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THE DISTRIBUTION DENSITY OF SQUARE VALUE PROBABILITIES FUNCTIONALITY FROM TRAJECTORIES OF WIENER PROCESS

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Abstract: In the article authors develop an approach to calculating the statistic development probability for composite functions of square values in Gaussian casual process trajectories. Calculating distribution density for additive composite functions is based on standard Wiener process trajectories. Authors have developed a density formula for uniformly convergent decomposition, with x = 0. The convergence is exponentially fast. The calculation of the approximated probability is presented: $\Pr\{J_T[w]>c\}$.

Keywords: stationary processes; casual Gaussian processes; Wiener process; the density of distribution of probabilities; additive functionality; return transformation of Laplace; calculating the successive approximations; accuracy assessment.

I. INTRODUCTION

In work the task about the calculating the density of distribution of probabilities of a random variable $J_T[w]$ where is considered $J_T[w]$

$$J[u]=\int_0^T |u(t)|^2\,dt \qquad (1)$$

- functionality in space of L2 [6] $[0,T]$.

Let $w(t);t\in[0,T],T>0$ - trajectories of standard Wiener process on [0, T] with a population mean of E $w^2(t)=t$. This task, as well as other tasks about the calculating the density of distributions of probabilities of casual values of additive square functionalities from trajectories of Gaussian casual processes, is classical.

From the analytical point of view, Gaussian casual processes are, apparently, the simplest casual processes. It is connected with the fact that the Gaussian form of private multipoint distributions of Gaussian processes allows to calculate obviously characteristic functionality of each such process and to apply to their research well-developed analysis methods in Hilbert space [1,2]. Now the class of Gaussian casual processes, at least stationary, is well studied. For Gaussian processes, the problem about the calculating the characteristic functions for random variables such is essentially solved. On the basis of this method a large number of solutions of the specific objectives having various appendices [3,4] is so far received). For example, the first such result for the simplest stationary Gaussian process (Ornstein-Ulenbek's process) was received still in [5]. However, the problem of restoration of the density of distribution on the basis of the turning-out formulas for characteristic functions remains the guaranteed accuracy which is still poorly investigated in sense of receiving the approximations with suitable for use in all range of change of a random variable. The usual approach to the solution of this task (see, for example, [6]) leads to approximate formulas for the density of distribution of f (x), $x \in [0,\infty)$ suitable for the assessment of probabilities of big evasion, i.e. in asymptotic area x $\to$ ∞ changes of values of a random variable.

In this work we investigate a problem of calculating the successive approximations of the density of distribution of probabilities of f (x) for casual values of functionality (1) in any piece [0, M], ∞> M> 0.

II. MATERIAL AND METHODS

As trajectories of standard Wiener process { w(t); t ≥0}, $w^2(t) = t\,1$ are continuous $\mathrm{E}\,w^2(t) = t$ with probability, honor probably for each of them the random variable is defined $J_T[w]$.

We will proceed [2] of the following formula for the making function of a random variable (1)

$$Q_T(\lambda) = \mathrm{E}\exp(-\lambda J_T[w]) = \left(ch\left(\lambda^{1/2}T\right)\right)^{-1/2},$$

$$Q_T(\lambda) = Q_1\left(\lambda T^2\right).$$

The density of distribution of f (x) random variable $J_T[w]$ is defined by the return transformation of Laplace

$$f(x) = \frac{1}{2\pi i}\int_{-i\infty+c}^{i\infty+c} e^{\lambda x}\left(ch\left(\lambda^{1/2}T\right)\right)^{-1/2} d\lambda \qquad (2)$$

where c>0, and a section in the plane λ is made by a negative part of the valid axis.

Let's enter density $g(x) = T^2 f\left(T^2 x\right)$ at $x \in [0,\infty)$ for which replacement of a variable of integration in (2) we receive

$$g(x) = \frac{1}{2\pi i}\int_{-i\infty+c}^{i\infty+c} e^{\lambda x}\left(ch\left(\lambda^{1/2}\right)\right)^{-1/2} d\lambda \;=$$

$$\frac{1}{\sqrt{2\pi} i}\int_{-i\infty+c}^{i\infty+c}\left(\frac{\exp\left(2\lambda x - \lambda^{1/2}\right)}{1+\exp\left(-2\lambda^{-1/2}\right)}\right)^{1/2} d\lambda\,. \qquad (3)$$

III. RESULTS AND DISCUSSION

Let's prove the following theorem [7, 8].THEOREM. The density of g (x) is represented as the following absolutely meeting row

$$g(x) = \sqrt{\frac{2}{\pi x^3}}\sum_{l=0}^{\infty}\frac{(-1)^l(2l)!}{4^l(l!)^2}\left(l+\frac{1}{4}\right)\exp\left(-\frac{1}{x}\left(l+\frac{1}{4}\right)^2\right) (4)$$

for which N-y rest

$$g_{N-1}(x) = \sqrt{\frac{2}{\pi x^3}}\sum_{l=N}^{\infty}\frac{(-1)^l(2l)!}{4^l(l!)^2}\left(l+\frac{1}{4}\right)\exp\left(-\frac{1}{x}\left(l+\frac{1}{4}\right)^2\right)$$

it is estimated by the size

$$\left|g_{N-1}(x)\right| \le \sqrt{\frac{2}{\pi x^3}}\frac{(2N)!}{4^N(N!)^2}\left(N+\frac{1}{4}\right)\exp\left(-\frac{1}{x}\left(N+\frac{1}{4}\right)^2\right) \quad (5)$$

PROOF: Let's put in (3) c=0 as the available features lie on a negative half shaft. We deform an integration contour in the contour of C consisting of consecutive passing of ways { s-iε; s ∈ $(-\infty;0]$}, { ε · eis; s ∈ $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$}, {-s+iε; s ∈ $(0;+\infty]$}. Such deformation is possible since.

$$\left|ch\left(\lambda^{1/2}\right)\right|^2 = ch\left(\lambda^{1/2}\right)ch\left(\left(\lambda^*\right)^{1/2}\right) = \frac{1}{2}\left(ch\left(2\,\mathrm{Re}\left(\lambda^{1/2}\right)\right) + ch\left(2i\,\mathrm{Im}\left(\lambda^{1/2}\right)\right)\right) >$$

$$> \frac{1}{2}\left(ch\left(2R^{1/2}\cos(\varphi/2)\right)-1\right) = sh^2\left(R^{1/2}\cos(\varphi/2)\right),$$

where $\lambda = Re^{i\varphi}$ and on an arch of a circle $\left\{\lambda; \varphi \in \left[\frac{\pi}{2};\pi\right)\right\}$ assessment for mo the module of sub integral expression in is carried out (3),

$$\left|\frac{\exp(\lambda x)}{\left(ch\left(\lambda^{1/2}\right)\right)^{1/2}}\right| \le \frac{\exp(xR\cos\varphi)}{\left(sh\left(R^{1/2}\cos(\varphi/2)\right)\right)^{1/2}},$$

guaranteeing at x> 0, $R^{1/2}\cos(\varphi/2) < \varepsilon$ the performance of a condition of Jordan on this arch at as much as small ε> 0, since $\cos\varphi < 0$. The same takes place for an arch $\left\{\lambda; \varphi \in \left(-\pi, \frac{\pi}{2}\right]\right\}$

In (3) we will make replacement of a variable of integration $\lambda^{1/2} = q$, then $\lambda = q^2, d\lambda = 2q dq$. At the same time the contour of C in the plane λ after the transition to a limit $\varepsilon \to 0$ to turn into a straight line { q=is, s ∈ R } into the planes q. After these transformations we have:

$$g(x) = \frac{\sqrt{2}}{\pi i} \int_{-i\infty}^{i\infty} q \left(\frac{\exp(2q^2 x - q)}{1 + \exp(-2q)} \right)^{1/2} dq\,.$$

Let's pass to integration on a variable s, q = is, dq = with ids. We receive

$$g(x) = i \frac{\sqrt{2}}{\pi} \int_{-\infty}^{+\infty} s \left(\frac{\exp(-2xs^2 - is)}{1 + \exp(-2is)} \right)^{1/2} ds \qquad (6)$$

In the last integral we will makeshift $s + i(4x)^{-1} \Rightarrow s$ of a variable of integration, therefore, we will receive:

$$g(x) = i \frac{\sqrt{2}}{\pi} \int_{-\infty}^{+\infty} s \left(\frac{\exp\left(-2x\left(s - i(4x)^{-1}\right)^2 - (8x)^{-1}\right)}{1 + \exp\left(-2i\left(s + i(4x)^{-1}\right) - (2x)^{-1}\right)} \right)^{1/2} ds =$$

$$= \frac{\sqrt{2}}{\pi} \exp\left(-(16x)^{-1}\right) \int_{-\infty}^{+\infty} \left(is + (4x)^{-1}\right) \left(\frac{\exp(-2xs^2)}{1 + \exp(-1/(2x))} \exp(-2is) \right)^{1/2} ds\,. \qquad (7)$$

In (7) we will spread out a denominator of sub integral expression in meeting at any x> 0 and any s ∈ R row

$$\left(1 + \exp\left(-\frac{1}{2x}\right)\exp(-2is)\right)^{-1/2} = \sum_{l=0}^{\infty} \frac{(-1)^l (2l-1)!!}{2^l\, l!} \exp\left(-\frac{l}{2x}\right) \exp(-2ils)$$

Its convergence is uniform in any strip [0, M] ×R the planes (x, s), M> 0.

Substituting the last expression in (7), we will receive

$$g(x) = \frac{\sqrt{2}}{\pi} \exp\left(-(16x)^{-1}\right) \int_{-\infty}^{+\infty} \left(is + (4x)^{-1}\right) \exp(-xs^2) \times$$

$$\times \sum_{l=0}^{\infty} \frac{(-1)^l (2l)!}{4^l (l!)^2} \exp(-2ils) \exp\left(-\frac{l}{2x}\right) ds =$$

$$= \frac{\sqrt{2}}{\pi} \exp\left(-(16x)^{-1}\right) \sum_{l=0}^{\infty} \frac{(-1)^l (2l)!}{4^l (l!)^2} \int_{-\infty}^{+\infty} \left(is + (4x)^{-1}\right) \exp(-xs^2) \times$$

$$\times \exp\left(-x\left(s + \frac{il}{x}\right)^2 - \frac{l(l+1/2)}{x}\right) ds.$$

The shift of operators of summation and integration is based on uniform convergence of a row on s at any fixed x.

In everyone composed the sums we will make a shift $s + \frac{il}{x} \Rightarrow s$ on an integration variable, we will receive

$$g(x) = \frac{\sqrt{2}}{\pi} \exp\left(-(16x)^{-1}\right) \int_{-\infty}^{+\infty} \sum_{l=0}^{\infty} \frac{(-1)^l (2l)!}{4^l (l!)^2} \left(is + \frac{l+1/4}{x}\right) \times$$

$$\times \exp\left(-xs^2 - \frac{l(l+1/2)}{x}\right) ds. \qquad (8)$$

Let's present integral in the form of the sum of two integrals according to the expression standing in a pre-exponential bracket. The integral corresponding to the composed is addressing in zero,

$$i \int_{-\infty}^{+\infty} s \sum_{l=0}^{\infty} \frac{(-1)^l (2l)!}{4^l (l!)^2} \exp(-xs^2) \exp\left(-\frac{l(l+1/2)}{x}\right) ds = 0$$

in a type of oddness of sub integral function. The integral corresponding to composed $\frac{l+1/4}{x}$ will be transformed as follows

$$\frac{1}{x} \left(\int_{-\infty}^{+\infty} \exp(-xs^2) ds \right) \sum_{l=0}^{\infty} \frac{(-1)^l (2l)!}{4^l (l!)^2} \left(l + \frac{1}{4}\right) \exp\left(-\frac{l(l+1/2)}{x}\right).$$

Substitution of this expression in (8) taking into account the value of integral of Poisson leads to formula (4).

As a row (4) sign-variable g_N does not surpass the rest of a row g_N first composed from among rejected. Therefore, the assessment takes place (5).

INVESTIGATION. The assessment takes place [9]

$$\left| g_{N-1}(x) \right| < \frac{3}{2e^2} N^{-5/2}\,. \qquad (9)$$

PROOF: Let's find assessment of the rest of a row (4). For this purpose, we will write down on the basis of (4) expression for g(x) in shape:

$$g(x) = \sqrt{\frac{2}{\pi}} \sum_{l=0}^{\infty} (-1)^l a_l h_l(x),$$

Where $a_l = \frac{(2l)!}{4^l (l!)^2} \left(l + \frac{1}{4}\right)$,

$$h_l(x) = x^{-3/2} \exp\left(-\frac{(l+1/4)^2}{x}\right),$$

also, we will find a maximum on x functions $h_N(x)$. Equating zero a derivative on x this function:

$$h'_N(x) = \frac{1}{x^2} h_N(x) \left(\left(N + \frac{1}{4}\right)^2 - \frac{3x}{2} \right) = 0,$$

We find the solution x_* of this equation - a point of the only maximum of a function $h_N(x)$.

$$x_* = \frac{2}{3}\left(N + \frac{1}{4}\right)^2,$$

$$h_N(x_*) = \left(\frac{3}{2e}\right)^{3/2}\left(N + \frac{1}{4}\right)^{-3}.$$

Therefore, rest N assessment

$$\left|g_{N-1}(x)\right| \le \sqrt{\frac{2}{\pi}}\, a_N\, h_N(x_*) = \frac{3}{2}\sqrt{\frac{3}{\pi e}}\left(\frac{(2N)!}{4^N (N!)^2}\right)\left(N + \frac{1}{4}\right)^{-2}.$$

Let's estimate from the above coefficient a_N, having made the received assessment of more transparent,

$$a_N = \frac{(2N-1)!!}{2^N N!} = \prod_{l=1}^{N}\left(\frac{2l-1}{2l}\right) = \prod_{l=1}^{N}\left(1 - \frac{1}{2l}\right) = \exp\left(\sum_{l=1}^{N}\ln\left(1 - \frac{1}{2l}\right)\right) <$$

$$< \exp\left(-\frac{1}{2}\sum_{l=1}^{N}\frac{1}{l}\right) < \exp\left(-\frac{1}{2}\left(1 + \ln N\right)\right) = \frac{e^{-1/2}}{\sqrt{N}},$$

in view of the justice of inequalities $\ln(1-x) < x$ at x> 0 and

$$\sum_{l=1}^{N}\frac{1}{l} > 1 + \int_1^N \frac{dx}{x} = 1 + \ln N.$$

Then, as $\sqrt{\frac{3}{\pi}} < 1$, takes place (9)

Let's estimate the accuracy of the approximations of the probability of PR $\{J_T[w] > c\}$received on the basis of functions g_N, N = 1 now$\{J_T[w] > c\}$ g_N. 2, … [6] As distribution density $f(x) = T^{-2} g(T^{-2}x)$,

$$\Pr\{J_T[w] > c\} = 1 - T^{-2}\int_0^c g(T^{-2}x)dx \equiv 1 - R(c),$$

Where, $R(c) = \int_0^{c/T^2} g(x)dx.$

Having designated the right part of inequality (5) by means of $Q_N(x)$, we have:

$$\left|g_{N-1}(x)\right| \le Q_N(x).$$

Let's define function now

$$P_N(c) \equiv 1 - R_N(c),\ R_N(c) = \int_0^{c/T^2}\left(g(x) - g_{N-1}(x)\right)dx.$$

Our task is receiving the top assessment for evasion $\left|\Pr\{J_T[w] > c\} - P_N(c)\right|$. Follows from inequality (5) that $-Q_N(x) \le g_{N-1}(x) \le Q_N(x)$. Integrating ranging from 0 to with / T 2, we receive

$$-\int_0^{c/T^2} Q_N(x)dx \le \int_0^{c/T^2} g_{N-1}(x)dx \le \int_0^{c/T^2} Q_N(x)dx$$

.

Therefore,

$$\left|R(c) - R_N(c)\right| = \left|\int_0^{c/T^2} g_{N-1}(x)dx\right| \le \int_0^{c/T^2} Q_N(x)dx,$$

it gives a required assessment

$$\left|\Pr\{J_T[w] > c\} - P_N(c)\right| = \left|R(c) - R_N(c)\right| \le \int_0^{c/T^2} Q_N(x)dx \qquad (10)$$

At last, to make an assessment (10) obvious, we will calculate the integral standing in the right part

$$\int_0^{c/T^2} Q_N(x)dx = a_N\sqrt{\frac{2}{\pi}}\int_0^{c/T^2}\exp\left(\frac{(N+1/4)^2}{x}\right)\frac{dx}{x^{3/2}}.$$

Replacement of a variable of integration $y = x^{-1/2}, dy = -\frac{dx}{2x^{3/2}}$ leads to a formula

$$\int_0^{c/T^2} Q_N(x)dx = \sqrt{\frac{8}{\pi}}\frac{a_N}{N+1/4}\int_{\frac{T(N+1/4)}{\sqrt{c}}}^{\infty}\exp\left(-y^2\right)dy = \frac{\sqrt{2}a_N}{N+1/4}Erfc\left[\frac{T(N+1/4)}{c^{1/2}}\right]$$

.

From here, using Erfc(x) <1, we find uniform in parameters with and T assessment

$$\int_0^{c/T^2} Q_N(x)dx \le \sqrt{\frac{2}{eN^3}}.$$

Thinner assessment considering the size of parameters with and T, turns out the use of standard inequality of [10] Erfc(x) $< \frac{\exp\left(-x^2\right)}{\sqrt{\pi}x}$,

$$\int_0^{c/T^2} Q_N(x)dx \le \sqrt{\frac{2c}{\pi}}\frac{a_N}{T(N+1/4)^2}\exp\left(-\frac{T^2(N+1/4)^2}{c}\right) <$$

$$< \sqrt{\frac{2c}{\pi e}}\left(TN^{5/2}\right)^{-1}\exp\left(-\frac{(TN)^2}{c}\right).$$

IV. CONCLUSION

The task of calculating the characteristic functions for random variables of type (1) in Gaussian casual processes are essentially solved [5]. The task of the density distribution of f on the basis of characteristic functions formulas is studied insufficiently. Such a task involves acquiring approximations with guaranteed accuracy, applicable for the whole random variable change diapason. The task of calculating successive approximations for density distribution of f(x) probabilities for all random composite functions (1) on every [0, M], ∞ >M > 0 interval has been considered. Accuracy assessment for probability approximations of PR $\{J_T[w] > c\}$ received on the basis of g_N, N = 1, 2… functions is presented.

V. SUMMARY

An impression in the form of absolutely meeting row for a density of distribution of probabilities is gained:

$$g(x) = \frac{1}{2\pi i} \int_{-i\infty+c}^{i\infty+c} e^{\lambda x} \left(ch\left(\lambda^{1/2}\right)\right)^{-1/2} d\lambda =$$

$$= \sqrt{\frac{2}{\pi x^3}} \sum_{l=0}^{\infty} \frac{(-1)^l (2l)!}{4^l (l!)^2} \left(l + \frac{1}{4}\right) \exp\left(-\frac{1}{x}\left(l + \frac{1}{4}\right)^2\right).$$

The top assessment for evasion is also received

$$\left|\Pr\{J_T[w] > c\} - P_N(c)\right|$$

$$\le \int_0^{c/T^2} Q_N(x)\,dx \le \sqrt{\frac{2}{eN^3}}.$$

More exact assessment which considers the size of parameters with and T has an appearance

$$\left|\Pr\{J_T[w] > c\} - P_N(c)\right| < \sqrt{\frac{2c}{\pi e}} \left(TN^{5/2}\right)^{-1} \exp\left(-\frac{(TN)^2}{c}\right).$$

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