THE SCHWARZ PROBLEM FOR DOUGLIS ANALYTIC FUNCTIONS

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Abstract. This paper considers the Schwarz problem that consists in finding a J-analytic function by its real part on the boundary. The Fredholm solvability of this problem is proved. The integral representation of J-analytic functions by Cauchy-type integrals with real density is obtained.

The classical Schwarz problem [1] consists in finding an analytic function by its real part given on the boundary of the domain considered. In this work, we consider an analogous problem for Douglis analytic functions. The latter are solutions $\phi = (\phi_1, \dots, \phi_l)$ of the first-order elliptic system

$$\frac{\partial \phi}{\partial y} - J\frac{\partial \phi}{\partial x} = 0, \quad J \in \mathbb{C}^{l\times l}, \tag{1}$$

where the eigenvalues ν of the matrix J lie in the upper half-plane $\operatorname{Im}\nu > 0$. This system was considered in [2] in detail, and its solutions are briefly called the J-analytic functions or the Douglis analytic functions.

Let a domain D be bounded by a smooth contour Γ composed of connected components $\Gamma_1, \dots, \Gamma_m$. The domain D can be finite, as well as infinite; we distinguish these cases by using the characteristic $\varkappa_D$ that assumes the values 1 and 0. For $\varkappa_D = 1$, we agree to assume that contour Γ_m contains all other components Γ_j. As was mentioned above, the Schwarz problem (Problem S in short) is defined by the boundary condition

$$\operatorname{Re}\phi^+ = f$$

and is considered in the Hölder classes $C^\mu(\overline{D})$, $0 < \mu < 1$, and Hardy classes $H^p(D)$, $1 < p < \infty$, which are introduced in [3]. For $\varkappa_D = 0$, the definition of these classes includes the condition $\phi(\infty) = 0$.

Let the contour Γ be positively oriented with respect to D (i.e., the domain remains to the left with respect to this orientation), and let $e(t) = e_1(t) = ie_2(t)$ be the unit tangent vector to the contour at the point t directed according to this orientation. By definition, $\Gamma \in C^{1,\mu+0}$ if $e \in C^{\mu+\varepsilon}(\Gamma)$ with a certain $\varepsilon > 0$. To Problem S, we put in correspondence the adjoint Problem $\tilde{S}$, which is considered in the corresponding adjoint classes $C^\mu(\overline{D})$,$H^q(D)$, $1/q = 1 - 1/p$, for $J^\top$-analytic functions and is defined by the boundary condition

$$\operatorname{Re} e_{J^\top}\tilde{\phi}^+ = f,$$

where $e_{J^\top} = e_1 \cdot 1 + e_2 J^\top$. By using the bilinear form

$$(f, g) = \int_\Gamma f(t)g(t)|dt|,$$

where $|dt|$ is the arc length element, the connection between these problems is the identity

$$(\phi^+, e_{J^\top}\tilde{\phi}^+) = 0.$$

Problem S is a particular case of the Riemann–Hilbert problem considered in [4], and the results corresponding to it can be formulated as follows.

Theorem 1. *Under the condition $\Gamma \in C^{1,\mu+0}$, Problem S is Fredholm in each of the classes $C^\mu(\overline{D})$ and $H^p(D)$, and its index is $\operatorname{Ind} S = l(2\varkappa_D - m)$. The inhomogeneous Problem S is solvable iff the orthogonality conditions $(f, \operatorname{Im} e_{J^\top}\tilde{\phi}^+) = 0$ to all solutions $\tilde{\phi}$ of the homogeneous adjoint problem hold.*

Moreover, any solution $\phi \in H^p$ of Problem S with right-hand side $f \in C^\mu(\Gamma)$ belongs to the class $C^\mu(\overline{D})$. Analogously, $f \in C^{1,\mu}(\Gamma)$ implies $\phi \in C^{1,\mu}(\overline{D})$. In particular, solutions of the homogeneous Problem S belong to the class $C^{1,\mu+0}(\overline{D})$. Analogous assertions also hold for Problem $\tilde{S}$ with the only difference being that their indices are opposite.

According to this theorem, the finite-dimensional kernel of the Schwarz problem is contained in the class $C^{1,\mu+0}(\overline{D})$. This kernel contains functions whose real parts are identically equal to zero. The following lemma shows that the functions of such a type are polynomials.

Lemma 1. *If the real part of a J-analytic function ϕ is identically equal to zero in a neighborhood of a certain point z_0, then ϕ is a polynomial.*

Proof. Let the above neighborhood be a disk D_0. In $\mathbb{C}^l$, let us consider the sequence of subspaces $X_0 \supseteq X_1 \supseteq \cdots$, which is inductively defined by the conditions $X_0 = \mathbb{C}^l$ and $X_k = \{\eta \in X_{k-1},\ \mathrm{Re}\, J\eta = 0\}$. Let $\eta \in X_k$ and $u(z) = \mathrm{Re}(z - z_0)_J\eta$. Then

$$\frac{\partial^k u}{\partial x^{k-s}\partial y^s} = \mathrm{Re}\, J^s\eta = 0, \quad 0 \le s \le k,$$

so that the function $u(x, y)$ is a polynomial of degree k. Since this function has a zero of order k at the point z_0, it follows that $u = 0$. Therefore, $\mathrm{Re}(z - z_0)_J\eta \equiv 0$ for $\eta \in X_k$.

Now let us write the Taylor series expansion of the function ϕ:

$$\phi(z) = \sum_{k=0}^{\infty}(z - z_0)_J\eta_k, \quad \eta_k = \frac{1}{k!}\phi^{(k)}(z_0). \tag{2}$$

By assumption,

$$\mathrm{Re}\left[\sum_{k=0}^{\infty}(z - z_0)_J\eta_k\right] \equiv 0$$

in the disk D_0. Sequentially differentiating this relation and passing to the limit as $z \to z_0$, we conclude that $\eta_k \in X_k$ for all k. The converse is also true: if series (2) uniformly converges in the disk D and the coefficients $\eta_k \in X_k$ for all k, then $\mathrm{Re}\,\phi \equiv 0$. Therefore, it remains to verify that starting from a certain number, all $X_k = 0$.

Assume the contrary, so that all X_k contain a certain subspace X. Then the class of functions (2) with coefficients $\eta_k \in X$ is infinite-dimensional, which contradicts the fact that the whole kernel of Problem S in the disk D_0 is finite-dimensional. $\square$

The proof of Theorem 1 is based on the reduction of the problem to an equivalent system of singular integral equations on the boundary Γ by using the representation of the function ϕ by the Cauchy-type integrals

$$(I_J\varphi)(z) = \frac{1}{2\pi i}\int_\Gamma (t - z)_J dt_J\varphi(t), \quad z \in D,$$

with real density $\varphi = (\varphi_1, \ldots, \varphi_l)$. More precisely, the following theorem, which was proved in [5], holds.

Theorem 2. *Let the domain D be bounded by the contour $\Gamma \in C^{1,\mu+0}$, and let the matrix J be triangular. Then any function $\phi \in H^p(D)$, $1 < p < \infty$, is uniquely represented in the form*

$$\phi = I\varphi + i\xi, \quad \xi \in \varkappa_D\mathbb{R}^l,$$

where the real l-vector-valued function $\varphi \in L^p(\Gamma)$ satisfies the conditions

$$\int_{\Gamma_j} \varphi(t)|dt| = 0, \quad 1 \le j \le m - \varkappa_D.$$

If $\phi \in C^{\mu}(\overline{D})$ in this representation, then $\varphi \in C^{\mu}(\Gamma)$. Analogously, $\phi \in C^{1,\mu}(\overline{D})$ implies $\varphi \in C^{1,\mu}(\Gamma)$.

In a number of cases, it is desirable to have an analogous result not imposing the additional triangular condition on the matrix J. The general situation of such a type is considered in [4] for the Cauchy-type integrals with density of the form $G^{-1}\varphi$, where the vector-valued function φ is real and the matrix function $G \in C^{\mu+0}(\Gamma)$ is invertible. Modifying the proof of the corresponding result of [4] on the representation of J-analytic functions by integrals of this type applied to the case $G = 1$, we obtain the following result.

Consider an open set $D' = \mathbb{C} \setminus \overline{D}$ consisting of domains D'_j, $1 \le j \le m$. Let us agree to choose the enumeration of these components in such a way that $\partial D'_j = \Gamma_j$. If the domain D is finite, then by the above convention, the contour Γ_m contains Γ_j, $j < m$, and is the boundary of the infinite domain D'_m. If the domain D is infinite, then all components D'_j are of equal rights. Let $\mathbb{R}^l(\Gamma)$ denote the class of real-valued l-vector-valued functions that are constant on Γ. The notation $\mathbb{C}^l(D')$ has the same meaning for complex l-vector-valued functions that are constant in domains D'_j.

In the open set D', let us consider the Schwarz problem S' defined by the boundary condition $\operatorname{Re}\psi^- = f$. Here, we take into account that the contour Γ is negatively oriented with respect to D', and in accordance with this, the boundary value of the function ψ defined in D' is denoted by ψ^-. By Theorem 1, any solution of this problem with right-hand side $f \in \mathbb{R}^l(\Gamma)$ belongs to the class $C^{1,\mu+0}$ (more precisely, to the class $C^{1,\mu+0}(\overline{D'_j})$) in each connected component D'_j, $1 \le j \le m$, of the open set D'. Obviously, any function $\phi \in \mathbb{C}^l(D')$ such that it vanishes in the domain D'_m for $\varkappa_D = 1$ is a solution of this problem. Therefore, the space

$$Y = \{\operatorname{Im}\psi^-,\ \operatorname{Re}\psi^- \in \mathbb{R}^l(\Gamma)\}$$

contains the subspace

$$\widetilde{\mathbb{R}}^l(\Gamma) = \{g \in \mathbb{R}^l(\Gamma),\ \varkappa_D g(\Gamma_m) = 0\}$$

of dimension $l(m - \varkappa_D)$.

Theorem 3. *Let the domain D be bounded by the contour $\Gamma \in C^{1,\mu+0}$, and let the finite-dimensional space Y consist of functions $\operatorname{Im}\psi^-$, where $\psi \in C^{\mu}(\overline{D'})$ and $\operatorname{Re}\psi^- \in \mathbb{R}^l(\Gamma)$. Then there exists a finite-dimensional space $X \subseteq C^{1,\mu+0}(\overline{D})$ of dimension*

$$\dim X = \dim Y - l(m - \varkappa_D) \tag{3}$$

such that any function $\phi \in H^p(D)$, $1 < p < \infty$, is uniquely represented in the form

$$\phi = I\varphi + \phi_0 + i\xi, \quad \phi_0 \in X, \quad \xi \in \varkappa_D \mathbb{R}^l,$$

where the real-valued l-vector-valued function $\varphi \in L^p(\Gamma)$ satisfies the conditions

$$(\varphi, g) = 0, \quad g \in Y. \tag{4}$$

If $\phi \in C^{\mu}(\overline{D})$ in this representation, then $\varphi \in C^{\mu}(\Gamma)$. Analogously, $\phi \in C^{1,\mu}(\overline{D})$ implies $\varphi \in C^{1,\mu}(\Gamma)$.

Let us show that Theorem 2 is a particular case of Theorem 3. As was noted above, $Y \supseteq \widetilde{\mathbb{R}}^l(\Gamma)$, and, therefore, Y can be decomposed into the direct sum $Y_0 \oplus \widetilde{\mathbb{R}}^l(\Gamma)$. Then, in accordance with (3), the dimensions of the spaces X and Y_0 coincide, and the orthogonality conditions (4) can be rewritten in the form

$$\int\limits_{\Gamma} \varphi(t)g(t)|dt| = 0, \quad g \in Y_0; \qquad \int\limits_{\Gamma_j} \varphi(t)|dt| = 0, \quad 1 \le j \le m - \varkappa_D.$$

It remains to verify that in the case where the matrix J is triangular, the subspace $Y_0 = 0$. This fact follows from the following lemma.

Lemma 2. *Let the domain D (finite or infinite) be bounded by a simple Lyapunov contour Γ, and let the matrix J be triangular. Let a J-analytic function $\phi \in C^\mu(\overline{D})$ be such that* $\mathrm{Re}\phi^+$ *is constant on Γ. Then ϕ is constant in the domain D, and, in particular, it is equal to* 0 *in the case where the domain D is infinite.*

We first perform the proof in the scalar case $l = 1$, when $J = \nu \in \mathbb{C}$ and ϕ is a solution of the equation

$$\frac{\partial \phi}{\partial y} - \nu \frac{\partial \phi}{\partial x} = 0. \tag{5}$$

Under the affine transformation $z = x + iy \to x + \nu y$, this equation passes to the Cauchy–Riemann equation, which defines analytic functions, so that the assertion of the lemma is obvious in this case.

Let us turn to the general case $l > 1$. For definiteness, let the matrix J be upper-triangular, i.e., its entries $J_{ik} = 0$ for $i > k$. Then, in the coordinate writing, system (1) takes the form

$$\frac{\partial \phi_j}{\partial y} - \sum_{k=j}^{l} J_{kj} \frac{\partial \phi_k}{\partial x} = 0, \quad 1 \le k \le l.$$

In particular, the latter equation contains only the function ϕ_l and coincides with (5) for $\nu = J_{ll}$. Note that the spectrum $\sigma(J)$ of the triangular matrix J consists of its diagonal entries. By what was already proved, the latter equation of this system implies that the function ϕ_l is constant. Hence the $(l-1)$th equation of this system passes to (5) with respect to ϕ_{l-1}, and $\nu = Jl - 1, l - 1$. Therefore, for the same reasons, the function ϕ_{l-1} is constant. Repeating these arguments, as a result, we conclude that all functions ϕ_k are constant.

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