finding a pair of functions $\Phi^\pm, \Phi^+ \in A(\mathbf{T}^m), \Phi^- \in B(\mathbf{T}^m)$, such that

$$\Phi^+(\xi) - \Phi^-(\xi) = g(\xi), \quad \xi \in \mathbf{T}^m, \tag{13}$$

where $g(\xi) \in L_2(\mathbf{T}^m)$ is given.

Lemma 2. *The operator $B_d : L_2(\mathbf{T}^m) \to A(\mathbf{T}^m)$ is a bounded projector. A function $u_d \in L_2(D_d)$ iff its Fourier transform $\tilde{u}_d \in A(\mathbf{T}^m)$.*

Theorem 3. *The jump problem has unique solution for arbitrary right-hand side from $L_2(\mathbf{T}^m)$.*

Example 4. If $m = 2$ and D is the first quadrant in a plane then a solution of a jump problem is given by formulas

$$\Phi^+(\xi) = \frac{1}{(4\pi i)^2} \lim_{\tau \to 0} \int\limits_{-\pi}^{\pi} \int\limits_{-\pi}^{\pi} \cot \frac{\xi_1 + i\tau_1 - t_1}{2} \cot \frac{\xi_2 + i\tau_2 - t_2}{2} g(t_1, t_2) dt_1 dt_2$$

$$\Phi^-(\xi) = \Phi^+(\xi) - g(\xi), \quad \tau = (\tau_1, \tau_2) \in D.$$

6.3.3 A general statement

It looks as follows. Finding a pair of functions $\Phi^\pm, \Phi^+ \in A(\mathbf{T}^m), \Phi^- \in B(\mathbf{T}^m)$, such that

$$\Phi^+(\xi) = G(\xi)\Phi^-(\xi) + g(\xi), \quad \xi \in \mathbf{T}^m, \tag{14}$$

where $G(\xi), g(\xi)$ are given periodic functions. If $G(\xi) \equiv 1$, we have the jump problem (3).

Like classical studies [4, 6], we want to use a special representation for an elliptic symbol to solve the problem (4).

6.3.4 Periodic wave factorization

Let us denote $H^s(D_d)$ a subspace of $H^s(\mathbf{Z}^m)$ consisting of functions of discrete variable $\tilde{x}$ for which their supports belong to $\overline{D_d}$, and $\widetilde{H}^s(D_d), \widetilde{H}^s(\mathbf{Z}^m)$ their Fourier images.

Lemma 3. *For $|s| < 1/2$, the operator B_d is a bounded projector $\widetilde{H}^s(\mathbf{Z}^m) \to \widetilde{H}^s(D_d)$, and a jump problem has unique solution $\Phi^+ \in \widetilde{H}^s(D_d), \Phi^- \in \widetilde{H}^s(\mathbf{Z}^m \setminus D_d)$ for arbitrary $g \in \widetilde{H}^s(\mathbf{Z}^m)$.*

Definition 3. Periodic wave factorization for elliptic symbol $\tilde{A}(\xi)$ is called its representation in the form
$$\tilde{A}_d(\xi) = \tilde{A}_{\neq}(\xi)\tilde{A}_{=}(\xi)$$

where the factors $A_{\neq}^{\pm 1}(\xi), A_{=}^{\pm 1}(\xi)$ admit bounded holomorphic continuation into domains $T(\pm \overset{*}{D})$.

Theorem 4. *If $|s| < 1/2$ and the elliptic symbol $\tilde{A}_d(\xi) \in S_\alpha(\mathbf{T}^m)$ admits periodic wave factorization, then the operator A_d is invertible in the space $H^s(D_d)$.*

7 Conclusion

As it was shown all aspects of this problem of solving the equation (5) are closely related and use similar ideas and methods. Author hopes that in future it will be possible to unit these considerations in a general theory of elliptic pseudo-differential equations on manifolds with non-smooth boundaries.

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