SYMBOL CALCULATIONS THE APPROXIMATE INTEGRALS OF MOTION FOR HAMILTONIAN SYSTEMS WITH N DEGREES OF FREEDOM

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On the base of the Birkhoff-Gustavson method the algorithm and program in MAPLE for symbol-numerical calculations of the normal form and integrals of motion for resonance and nonresonance Hamiltonian systems with any number of degrees of freedom are developed. Examples of this program's work for some Hamiltonian systems are presented.

1. INTRODUCTION

Since many dynamic systems of the classical mechanics can not be presented in an analytical form (for example, [1–3]), different approximate analytical methods [4–8] and many direct numerical ones for the direct calculations [9–11] were developed.

At present time the perspective direction is the development of hybrid or combine methods, in which first the symbolic calculations are performed with the consequent numerical computing using contemporary mathematical packages, for example, MAPLE, REDUCE, MATHEMATICA and others [12–14].

In the present paper the developed algorithm and program for calculation of normal form and integrals of motion are presented. Examples for some Hamiltonian systems with n degrees of freedom are also given.

2. THE BIRKHOFF-GUSTAVSON METHOD

Let us the classical system with n degrees of freedom is given, which Hamiltonian is presented in the polynomial form of the expansion

$$H = \sum_{k=1}^{n} \frac{\omega_k}{2}(p_k^2 + q_k^2) + \\ + \sum_{S=3}^{\infty} \sum_{\substack{l_1+\ldots+l_n+\\+m_1+\ldots+m_n=S}} C_{l_1,\ldots,l_n,m_1,\ldots,m_n} \times \\ \times p_1^{l_1} p_2^{l_2} \ldots p_n^{l_n} \cdot q_1^{m_1} q_2^{m_2} \ldots q_n^{m_n}, \quad (1)$$

where $p = (p_1, p_2, \ldots, p_n)$ and $q = (q_1, q_2, \ldots, q_n)$ are canonically conjugated variables, $C_{l_1,\ldots,l_n,m_1,\ldots,m_n}$ are known coefficients. Frequencies ω_k are resonance or nonresonance quantities. For resonance frequencies there are r-relations:

$$\sum\nolimits_{k=1}^{k=n} a_{ik}\omega_k = 0, i = 1, 2, 3\ldots, r, \quad (2)$$

where quantities a_{ik} are integer numbers.

As known [15], the classical Hamilton function $H(q,p)$ is in the normal form if there is the relation

$$D\,G(\xi,\eta) = 0, \quad (3a)$$

where the expression

$$D = \sum_{k=1}^{n} \omega_k \left(\eta_k \frac{\partial}{\partial \xi_k} - \xi_k \frac{\partial}{\partial \eta_k} \right) \quad (3b)$$

is so called the normal form of the differential operator.

As known [15], one has to solve basic equation

$$D(q,\eta)\, W^{(S)}(q,\eta) = -H^{(S)}(q,\eta) + \Gamma^{(S)}(q,\eta), \quad (4)$$

where $W^{(S)}$ is a homogeneous polynomial of the S degree in generating function

$$F(q,\eta) = q \cdot \eta + W(q,\eta), \quad (5)$$

that reduces to normal form any term of initial Hamiltonian (1) of the S degree. Terms of Hamilton function with higher degree than S are calculated according to formula

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$$\Gamma^{(i)}(\xi,\eta) = H^{(i)}(\xi,\eta) + \sum_{|k|} \frac{1}{k!}\left(\left(\frac{\partial W^{(s)}}{\partial \xi}\right)^k \left(\frac{\partial^{|k|} H^{(l)}}{\partial \eta^k}\right) - \left(\frac{\partial W^{(s)}}{\partial \eta}\right)^k \left(\frac{\partial^{|k|} \Gamma^{(l)}}{\partial \xi^k}\right)\right), \tag{6}$$

where $i = S+1, S+2, \ldots$ and $l - |k| + |k|\ (S-1) = i$, $1 \le |k| \le l < i$, $l \ge 2$, $s \ge 3$, $|k| = |k_1| + |k_2| + \ldots + |k_n|$, $k! = k_1! \cdot k_2! \cdot \ldots \cdot k_n!$.

In paper [15] it was proved that there exist $(n-r)$ approximated, independent integrals of motion of type

$$I(\xi,\eta) = \sum_{i=1}^{n} \frac{\mu_i}{2}(\xi_i^2 + \eta_i^2), \tag{7}$$

where $\mu = (\mu_1, \mu_2, \ \ldots \ , \mu_n)$ is a n-dimensional real vector, there is a relation

$$\sum_{k=1}^{n} a_{ik}\mu_k = 0, i = 1, 2, 3, \ldots, r. \tag{8}$$

Performing the back transformation from $(\xi,\eta) \to (q,p)$, one obtains the integrals of motion in initial variables $p = (p_1, p_2, \ldots, p_n)$, $q = (q_1, q_2, \ldots, q_n)$.

3. DESCRIPTION OF ALGORITHM

Input:
n – number of degree of freedom;
SMAX – desired maximal degree of normal form and integrals of motion; jmax – maximal degree of initial Hamiltonian function, that quadric part is in normal form, i.e. v[2] + v[3]+ … +v[j]+ … +v[jmax] is the potential part of initial Hamiltonian function;
if gip $\neq 0$, then normal form is calculated in action-angle variables;
if intg2 $\neq 0$, then integrals of motion are calculated;
if intg2 $\neq 0$ and test1 $\neq 0$, then Poisson bracket pb = {H, intg2} should not be equal 0;
if intg2 $\neq 0$ and test2 $\neq 0$, then Htest2 has to be equal initial Hamiltonian function.

Output:
$g_{SMAX} = g^{(2)} + g^{(3)} + \ \ldots \ + g^{(SMAX)}$ – normal form;
$w_{SMAX} = w^{(3)} + w^{(4)} + \ \ldots \ + w^{(SMAX)}$ – the polynomial of the generating function, where $g^{(S)}$ and $w^{(S)}$ are homogeneous polynomials of S degree $p_1^{l_1} p_2^{l_2} \ldots p_n^{l_n} \cdot q_1^{m_1} q_2^{m_2} \ldots q_n^{m_n}$, $S = l_1 + l_2 + \ \ldots \ + l_n + m_1 + m_2 + \ \ldots \ + m_n$;
intg[i] – integrals of motion in variables $(p_1^{l_1}, p_2^{l_2}, \ldots, p_n^{l_n})$, $(q_1^{m_1}, q_2^{m_2}, \ldots, q_n^{m_n})$.

4. EXAMPLES

Let us present results for some Hamiltonian systems.

4.1. Resonant Hamiltonian system $(\omega_1 : \omega_2 : \omega_3 = 1 : 1 : 1)$ with $n = 3$ degrees of freedom [16]:

$$H = \frac{1}{2}\omega_1\left(p_1^2 + q_1^2\right) + \frac{1}{2}\omega_2\left(p_2^2 + q_2^2\right) + \frac{1}{2}\omega_3\left(p_3^2 + q_3^2\right) + q_1 q_3^2 + q_2 q_3^2. \tag{9}$$

Normal form with order of $SMAX = 4$ was obtained:

$$G_4 = G^{(2)} + G^{(3)} + G^{(4)}, \tag{10}$$

$$G^{(2)} = \frac{1}{2}\eta_1^2 + \frac{1}{2}\xi_1^2 + \frac{1}{2}\eta_2^2 + \frac{1}{2}\xi_2^2 + \frac{1}{2}\eta_3^2 + \frac{1}{2}\xi_3^2, \tag{11a}$$

$$G^{(3)} = 0, \tag{11b}$$

$$\begin{aligned} G^{(4)} = & -\eta_3\xi_2\xi_3\eta_1 - \eta_3\xi_2\xi_3\eta_2 - \eta_3\xi_1\xi_3\eta_2 - \eta_3\xi_1\xi_3\eta_1 + \frac{1}{6}\eta_3^2\xi_1\xi_2 + \frac{1}{6}\xi_3^2\eta_1\eta_2 - \frac{5}{6}\xi_3^2\xi_1\xi_2 - \\ & -\frac{5}{6}\eta_3^2\eta_1\eta_2 + \frac{1}{12}\eta_3^2\xi_1^2 + \frac{1}{12}\eta_3^2\xi_2^2 + \frac{1}{12}\xi_3^2\eta_1^2 + \frac{1}{12}\xi_3^2\eta_2^2 - \frac{5}{12}\xi_3^2\xi_1^2 - \frac{5}{12}\xi_3^2\xi_2^2 - \frac{5}{12}\eta_3^2\eta_1^2 - \\ & -\frac{5}{12}\eta_3^2\eta_2^2 - \frac{5}{12}\eta_3^2\xi_3^2 - \frac{5}{24}\eta_3^4 - \frac{5}{24}\xi_3^4, \end{aligned} \tag{11c}$$

and integral of motion:

$$\begin{aligned} I = & \, p_3 q_2 q_3 p_1 + p_3 q_2 q_3 p_2 + p_3 q_1 q_3 p_2 + p_3 q_1 q_3 p_1 + q_1 q_3^2 + q_2 q_3^2 - \frac{1}{6}p_3^2 q_1 q_2 - \frac{1}{6}q_3^2 p_1 p_2 + \\ & +\frac{5}{6}q_3^2 q_1 q_2 + \frac{5}{6}p_3^2 p_1 p_2 - \frac{1}{12}p_3^2 q_1^2 - \frac{1}{12}p_3^2 q_2^2 - \frac{1}{12}q_3^2 p_1^2 - \frac{1}{12}q_3^2 p_2^2 + \frac{5}{12}q_3^2 q_1^2 + \frac{5}{12}q_3^2 q_2^2 + \\ & +\frac{5}{12}p_3^2 p_1^2 + \frac{5}{12}p_3^2 p_2^2 + \frac{5}{12}p_3^2 q_3^2 + \frac{1}{2}p_1^2 + \frac{1}{2}q_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2}q_2^2 + \frac{1}{2}p_3^2 + \frac{1}{2}q_3^2 + \frac{5}{24}p_3^4 + \frac{5}{24}q_3^4. \end{aligned} \tag{12}$$

4.2. Resonant Hamiltonian system ($\omega_1 : \omega_2 : \omega_3 = 1 : 1 : 1$) with $n = 3$ degrees of freedom:

$$H = \frac{1}{2}\omega_1\left(p_1^2 + q_1^2\right) + \frac{1}{2}\omega_2\left(p_2^2 + q_2^2\right) + \frac{1}{2}\omega_3\left(p_3^2 + q_3^2\right) + c\left(q_1^2 q_2^2 + q_2^2 q_3^2 + q_1^2 q_3^2\right). \tag{13}$$

Normal form with order of $SMAX = 4$ was obtained:

$$G_4 = G^{(2)} + G^{(3)} + G^{(4)}, \tag{14}$$

$$G^{(2)} = \frac{1}{2}\eta_1^2 + \frac{1}{2}\xi_1^2 + \frac{1}{2}\eta_2^2 + \frac{1}{2}\xi_2^2 + \frac{1}{2}\eta_3^2 + \frac{1}{2}\xi_3^2, \tag{15a}$$

$$G^{(3)} = 0, \tag{15b}$$

$$\begin{aligned} G^{(4)} = {} & \frac{1}{2}c\eta_1\xi_1\eta_3\xi_3 + \frac{1}{2}c\eta_1\xi_1\eta_2\xi_2 + \frac{1}{2}c\eta_2\xi_2\eta_3\xi_3 + \frac{3}{8}c\xi_1^2\xi_3^2 + \frac{3}{8}c\xi_1^2\xi_2^2 + \\ & + \frac{3}{8}c\xi_2^2\xi_3^2 + \frac{1}{8}c\xi_1^2\eta_3^2 + \frac{3}{8}c\eta_1^2\eta_2^2 + \frac{3}{8}c\eta_2^2\eta_3^2 + \frac{3}{8}c\eta_1^2\eta_3^2 + \frac{1}{8}c\eta_1^2\xi_2^2 + \frac{1}{8}c\xi_1^2\eta_2^2 + \\ & + \frac{1}{8}c\eta_2^2\xi_3^2 + \frac{1}{8}c\xi_2^2\eta_3^2 + \frac{1}{8}c\eta_1^2\xi_3^2, \end{aligned} \tag{15c}$$

and integral of motion:

$$\begin{aligned} I = {} & \frac{1}{2}q_1^2 + \frac{1}{2}p_1^2 + \frac{1}{2}q_2^2 + \frac{1}{2}p_2^2 + \frac{1}{2}q_3^2 + \frac{1}{2}p_3^2 - \frac{1}{2}c\,p_1q_1p_2q_2 + \frac{5}{8}c\,q_1^2q_2^2 + \frac{5}{8}c\,q_2^2q_3^2 + \\ & + \frac{5}{8}c\,q_1^2q_3^2 - \frac{1}{8}c\,q_1^2p_2^2 - \frac{3}{8}c\,p_1^2p_2^2 - \frac{1}{8}c\,p_1^2q_2^2 - \frac{1}{8}c\,q_1^2p_3^2 - \frac{3}{8}c\,p_1^2p_3^2 - \frac{3}{8}c\,p_2^2p_3^2 - \\ & - \frac{1}{8}c\,q_2^2p_3^2 - \frac{1}{8}c\,p_1^2q_3^2 - \frac{1}{8}c\,p_2^2q_3^2 - \frac{1}{2}c\,p_1q_1p_3q_3 - \frac{1}{2}c\,p_2q_2p_3q_3. \end{aligned} \tag{16}$$

4.3. Resonant Hamiltonian system ($\omega_1 : \omega_2 : \omega_3 : \omega_4 = 1 : 1 : 1 : 1$) with $n = 4$ degrees of freedom [17]:

$$H = 1/2\sum_{k=1}^{k=4}\omega_k\left(p_k^2 + q_k^2\right) + 1/2\left(\sum_{k=1}^{k=4} q_k^2/\omega_k\right)^2 - \left(q_1q_3/\sqrt{\omega_1\omega_3} + q_2q_4/\sqrt{\omega_2\omega_4}\right)^2. \tag{17}$$

Normal form with order of $SMAX = 4$ was obtained:

$$G_4 = G^{(2)} + G^{(3)} + G^{(4)}, \tag{18}$$

$$G^{(2)} = \frac{1}{2}\eta_1^2 + \frac{1}{2}\xi_1^2 + \frac{1}{2}\eta_2^2 + \frac{1}{2}\xi_2^2 + \frac{1}{2}\eta_3^2 + \frac{1}{2}\xi_3^2 + \frac{1}{2}\eta_4^2 + \frac{1}{2}\xi_4^2, \tag{19a}$$

$$G^{(3)} = 0, \tag{19b}$$

$$\begin{aligned} G^{(4)} = {} & \frac{3}{4}\xi_1\xi_3\xi_2\xi_4 + \frac{3}{8}\xi_1^2\xi_2^2 + \frac{3}{8}\xi_1^2\xi_4^2 + \frac{3}{8}\xi_2^2\xi_3^2 + \frac{3}{8}\xi_3^2\xi_4^2 + \frac{3}{16}\xi_1^4 + \frac{3}{16}\xi_2^4 + \frac{3}{16}\xi_3^4 + \frac{3}{16}\xi_4^4 - \\ & - \frac{3}{4}\eta_1\eta_3\eta_2\eta_4 + \frac{1}{2}\eta_1\xi_1\eta_2\xi_2 + \frac{1}{2}\eta_1\xi_1\eta_4\xi_4 + \frac{1}{2}\eta_2\xi_2\eta_3\xi_3 + \frac{1}{2}\eta_3\xi_3\eta_4\xi_4 - \frac{1}{4}\eta_1\eta_3\xi_2\xi_4 - \\ & - \frac{1}{4}\eta_1\xi_3\eta_2\xi_4 - \frac{1}{4}\eta_1\xi_3\xi_2\eta_4 - \frac{1}{4}\xi_1\eta_3\eta_2\xi_4 - \frac{1}{4}\xi_1\eta_3\xi_2\eta_4 - \frac{1}{4}\xi_1\xi_3\eta_2\eta_4 + \frac{3}{8}\eta_1^2\eta_2^2 + \frac{3}{8}\eta_1^2\eta_4^2 + \\ & + \frac{3}{8}\eta_2^2\eta_3^2 + \frac{3}{8}\eta_3^2\eta_4^2 + \frac{3}{8}\eta_1^2\xi_1^2 + \frac{1}{8}\eta_1^2\xi_2^2 + \frac{1}{8}\xi_1^2\eta_2^2 + \frac{1}{8}\eta_1^2\xi_4^2 + \frac{1}{8}\xi_1^2\eta_4^2 + \frac{3}{8}\eta_2^2\xi_2^2 + \frac{1}{8}\eta_2^2\xi_3^2 + \\ & + \frac{1}{8}\xi_2^2\eta_3^2 + \frac{3}{8}\eta_3^2\xi_3^2 + \frac{1}{8}\eta_3^2\xi_4^2 + \frac{1}{8}\xi_3^2\eta_4^2 + \frac{3}{8}\eta_4^2\xi_4^2 + \frac{3}{16}\eta_1^4 + \frac{3}{16}\eta_2^4 + \frac{3}{16}\eta_3^4 + \frac{3}{16}\eta_4^4, \end{aligned} \tag{19c}$$

and integral of motion:

$$\begin{aligned} I = {} & -\frac{5}{4}q_1q_3q_2q_4 + \frac{1}{2}p_1^2 + \frac{1}{2}q_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2}q_2^2 + \frac{1}{2}p_3^2 + \frac{1}{2}q_3^2 + \frac{1}{2}p_4^2 + \frac{1}{2}q_4^2 + \frac{5}{8}q_1^2q_2^2 + \\ & + \frac{5}{8}q_1^2q_4^2 + \frac{5}{8}q_2^2q_3^2 + \frac{5}{8}q_3^2q_4^2 + \frac{5}{16}q_1^4 + \frac{5}{16}q_2^4 + \frac{5}{16}q_3^4 + \frac{5}{16}q_4^4 + \frac{3}{4}p_1p_3p_2p_4 - \frac{1}{2}p_1q_1p_2q_2 - \\ & - \frac{1}{2}p_1q_1p_4q_4 - \frac{1}{2}p_2q_2p_3q_3 - \frac{1}{2}p_3q_3p_4q_4 + \frac{1}{4}p_1p_3q_2q_4 + \frac{1}{4}p_1q_3p_2q_4 + \frac{1}{4}p_1q_3q_2p_4 + \\ & + \frac{1}{4}q_1p_3p_2q_4 + \frac{1}{4}q_1p_3q_2p_4 + \frac{1}{4}q_1q_3p_2p_4 - \frac{3}{8}p_1^2p_2^2 - \frac{3}{8}p_1^2p_4^2 - \frac{3}{8}p_2^2p_3^2 - \frac{3}{8}p_3^2p_4^2 - \\ & - \frac{3}{8}p_1^2q_1^2 - \frac{1}{8}p_1^2q_2^2 - \frac{1}{8}q_1^2p_2^2 - \frac{1}{8}p_1^2q_4^2 - \frac{1}{8}q_1^2p_4^2 - \frac{3}{8}p_2^2q_2^2 - \frac{1}{8}p_2^2q_3^2 - \frac{1}{8}q_2^2p_3^2 - \\ & - \frac{3}{8}p_3^2q_3^2 - \frac{1}{8}p_3^2q_4^2 - \frac{1}{8}q_3^2p_4^2 - \frac{3}{8}p_4^2q_4^2 - \frac{3}{16}p_1^4 - \frac{3}{16}p_2^4 - \frac{3}{16}p_3^4 - \frac{3}{16}p_4^4. \end{aligned} \tag{20}$$

5. CONCLUSIONS

Presented program allows us to carry out on computers the cumbersome analytical calculations related to reducing the classical Hamiltonian to the Birkhoff-Gustavson normal form and obtaining the integrals of motion for n-dimensional Hamiltonian systems. As input to our program it is enough to give the parameters of the Hamiltonian, frequency relations and to assign the SMAX degree of the needed approximation. The program can be used in any scientific centers, where the investigations of nonlinear processes are conducted, as well as in educational purposes for writing yearly projects and graduating thesis by students.

References

1. V.I. Arnold. *Mathematical methods of classical mechanics.* M.: "Nauka", 1974, 432 p. (in Russian).
2. A.M. Perelomov. *Integrable systems of classical mechanics and Lee groups.* Izhevsk: NNC "Regular and chaotic dynamics", 2002, 238 p. (in Russian).
3. A. Goriely. *Integrability and singularity* . M.-Izhevsk: NNC "Regular and chaotic dynamics", 2006, 316 p. (in Russian).
4. G.E.O. Giacaglia. *Perturbation theory methods for nonlinear systems.* M.: "Nauka", 1979, 320 p. (in Russian).
5. N.N. Bogolubov, Yu.A. Mitropolsky. *Asymptotic methods in theory of nonlinear vibrations.* M.: "Nauka", 1974, 504 p. (in Russian).
6. A.H. Nayfeh. *Perturbation methods* . M.: "Mir", 1976, 456 p. (in Russian).
7. A.H. Nayfeh. *Introduction in perturbation techniques.* M.: "Mir", 1984, 535 p. (in Russian).
8. V. Basios, N.A. Chekanov, B.L. Markovski, V.A. Rostovtsev, S.I. Vinitsky. GITA: a REDUCE program for the normalization of polynomial Hamiltonians // *Comp. Phys. Commun.* 1995, v. 90, p. 355-368.
9. I.S. Berezin, N.P. Zhidkov. *Computation methods.* M.: Gos. phys.-mat. lit., 1962, v. 2, 640 p. (in Russian).
10. G. Forsythe, M. Malcolm, C. Moler. *Computer methods for mathematical computations.* M.: "Mir", 1980, 277 p. (in Russian).
11. J.H. Wilkinson, C. Reinsch. *Handbook for automatic computation linear algebra.* M.: "Mashinostroenie", 1976, 392 p. (in Russian).
12. V.P. Diakonov. *Maple 8 in mathematics, physics and education.* M.: "SOLON-Press", 2003, 656 p. (in Russian).
13. A.C. Hearn. *REDUCE User's Manual.* Santa Monica, CA, USA, 2004, 215 p.
14. A.G. Akritas. *Elements of computer algebra with applications.* M: "Mir", 1994, 544 p. (in Russian).
15. F.G. Gustavson. On construction formal integral of a Hamiltonian system near an equilibrium point // *Astronom. J.* 1966, v. 71, N 8, p. 670-686.
16. J. Hietarinta. Direct methods for the search of the second invariant // *Phys. Rep.* 1987, v. 147, p. 87-154.
17. A. Fordy, S. Vojciechowski, I. Marshall. A family of integrable quartic potentials related to symmetric spaces // *Phys. Lett.* 1986, v. 113A, p. 395-400.

СИМВОЛЬНЫЕ ВЫЧИСЛЕНИЯ ПРИБЛИЖЕННЫХ ИНТЕГРАЛОВ ДВИЖЕНИЯ ДЛЯ ГАМИЛЬТОНОВЫХ СИСТЕМ С N СТЕПЕНЯМИ СВОБОДЫ

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На основе метода нормальных форм Биркгофа-Густавсона разработаны алгоритм и программа символьно-численного построения нормальной формы и формальных интегралов движения для резонансных и нерезонансных гамильтоновых систем с произвольным числом степеней свободы в среде MAPLE. Представлены примеры работы программы для некоторых гамильтоновых систем.

СИМВОЛЬНІ ОБЧИСЛЕННЯ НАБЛИЖЕНИХ ІНТЕГРАЛІВ РУХУ ДЛЯ ГАМІЛЬТОНОВИХ СИСТЕМ З N СТУПЕНЯМИ СВОБОДИ

В.Є. Богачов, М.О. Чеканов

На основі методу нормальних форм Біркгофа-Густавсона розроблені алгоритм і програма символьно-числової побудови нормальної форми і формальних інтегралів руху для резонансних і нерезонансних гамільтонових систем з довільним числом ступенів свободи в середовищі MAPLE. Представлено приклади роботи програми для деяких гамільтонових систем.