ISSN 0012-2661, Differential Equations, 2020, Vol. 56, No. 10, pp. 1324–1334. © Pleiades Publishing, Ltd., 2020.
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PARTIAL DIFFERENTIAL EQUATIONS

Boundary Value Problems for Elliptic Pseudodifferential Equations in a Multidimensional Cone

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Received March 24, 2020; revised March 24, 2020; accepted May 14, 2020

Abstract—We consider model boundary value problems for elliptic pseudodifferential equations in multidimensional cones. A result on the unique solvability and representation of solutions of some boundary value problems in suitable Sobolev–Slobodetskii spaces is obtained. A priori estimates of solutions are given.

DOI: 10.1134/S0012266120100092

INTRODUCTION

At present, there is no unified and generally accepted point of view on the construction of the theory of boundary value problems for elliptic pseudodifferential equations on manifolds with nonsmooth boundary [1–5]. The local principle is used by everyone, and disagreements begin where the singularity of the boundary itself and the type of the local operator in the vicinity of this singularity are described. Of course, this gives rise to different conditions for the Fredholm property, and the results obtained under different initial conditions are incomparable. In this case, such criteria as "visibility," "applicability," and "usefulness" probably come to the foreground. From this point of view, in our opinion, the constructions of the present author considered earlier and proposed in this paper are in no way inferior to other versions of the theory, but, on the contrary, much better match the specified criteria.

The present work deals with rendering concrete some of the constructions carried out by the present author to study the solvability of elliptic pseudodifferential equations and boundary value problems in cones [6–17]. In a number of his studies, there is a certain transformation operator, the knowledge of the explicit form of which makes it possible to write down the exact form of the solution of a special boundary value problem. Here we will describe this operator for a circular cone in the m-dimensional Euclidean space.

1. PRELIMINARIES. MODEL DOMAINS

By model (canonical) domains we mean special domains of the Euclidean space $\mathbb{R}^m$. The first model domain was the entire space $\mathbb{R}^m$ as a local model of a smooth compact manifold. The second model domain is the half-space $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x = (x_1, \dots, x_m), x_m > 0\}$. In this case, it proved possible to obtain exhaustive results on the statement and solvability of boundary value problems for elliptic pseudodifferential equations on compact manifolds with smooth boundary [18], which ended with the index theory [19]. Since then, the $\mathbb{R}^m_+$ model has been "canonized" even for the case of a nonsmooth boundary, and all local studies have been associated with the interpretation of the conic model as the direct product $B \times \mathbb{R}_+$, where B is the cone base. On this path, a large number of interesting and impressive results [1–3], mostly of abstract and theoretical nature, were obtained, in which the fundamental point was the identification of a local operator near a singular point. As soon as a local description appeared, it became possible to study the Fredholm property of boundary value problems, including index theorems.

The present author sticks to a different point of view on these issues, which he started to develop in the 1990s [4, 5]. The simplest circular cone

$$C^a_+ = \{x \in \mathbb{R}^m : x = (x', x_m), \quad x_m > a|x'|, \quad a > 0\}$$

delivers a new type of singularity, irreducible to the case of half-space. As became clear later, studying the solvability of pseudodifferential equations in such a cone is closely related to the multidimensional linear conjugation problem, which is one of the (rather many) versions of the classical Riemann boundary value problem [20, p. 95; 21, p. 140]. The idea of multidimensional factorization has enabled a complete description of the solvability pattern [4, p. 27; 5, p. 32] for the model elliptic equation on the plane,

$$(Au)(x) = v(x), \quad x \in C_+^a, \tag{1}$$

where A is a pseudodifferential operator with symbol $A(\xi)$ satisfying the condition

$$c_1(1+|\xi|)^\alpha \le |A(\xi)| \le c_2(1+|\xi|)^\alpha, \quad c_1, c_2 = \text{const}\,, \tag{2}$$

and a solution u is sought in the Sobolev–Slobodetskii space $H^s(C_+^a)$.

By definition, the space $H^s(C_+^a)$ consists of those generalized functions of the space $H^s(\mathbb{R}^m)$ whose supports are contained in $\overline{C_+^a}$ [18, p. 37]. The norm $\|\cdot\|_s$ in the space $H^s(C_+^a)$ is induced by the norm of the space $H^s(\mathbb{R}^m)$,

$$\|u\|_s = \left(\int\limits_{\mathbb{R}^m} |\tilde{u}(\xi)|^2 (1+|\xi|)^{2s}\, d\xi \right)^{1/2},$$

where $\tilde{u}$ is the Fourier transform of the function u; i.e.,

$$\tilde{u}(\xi) \equiv (Fu)(\xi) = \int\limits_{\mathbb{R}^m} e^{ix\xi} u(x)\, dx.$$

The right-hand side v of Eq. (1) is selected from the space $H_0^{s-\alpha}(C_+^a)$, which consists of generalized functions in $S'(C_+^a)$ that admit continuation to the entire space $H^{s-\alpha}(\mathbb{R}^m)$. The norm on the space $H_0^{s-\alpha}(C_+^a)$ is determined by the formula

$$\|v\|_{s-\alpha}^+ = \inf \|\ell v\|_{s-\alpha},$$

where the infimum is taken over all continuations ℓ. The symbol $\widetilde{H}^s$ will stand for the Fourier transform of the respective space H^s.

The model operator and the canonical domain arise when "freezing the coefficients" and "straightening the boundary"; in the case of a cone, the latter implies that, in a neighborhood of the singular point, the surface of the boundary neighborhood smoothly transforms into a conic surface. In this case, to derive conditions for the original operator to be Fredholm, in accordance with the local principle, it is necessary to describe conditions for the invertibility of the model operator in the canonical domain. These issues were covered in the papers by the present author [6–10, 13–16]. All the constructions indicated in these papers contain some parameters that characterize the dimensions of the cone. (For the cone C_+^a, this is the parameter a.) For example, the cone C_+^a becomes a half-space as $a \to 0$, while it degenerates into a ray as $a \to +\infty$. This is obvious geometrically, but it remains unclear how to define pseudodifferential operators on such limit structures and pose boundary value problems for these operators. The present author undertook the first attempts to elucidate these questions several years ago [11, 12, 17], with the motivation being the article [22], which sets forth the theory of boundary value problems (for differential operators) on the corresponding manifolds.

Since there is no formula for the general solution in the multidimensional case, we consider here some specific cones and, based on our theory of boundary value problems in cones [4, 5], describe the solution of the special boundary value problem, as well provide an explicit formula for its solution.

2. EQUATION IN THE CONE, WAVE FACTORIZATION, AND GENERAL SOLUTION

To study the solvability of Eq. (1), we use the concept of wave factorization of an elliptic symbol. Let us give the definition in a rather general form, because the results produced below can readily be extended to more complicated singularities.

Let C^{m-k} be a convex cone lying in the real $(m-k)$-dimensional linear space and not containing any straight line entirely. A *radial tubular domain* $T(C^{m-k})$ *over the cone* C^{m-k}[23, p. 206] is a subset of the complex space $\mathbb{C}^{m-k}$ of the form

$$T(C^{m-k}) = \left\{ z \in \mathbb{C}^{m-k} : z = x + iy, \quad x \in \mathbb{R}^{m-k}, \quad y \in C^{m-k} \right\}.$$

The *dual cone of the cone* C^{m-k} is the cone [23, p. 257]

$$\overset{*}{C}{}^{m-k} = \left\{ x \in \mathbb{R}^{m-k} : x \cdot y > 0, \quad y \in C^{m-k} \right\}.$$

For a vector $\xi \in \mathbb{R}^m$, we set $\xi = (\xi', \xi'')$, where $\xi' \in \mathbb{R}^k$ and $\xi'' \in \mathbb{R}^{m-k}$.

Definition. For a function $A(\xi)$ defined for almost all $\xi \in \mathbb{R}^m$, the *k-wave factorization with respect to the cone* C^{m-k} is its representation in the form of the product

$$A(\xi) = A_{\neq}(\xi) A_{=}(\xi),$$

in which the factors $A_{\neq}(\xi)$ and $A_{=}(\xi)$ satisfy the following conditions:

1. The functions $A_{\neq}(\xi)$ and $A_{=}(\xi)$ are defined for all values of $\xi \in \mathbb{R}^m$ except maybe for points belonging to the set $\mathbb{R}^k \times \partial(\overset{*}{C}{}^{m-k} \cup (-\overset{*}{C}{}^{m-k}))$.
2. For almost all $\xi' \in \mathbb{R}^k$, the functions $A_{\neq}(\xi)$ and $A_{=}(\xi)$ admit analytic continuation into the radial tubular domains $T(\overset{*}{C}{}^{m-k})$ and $T(-\overset{*}{C}{}^{m-k})$, respectively, and in this case there exists a $\varkappa_k \in \mathbb{R}$ such that for each $\tau \in \overset{*}{C}{}^{m-k}$ one has the estimates

$$c_1(1 + |\xi| + |\tau|)^{\varkappa_k} \le |A_{\neq}(\xi', \xi'' + i\tau)| \le c_4(1 + |\xi| + |\tau|)^{\varkappa_k},$$
$$c_3(1 + |\xi| + |\tau|)^{\alpha-\varkappa_k} \le |A_{=}(\xi', \xi'' - i\tau)| \le c_2(1 + |\xi| + |\tau|)^{\alpha-\varkappa_k},$$

where c_1, c_2, c_3, and c_4 are positive constants.

The number $\varkappa_k$ is called the *index of the k-wave factorization*.

Let us start our consideration from the simplest cone C_+^a.

It was shown in the papers [4, 5] that the presence of a k-wave factorization for the symbol $A(\xi)$ allows one to completely describe the solvability pattern for Eq. (1). Obviously, for the cone C_+^a one needs the presence of a 0-wave factorization or simply a wave factorization. In what follows, we assume that such a factorization exists. (The examples and classes of symbols are given in [4, 5].) A solution of Eq. (1) with right-hand side in $H_0^{s-\alpha}(C_+^a)$ is sought in the Sobolev–Slobodetskii space $H^s(C_+^a)$ [4, 5]. Below, in the case under consideration, we prove a theorem that describes the structure of solution, pose the boundary value problem, and find its solution.

The present paper treats only such equations (1) for which the index $\varkappa$ of the wave factorization of their symbol $A(\xi)$ with respect to the cone C_+^a satisfies the condition $\varkappa - s = n + \delta$, where $n \in \mathbb{N}$ and $|\delta| < 1/2$.

As before [6, 7], by $T_a : \mathbb{R}^m \to \mathbb{R}^m$ we denote the transformation that sends ∂C_+^a into the hyperplane $x_m = 0$; i.e.,

$$t_1 = x_1, \quad \ldots, \quad t_{m-1} = x_{m-1}, \quad t_m = x_m - a|x'|, \quad x' = (x_1, \ldots, x_{m-1}).$$

We introduce the transformation operator

$$F T_a F^{-1} \equiv V_a$$

and use it to describe the structure of the solution of Eq. (1).

We will also need the special multidimensional singular integral operator

$$(G_m u)(x) = a_m \lim_{\tau \to 0+} \int\limits_{\mathbb{R}^m} \frac{u(y', y_m)\, dy'\, dy_m}{(|x' - y'|^2 - a^2(x_m - y_m + i\tau)^2)^{m/2}},$$

where a_m is a known constant whose explicit value is irrelevant; recall that this operator is a multidimensional analog of the Cauchy type integral, or, more precisely, the Hilbert transform, and it is used to solve [4, 5] one of the versions of the multidimensional Riemann problem.

Theorem 1. *The general solution of Eq.* (1) *in terms of Fourier transforms is given by the formula*

$$\tilde{u}(\xi) = A_{\neq}^{-1}(\xi)Q_n(\xi)G_m Q_n^{-1}(\xi)A_{=}^{-1}(\xi)\widetilde{\ell v}(\xi) + A_{\neq}^{-1}(\xi)V_{-a}F\left(\sum_{k=1}^{n} c_k(x')\delta^{(k-1)}(x_m)\right),$$

where $c_k(x') \in H^{s_k}(\mathbb{R}^{m-1})$ *are arbitrary functions,* $s_k = s - \varkappa + k - 1/2$, $k = 1,\dots,n$, ℓv *is an arbitrary continuation of the function* v *to a function in* $H^{s-\alpha}(\mathbb{R}^m)$, *and* $Q_n(\xi)$ *is an arbitrary polynomial satisfying the condition*

$$c_1(1+|\xi|)^n \le Q_n(\xi) \le c_2(1+|\xi|)^n, \quad c_1, c_2 = \text{const}. \tag{3}$$

One has the a priori estimate

$$\|u\|_s \le C\left(\|v\|_{s-\alpha}^{+} + [c_k]_{s_k}\right),$$

where $[\,\cdot\,]_{s_k}$ *stands for the* $H^{s_k}(\mathbb{R}^{m-1})$*-norm and* C *is a constant.*

Proof. By ℓv we denote the continuation of the right-hand side v of Eq. (1) to a function from $H^{s-\alpha}(\mathbb{R}^m)$ and set

$$u_-(x) = (\ell v)(x) - (Au)(x),$$

so that

$$u_-(x) = 0, \quad x \notin C_+^a.$$

Then Eq. (1) is written in the form

$$(Au)(x) + u_-(x) = (\ell v)(x), \quad x \in \mathbb{R}^m;$$

applying the Fourier transform to the latter, we obtain

$$A(\xi)\tilde{u}(\xi) + \tilde{u}_-(\xi) = \widetilde{\ell v}(\xi),$$

which, after wave factorization of the symbol, leads to the relation

$$A_{\neq}(\xi)\tilde{u}(\xi) + A_{=}^{-1}(\xi)\tilde{u}_-(\xi) = A_{=}^{-1}(\xi)\widetilde{\ell v}(\xi). \tag{4}$$

According to our assumptions, the inclusion $A_{=}^{-1}(\xi)\widetilde{\ell v}(\xi) \in \widetilde{H}^{-n-\delta}(\mathbb{R}^m)$ holds, because we have $s - \varkappa - (\varkappa - \alpha) = s - \varkappa = -n - \delta$. Choose an arbitrary polynomial $Q_n(\xi)$ of degree n satisfying inequality (3) to arrive, after multiplying both sides of relation (4) by $Q_n^{-1}(\xi)$, at the relation

$$Q_n^{-1}(\xi)A_{\neq}(\xi)\tilde{u}(\xi) + Q_n^{-1}(\xi)A_{=}^{-1}(\xi)\tilde{u}_-(\xi) = Q_n^{-1}(\xi)A_{=}^{-1}(\xi)\widetilde{\ell v}(\xi). \tag{5}$$

It can be seen that $Q_n^{-1}(\xi)A_{=}^{-1}(\xi)\widetilde{\ell v}(\xi) \in \widetilde{H}^{-\delta}(\mathbb{R}^m)$, and consequently, we can write [4, 5]

$$Q_n^{-1}(\xi)A_{=}^{-1}(\xi)\widetilde{\ell v}(\xi) = v_+(\xi) = v_-(\xi),$$

where

$$v_+(\xi) = G_m Q_n^{-1}(\xi)A_{=}^{-1}(\xi)\widetilde{\ell v}(\xi) \quad \text{and} \quad v_-(\xi) = (I - G_m)Q_n^{-1}(\xi)A_{=}^{-1}(\xi)\widetilde{\ell v}(\xi),$$

with $v_+ \in \widetilde{H}^{-\delta}(C_+^a)$, and $v_- \in \widetilde{H}^{-\delta}(\mathbb{R}^m \setminus \overline{C_+^a})$. Now we write relation (5) in the form

$$Q_n^{-1}(\xi)A_{\neq}(\xi)\tilde{u}(\xi) + Q_n^{-1}(\xi)A_{=}^{-1}(\xi)\tilde{u}_-(\xi) = v_+(\xi) + v_-(\xi)$$

and further

$$A_{\neq}(\xi)\tilde{u}(\xi) - Q_n(\xi)v_+(\xi) = Q_n(\xi)v_-(\xi) - A_{=}^{-1}(\xi)\tilde{u}_-(\xi). \tag{6}$$

The following inclusions are obvious for the terms on the left-hand side in relation (6):

$$A_{\neq}(\xi)\tilde{u}(\xi) \in \widetilde{H}^{s-\varkappa}(C_+^a) \quad \text{and} \quad Q_n(\xi)v_+(\xi) \in \widetilde{H}^{-n-\delta}(C_+^a),$$

and since $s - \varkappa = -n - \delta$, we see that it belongs to the space $\widetilde{H}^{-n-\delta}(C_+^a)$. In a similar manner, we verify that the right-hand side of relation (6) belongs to the space $\widetilde{H}^{-n-\delta}(\mathbb{R}^m \setminus \overline{C_+^a})$.

Let us apply the inverse Fourier transform to both sides of relation (6),

$$F^{-1}(A_{\neq}(\xi)\tilde{u}(\xi) - Q_n(\xi)v_+(\xi)) = F^{-1}(Q_n(\xi)v_-(\xi) - A_{=}^{-1}(\xi)\tilde{u}_-(\xi)). \tag{7}$$

In this relation, the left-hand side belongs to the space $H^{-n-\delta}(C_+^a)$ and the right-hand side, to the space $H^{-n-\delta}(\mathbb{R}^m \setminus \overline{C_+^a})$, but the function satisfying both of these conditions can be only a (generalized) function concentrated on the surface of the cone C_+^a. If we now apply the operator T_a to both sides of relation (7), then the aforementioned function will be concentrated on the hyperplane $x_m = 0$. The form of such a function is well known [18, 24], and we obtain

$$T_a F^{-1}(A_{\neq}(\xi)\tilde{u}(\xi) - Q_n(\xi)v_+(\xi)) = \sum_{k=1}^{n} c_k(x')\delta^{(k-1)}(x_m).$$

Applying the Fourier transform to both sides of this relation, we have

$$V_a(A_{\neq}(\xi)\tilde{u}(\xi) - Q_n(\xi)v_+(\xi)) = \sum_{k=1}^{n} \tilde{c}_k(\xi')\xi_m^{k-1}, \tag{8}$$

where $V_a = FT_aF^{-1}$. The operator V_a is invertible, and $V_a^{-1} = V_{-a}$ [6]. Letting the operator V_{-a} act on both sides of relation (8), we obtain

$$\tilde{u}(\xi) = A_{\neq}^{-1}(\xi)Q_n(\xi)G_mQ_n^{-1}(\xi)A_{=}^{-1}(\xi)\tilde{\ell v}(\xi) + A_{\neq}^{-1}(\xi)V_{-a}F\left(\sum_{k=1}^{n} c_k(x')\delta^{(k-1)}(x_m)\right).$$

An a priori estimate can be produced in the same way as in [4, 18] with allowance for the operator V_a being isometric. The proof of the theorem is complete.

2.1. Operator V_a. The Case of a Circular Cone

We only consider here the multidimensional cone C_+^a and describe a boundary value problem with a unique solution for this cone. Note that the case of $m = 2$ has been described in [8] and that of a tetrahedral angle in the space $\mathbb{R}^3$, in [7, 9].

Based on the key relation $FT_a^{-1} = V_{-a}F$, we conclude that it is more convenient to work with the operator V_a in terms of Fourier transforms. Let us start from the left-hand side. Fix a $u \in S(\mathbb{R}^m)$ and perform the following computations:

$$\begin{aligned}(FT_a^{-1}u)(\xi) &= \int\limits_{\mathbb{R}^m} e^{iy\xi}(T_a^{-1}u)(y)\,dy = \int\limits_{\mathbb{R}^m} e^{iy\xi'}u(y', y_m + a|y'|)\,dy \\ &= \int\limits_{\mathbb{R}^m} e^{ix\xi}e^{-i\xi_m a|x'|}u(x', x_m)\,dx'\,dx_m = \int\limits_{\mathbb{R}^m} e^{ix'\xi'}e^{-i\xi_m a|x'|}\hat{u}(x', \xi_m)\,dx',\end{aligned}$$

where $\hat{u}(x', \xi_m)$ stands for the Fourier transform of the function $u(x', x_m)$ in the variable x_m. Set

$$F_{x'\to\xi'}\left(e^{-i\xi_m a|x'|}\right) \equiv K_a(\xi', \xi_m); \tag{9}$$

after this, considering the properties of Fourier transform, we obtain an integral representation of the operator V_{-a},

$$\left(FT_a^{-1}u\right)(\xi) = \int\limits_{\mathbb{R}^m} K_a(\xi' - \eta', \xi_m)\tilde{u}(\eta', \xi_m)\,d\eta',$$

where $K_a(\xi', \xi_m)$ is the Fourier transform of the corresponding generalized function.

Remark 1. In formula (9), the function $e^{-i\xi_m a|x'|}$ is nonintegrable and needs to be pre-regularized; in other words, the left-hand side of formula (9) should be understood as the limit

$$\lim_{\tau\to 0} F_{x'\to\xi'}\left(e^{-iz_m a|x'|}\right), \quad z_m = \xi_m - i\tau, \quad \tau > 0,$$

and accordingly,

$$\left(FT_a^{-1}u\right)(\xi) = \lim_{\tau\to 0+} \int\limits_{\mathbb{R}^m} K_a(\xi' - \eta', z_m)\tilde{u}(\eta', \xi_m)\, d\eta'.$$

2.1.1. Calculating the kernel K_a. By the preceding, we have

$$F_{x'\to\xi'}\left(e^{-i\xi_m a|x'|}\right) = \int\limits_{\mathbb{R}^{m-1}} e^{ix'\xi'} e^{-i\xi_m a|x'|}\, dx',$$

and to calculate the last integral, we can use the corresponding formulas in [25, p. 13 of the Russian translation]; we reproduce these computations here for the presentation to be complete.

Lemma. *The kernel K_a has the form*

$$K_a(\xi', z_m) = \frac{iaz_m 2^{m-1}\pi^{(m-2)/2}\Gamma(m/2)}{(\xi_1^2 + \xi_2^2 + \ldots + \xi_{m-1}^2 - a^2 z_m^2)^{m/2}}, \tag{10}$$

where $z_m = \xi_m - i\tau$, $\tau > 0$.

Proof. Let us use some technical tricks in [25, Ch. 1, Sec. 1 of the Russian translation] modified for our case. The computation of the integral (9) with real indices is based on two formulas for integrals, which, as can readily be verified, also hold for the corresponding sets of complex numbers. One of these integrals is the Fourier transform of the function $e^{-\alpha|x'|^2}$, $\alpha > 0$,

$$\int\limits_{\mathbb{R}^{m-1}} e^{ix'\xi'} e^{-\alpha|x'|^2}\, dx' = \prod_{j=1}^{m-1} \int\limits_{-\infty}^{+\infty} e^{ix_j\xi_j} e^{-\alpha x_j^2}\, dx_j = \prod_{j=1}^{m-1} \sqrt{\frac{\pi}{\alpha}} e^{-\xi_j^2/(4\alpha)} = \left(\frac{\pi}{\alpha}\right)^{(m-1)/2} e^{-|\xi'|^2/(4\alpha)},$$

which can be found in virtually any handbook on Fourier transform. An important role in the above computation is played by Poisson's integral

$$\int\limits_{-\infty}^{+\infty} e^{ix\xi} e^{-\alpha x^2}\, dx = \sqrt{\frac{\pi}{\alpha}} e^{-\xi^2/(4\alpha)}.$$

The last two formulas remain valid for complex values of α under the condition that $\operatorname{Re}\alpha > 0$.

Further computation is aided by the integral [25, p. 18 of the Russian translation]

$$e^{-\beta} = \frac{1}{\sqrt{\pi}} \int\limits_0^{+\infty} \frac{e^{-u}}{\sqrt{u}} e^{-\beta^2/(4u)}\, du, \quad \beta > 0.$$

Now, taking into account these formulas, we compute the integral

$$\int\limits_{\mathbb{R}^{m-1}} e^{ix'\xi'} e^{-\alpha|x'|}\, dx', \quad \alpha > 0.$$

The computation is fully similar to [25, p. 17 of the Russian translation]; we provide it here, in particular, to produce explicit expressions for some constants. We have

$$\int\limits_{\mathbb{R}^{m-1}} e^{ix'\xi'} e^{-\alpha|x'|}\,dx' = \frac{1}{\sqrt{\pi}} \int\limits_0^{+\infty} \frac{e^{-u}}{\sqrt{u}} \left(\int\limits_{\mathbb{R}^{m-1}} e^{-\alpha^2|x'|^2/(4u)} e^{ix't\xi'}\,dx' \right) du$$
$$= \frac{1}{\sqrt{\pi}} \int\limits_0^{+\infty} \frac{e^{-u}}{\sqrt{u}} \left(\frac{\alpha^2}{4u}\right)^{-(m-1)/2} \pi^{(m-1)/2} e^{-|\xi'|^2 u/\alpha^2}\,du$$
$$= 2^{m-1}\alpha^{-m+1}\pi^{(m-2)/2} \int\limits_0^{+\infty} u^{(m-2)/2} e^{-u(|\xi'|^2+\alpha^2)/\alpha^2}\,du.$$

After the change of variables $t = u(|\xi'|^2+\alpha^2)/\alpha^2$, we obtain

$$\int\limits_{\mathbb{R}^{m-1}} e^{ix'\xi'} e^{-\alpha|x'|}\,dx' = 2^{m-1}\pi^{(m-2)/2} \frac{\alpha}{(|\xi'|^2+\alpha^2)^{m/2}} \int\limits_0^{+\infty} t^{(m-2)/2} e^{-t} dt = \frac{\alpha 2^{m-1}\pi^{(m-2)/2}\Gamma(m/2)}{(|\xi'|^2+\alpha^2)^{m/2}},$$

where Γ is the Euler gamma function. Denoting $c_m = 2^{m-1}\pi^{(m-2)/2}\Gamma(m/2)$, we ultimately write

$$\int\limits_{\mathbb{R}^{m-1}} e^{ix'\xi'} e^{-\alpha|x'|}\,dx' = c_m \frac{\alpha}{(|\xi'|^2+\alpha^2)^{m/2}}.$$

Now, using the fact that the integral on the left-hand side in this relation exists for complex values of α such that $\operatorname{Re}\alpha > 0$, we write down the relation for the complex values of $\gamma + i\beta$ $(\gamma, \beta \in \mathbb{R})$,

$$\int\limits_{\mathbb{R}^{m-1}} e^{ix'\xi'} e^{-(\gamma+i\beta)|x'|}\,dx' = c_m \frac{\gamma+i\beta}{(|\xi'|^2+(\gamma+i\beta)^2)^{m/2}},$$

and for γ and β we choose the numbers $\gamma = a\tau$, $a > 0$, and $\tau > 0$, $\beta = a\xi_m$. Then

$$-(\gamma+i\beta) = -a\tau - ia\xi_m = -ia(\xi_m - i\tau),$$

and we can write

$$\int\limits_{\mathbb{R}^{m-1}} e^{ix'\xi'} e^{-i(\xi_m-i\tau)a|x'|}\,dx' = c_m \frac{ia(\xi_m-i\tau)}{(|\xi'|^2-a^2(\xi_m-i\tau)^2)^{m/2}}.$$

Thus, setting $z_m = \xi_m - i\tau$, we obtain formula (10). The proof of the lemma is complete.

Consequently, in the case of a straight circular cone, the action of the operator V_{-a} can be represented as

$$(V_{-a}\tilde{u})(\xi) = \lim_{\tau\to 0+} \frac{1}{(2\pi)^{m-1}} \int\limits_{\mathbb{R}^{m-1}} \frac{iaz_m 2^{m-1}\pi^{(m-2)/2}\Gamma(m/2)\tilde{u}(\eta',\xi_m)\,d\eta'}{((\xi_1-\eta_1)^2+(\xi_2-\eta_2)^2+\ldots+(\xi_{m-1}-\eta_{m-1})^2-a^2z_m^2)^{m/2}}. \quad (11)$$

In formula (11), we have made use of one of the key properties of the Fourier transform (the factor multiplying the integral).

Apparently, it is natural to call formula (10) a *complex analog of Poisson's kernel* [25].

Remark 2. In the paper [26], the present author described the transformation operator for a broad class of cones from the standpoint of the theory of generalized functions [24] and provided formulas similar to (10) and (11) in which the numerators are written incorrectly, a fact that is easy to check by means of passage to the limit as $a \to 0$ in them (see below).

2.1.2. Two passages to the limit. First, we assume that $u \in S(\mathbb{R}^m)$ in formula (11). Since $T_0 \equiv I$, we have

$$(V_0\tilde{u})(\xi) = \lim_{a\to 0}\lim_{\tau\to 0+}\frac{1}{(2\pi)^{m-1}}\int\limits_{\mathbb{R}^{m-1}}\frac{iaz_m 2^{m-1}\pi^{(m-2)/2}\Gamma(m/2)\tilde{u}(\eta',\xi_m)\,d\eta'}{(|\xi'-\eta'|^2-a^2z_m^2)^{m/2}} = \tilde{u}(\xi',\xi_m).$$

Further, it can readily be seen that the parameters a and ξ_m can be swapped in the kernel (10), so that

$$\lim_{\xi_m\to 0}(V_a\tilde{u})(\xi) = \lim_{\xi_m\to 0}\lim_{\tau\to 0+}\frac{1}{(2\pi)^{m-1}}\int\limits_{\mathbb{R}^{m-1}}\frac{iaz_m 2^{m-1}\pi^{(m-2)/2}\Gamma(m/2)\tilde{u}(\eta',\xi_m)\,d\eta'}{(|\xi'-\eta'|^2-a^2z_m^2)^{m/2}} = \tilde{u}(\xi',0).$$

The first limit transition demonstrates the naturality of the term "Poisson's kernel," while the second will be needed in what follows when studying a specific boundary value problem.

2.2. General Solution and Boundary Conditions

Let $v \equiv 0$. Then the assertion in Theorem 1 becomes considerably simpler and the general solution of the equation looks as follows:

$$\tilde{u}(\xi) = A_{\neq}^{-1}(\xi)V_{-a}\left(\sum_{k=1}^{n}\tilde{c}_k(\xi')\xi_m^{k-1}\right),$$

or, after transformation, as

$$\tilde{u}(\xi) = A_{\neq}^{-1}(\xi)\left(\sum_{k=1}^{n}\tilde{C}_k(\xi,\xi_m)\xi_m^{k-1}\right), \tag{12}$$

where $\tilde{C}_k(\xi,\xi_m) = (V_{-a}\tilde{c}_k)(\xi',\xi_m)$.

Thus, we need additional conditions to unambiguously determine the functions $\tilde{C}_k(\xi',\xi_m)$. If we determine the functions $\tilde{C}_k(\xi',\xi_m)$, then the functions $\tilde{c}_k(\xi')$ will be uniquely determined by virtue of the operator V_{-a} being invertible.

2.3. Integral Boundary Condition

Formula (12) contains n arbitrary functions, and if we fix them, then the solution becomes unique. To unambiguously determine these functions, we should equip the equation with an additional (boundary) condition to arrive, as a result, at some boundary value problem.

2.3.1. The case of $n = 1$. In this case, formula (12) acquires the form

$$\tilde{u}(\xi) = A_{\neq}^{-1}(\xi)\tilde{C}_1(\xi,\xi_m) = A_{\neq}^{-1}(\xi)(V_{-a}\tilde{c}_1)(\xi',\xi_m) \tag{13}$$

and contains only one arbitrary function.

This problem looks the easiest with the integral boundary condition

$$\int\limits_0^{+\infty} u(x',x_m)\,dx_m = g(x'), \tag{14}$$

which, in terms of Fourier transforms, acquires the form

$$\tilde{u}(\xi',0) = \tilde{g}(\xi'). \tag{15}$$

If we take into account condition (15) in formula (13), then we obtain

$$\tilde{u}(\xi',0) = A_{\neq}^{-1}(\xi',0)\tilde{C}_1(\xi',0) = A_{\neq}^{-1}(\xi',0)(V_{-a}\tilde{c}_1)(\xi',0),$$

based on which one can readily determine the function $\tilde{c}_1(\xi')$, because $(V_a\tilde{c}_1)(\xi',0) = \tilde{c}_1(\xi')$.

Thus, when written in terms of Fourier transforms, the solution of the boundary value problem (14) for the homogeneous equation

$$(Au)(x) = 0, \quad x \in C_+^a, \tag{16}$$

looks as follows:

$$\tilde{u}(\xi) = A_{\neq}^{-1}(\xi) \lim_{\tau \to 0+} \frac{1}{(2\pi)^{m-1}} \int\limits_{\mathbb{R}^{m-1}} \frac{c_m iaz_m A_{\neq}(\eta', 0)\tilde{g}(\eta'))\, d\eta'}{(|\xi' - \eta'|^2 - a^2 z_m^2)^{m/2}}. \tag{17}$$

Let us state the last result as a theorem.

Theorem 2. *Let $\varkappa - s = 1 + \delta$, $|\delta| < 1/2$. Then the boundary value problem* (14) *with any right-hand side $g \in H^{s+1/2}(\mathbb{R}^{m-1})$ for the homogeneous equation* (16) *has a unique solution in the space $H^s(C_+^a)$, which is given by formula* (17).

Remark 3. We can consider a more general version of the cone $\mathbb{R}^k \times C_+^a$ in a $(k+m)$-dimensional space and obtain an analog of the above-proved lemma, while the k-wave factorization will allow us to derive an analog of Theorem 2.

2.3.2. The case of $m = 2$. In the two-dimensional case, formula (17) can be simplified considerably with the help of the theory of boundary value problems for analytic functions and the theory of one-dimensional singular integral equations [20, 21].

Consider the integral in (17) (we have taken into account that $c_2 = 2$, $z_2 = \xi_2 - i\tau$, $\tau > 0$),

$$F(\xi_1, z_2) = \frac{1}{\pi} \int\limits_{-\infty}^{+\infty} \frac{iaz_2 f(\eta_1)\, d\eta_1}{(\xi_1 - \eta_1)^2 - a^2 z_2^2},$$

which can be represented as a sum of two integrals with the help of the decomposition

$$\begin{aligned} \frac{1}{\pi} \frac{iaz_2}{(\xi_1 - \eta_1)^2 - a^2 z_2^2} &= -\frac{1}{2\pi i}\left[\frac{1}{\xi_1 - a\xi_2 - (\eta_1 - ia\tau)} - \frac{1}{\xi_1 + a\xi_2 - (\eta_1 + ia\tau)}\right] \\ &= \frac{1}{2\pi i}\left[\frac{1}{\eta_1 - ((\xi_1 - a\xi_2) + ia\tau)} - \frac{1}{\eta_1 - ((\xi_1 + a\xi_2) - ia\tau)}\right]. \end{aligned}$$

Consider a Cauchy type integral on the real line ($z_2 = \xi_2 + i\tau$)

$$\Phi(\xi_1, z_2) = \frac{1}{2\pi i} \int\limits_{-\infty}^{+\infty} \frac{u(\xi_1, \eta)\, d\eta}{\eta - z_2}$$

and denote its limit values from the upper and lower half-planes by $\Phi^+(\xi_1, \xi_2)$ and $\Phi^-(\xi_1, \xi_2)$, respectively. The formulas for the limit values of a Cauchy type integral are well known [20, p. 47],

$$\Phi^{\pm}(\xi_1, z_2) = \pm\frac{1}{2} u(\xi_1, \xi_2) + \frac{1}{2\pi i}\, \text{v.p.} \int\limits_{-\infty}^{+\infty} \frac{u(\xi_1, \eta)\, d\eta}{\eta - z_2}.$$

Taking into account these formulas for the function $F(\xi_1, z_2)$, we find its limit value

$$\lim_{\tau \to 0+} F(\xi_1, z_2) = F^+(\xi_1, \xi_1 - a\xi_2) - F^-(\xi_1, \xi_1 + a\xi_2).$$

Thus, in the two-dimensional case, formula (17) acquires the form

$$\tilde{u}(\xi_1,\xi_2) = \frac{A_{\neq}(\xi_1 - a\xi_2, 0)\tilde{g}(\xi_1 - a\xi_2) + A_{\neq}(\xi_1 + a\xi_2, 0)\tilde{g}(\xi_1 + a\xi_2)}{2A_{\neq}(\xi_1,\xi_2)}$$
$$+ \frac{1}{A_{\neq}(\xi_1,\xi_2)}\,\text{v.p.}\frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty}\frac{A_{\neq}(\eta,0)\tilde{g}(\eta)\,d\eta}{\eta-(\xi_1 - a\xi_2)} - \frac{1}{A_{\neq}(\xi_1,\xi_2)}\,\text{v.p.}\,\frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty}\frac{A_{\neq}(\eta,0)\tilde{g}(\eta)\,d\eta}{\eta-(\xi_1 + a\xi_2)}. \qquad (18)$$

Corollary. *Under the assumptions of Theorem* 2, *for* $m = 2$ *formula* (17) *acquires the form* (18).

Note that formula (18) was produced using a different method in the paper [8].

CONCLUSIONS

In the present paper, we have considered one of the simplest multidimensional versions of the boundary value problem in a cone; however, the general approach used can be applied to other "conic" situations [8, 9]. We envisage further research of multidimensional conic situations, because the great variety of those necessitates detailed investigation in each particular case.

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