The Hardy Space of Solutions of the Generalized Beltrami System

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We consider the first-order system

$$\frac{\partial \phi}{\partial y} - J\frac{\partial \phi}{\partial x} = F \tag{1}$$

on the plane, where $J \in \mathbb{C}^{l\times l}$ is a constant matrix whose eigenvalues lie in the upper half-plane $\operatorname{Im}\nu > 0$. In the scalar case $l = 1$, Eq. (1) with an (in general, continuous) coefficient $J(z)$, $\operatorname{Im} J > 0$, is referred to as the Beltrami equation [1, p. 72].

The matrix function

$$E(z) = \frac{1}{2\pi i} z_J^{-1} \tag{2}$$

[here and throughout the following, we use the matrix notation $z_J = x \times 1 + y \times J$ for $z = (x + yi) \in \mathbb{C}$] is a fundamental solution of the generalized Beltrami system (1). In other words, for any continuously differentiable compactly supported function $F(z)$, the integral

$$(TF)(z) = \frac{1}{2\pi i}\int\limits_{\mathbb{C}} (t-z)_J^{-1} F(t)dt_1dt_2 \tag{3}$$

specifies a classical solution of Eq. (1).

Indeed, TF is a continuously differentiable function, and its derivatives are given by the formulas

$$\frac{\partial (TF)}{\partial x} = \frac{1}{2\pi i}\int\limits_{\mathbb{C}} t_J^{-1}\frac{\partial F}{\partial x}(z+t)dt_1dt_2, \qquad \frac{\partial (TF)}{\partial y} = \frac{1}{2\pi i}\int\limits_{\mathbb{C}} t_J^{-1}\frac{\partial F}{\partial y}(z+t)dt_1dt_2. \tag{4}$$

Consider the two-dimensional singular integral

$$(SF)(z) = \frac{1}{2\pi i}\int\limits_{\mathbb{C}} t_J^{-2} F(z+t)dt_1dt_2 \tag{5}$$

treated as the limit of integrals over $\{|t| \ge \varepsilon\}$ as $\varepsilon \to 0$. Since

$$\int\limits_{|t|=1} t_J^{-2} ds_t = 0, \tag{6}$$

it follows that the necessary condition for the existence of such integrals is satisfied. To verify relation (6), it is most convenient to use the function

$$\chi(\nu) = \int\limits_0^{2\pi} (\cos\theta + \nu \sin\theta)^{-2} d\theta,$$

which is analytic in the upper half-plane $\operatorname{Im}\nu > 0$. On the one hand, the integral on the left-hand side in relation (6) coincides with the value $\chi(J)$ of this function on the matrix J. On the other hand, the function χ of the matrix and all of its derivatives vanish at the point $\nu = i$, whence we have $\chi(J) = 0$. Let us rewrite the integral (3) as the limit of integrals over $\{|t| \geq \varepsilon\}$ as $\varepsilon \to 0$. Then, in the usual way, we obtain the relations

$$\frac{\partial(TF)}{\partial x} = (SF)(z) + \sigma_1 F(z), \qquad \frac{\partial(TF)}{\partial y} = J(SF)(z) + \sigma_2 F(z), \tag{7}$$

where the coefficients $\sigma_k \in \mathbb{C}^{l\times l}$ are given by the formulas

$$\sigma_k = \frac{1}{2\pi i}\int\limits_{|t|=1} t_J^{-1} n_k ds_t, \quad k = 1, 2,$$

here $n = (n_1, n_2)$ stands for the unit inward normal of the cycle $|t| = 1$, so that

$$\sigma_1 = -\frac{1}{2\pi i}\int\limits_0^{2\pi}(\cos\theta + J\sin\theta)^{-1}\cos\theta\, d\theta, \qquad \sigma_2 = -\frac{1}{2\pi i}\int\limits_0^{2\pi}(\cos\theta + J\sin\theta)^{-1}\sin\theta\, d\theta.$$

One can readily see that

$$\sigma_2 - J\sigma_1 = -\frac{1}{2\pi i}\int\limits_0^{2\pi}(-\sin\theta + J\cos\theta)(\cos\theta + J\sin\theta)^{-1} d\theta = 1. \tag{8}$$

Indeed, the left-hand side is the value on J of the function

$$\chi_0(\nu) = \frac{1}{2\pi i}\int\limits_0^{2\pi}\frac{-\sin\theta + \nu\cos\theta}{\cos\theta + \nu\sin\theta} d\theta,$$

which is analytic in the half-plane $\operatorname{Im}\nu > 0$. Simple computations show that $\chi_0(i) = 1$, $\chi_0^{(k)}(i) = 0$, $k = 1, 2, \ldots$, and hence $\chi_0(\nu) = 1$.

By combining (7) with (8), we find that the function TF indeed satisfies Eq. (1).

The singular operator (5) belongs to the Calderón–Zygmund type. By [2, p. 52 of the Russian translation], it is bounded in the space $L^p(\mathbb{C})$, $p > 1$; moreover, if $F \in L^p$, then the integral exists for almost all z. This, together with (7) and (8), implies the following result.

Theorem 1. *Let a domain $D \subseteq \mathbb{C}$ lie in a finite part of the plane, and let $F \in L^p(D)$, $p > 1$. Then the integral* (3) *defines a function TF that lies in the Sobolev space $W^{1,p}(D)$ and whose derivatives are given by* (7). *Moreover,*

$$|TF|_{W^{1,p}(D)} \leq C|F|_{L^p(D)}, \tag{9}$$

where $C > 0$ is a constant depending only on p and D.

Proof. The estimates

$$|(TF)_x|_{L^p} + |(TF)_y|_{L^p} \leq C|F|_{L^p}$$

form the contents of the Calderón–Zygmund theorem for the singular operator S. The estimate $|TF|_{L^p(D)} \leq C|F|_{L^p(D)}$ for the integral (3) can readily be derived from the Hölder inequality. These estimates imply (9).

Let D be the domain bounded by a piecewise smooth contour Γ. Consider a sequence of contours $\Gamma_n \subseteq D$, $n = 1, 2, \dots$, approximating Γ in the following sense: for each n, there exists a piecewise continuous differentiable homeomorphic mapping $\alpha_n : \Gamma \to \Gamma_n$ such that

$$|\alpha_n(t) - t|_{\mathbb{C}(\Gamma)} + |\alpha_n' - 1|_{\mathbb{C}(\Gamma)} \to 0 \quad \text{as} \quad n \to \infty. \tag{10}$$

In particular, if the contour Γ_n bounds the domain D_n, then every compact set $K \subseteq D$ lies in all D_n for sufficiently large n.

Let $\phi \in W^{1,p}_{\text{loc}}(D)$; i.e., this function belongs to the class $W^{1,p}$ in each domain D_n, $n = 1, 2, \dots$ Then, by the embedding theorem [3], the estimate

$$|\phi|_{L^p(\Gamma_n)} \leq C_n |\phi|_{W^{1,p}(D_n)} \tag{11}$$

is valid for each n.

We introduce the following notion. A function $\phi \in W^{1,p}_{\text{loc}}(D)$ belongs to the Hardy class $H^p_J(D)$ if

$$\frac{\partial \phi}{\partial y} - J\frac{\partial \phi}{\partial x} \in L^p(D) \tag{12}$$

$$\sup_n |\phi|_{L^p(\Gamma_n)} < \infty. \tag{13}$$

Note that

$$W^{1,p}(D) \subseteq H^p_J(D). \tag{14}$$

Indeed, in this case, the norm on the right-hand side in the estimate (11) can be computed in $W^{1,p}(D)$; therefore, it is only necessary to show that the constants C_n occurring in this estimate are uniformly bounded. By using condition (10), one can readily obtain this fact from the proof of the embedding $W^{1,p} \subseteq L^p(\Gamma)$ (e.g., see [3, p. 420]).

It follows from (14) and Theorem 1 that T is a bounded operator in the spaces $L^p(D) \to H^p_J(D)$ and each function $\phi \in H^p$ can be represented in the form

$$\phi = TF + \phi_0, \tag{15}$$

where $F \in L^p(D)$ and ϕ_0 satisfies the homogeneous equation (1). Solutions ϕ of this homogeneous equation (1) were dubbed Douglis analytic functions or, briefly, J-analytic functions in [4].

An example of such functions is given by the Cauchy type integral

$$(I\varphi)(z) = \frac{1}{\pi i} \int_\Gamma (t - z)_J^{-1} dt_J \varphi(t), \quad z \in D, \tag{16}$$

where dt_J stands for the matrix differential $dt_1 + J\,dt_2$, $t = t_1 + it_2 \in \Gamma$, and, to be definite, we assume that the contour Γ has the positive sense with respect to D (i.e., the domain D lies to the left when moving along it).

By [5], the integral (16) with a vector function $\varphi \in L^p(\Gamma)$ defines a function $I\varphi \in H^p_J(D)$. More precisely, the following assertion is valid.

Theorem 2. (a) *The operator I is a bounded operator in the spaces $L^p(D) \to H^p_J(D)$.*

(b) *Let $\varphi \in L^p(\Gamma)$, $p > 1$. Then for the integral, there exist angular limit values $\phi^+(t_0)$ for almost all $t_0 \in \Gamma$, and the Sokhotskii–Plemelj formula*

$$\phi^+(t_0) = \varphi(t_0) + \frac{1}{\pi i} \int_\Gamma (t - t_0)_J^{-1} dt_J \varphi(t)$$

is valid, where the integral on the right-hand side is a singular integral treated in the sense of the Cauchy principal value. In addition, $|\phi^+|_{L^p} \leq C|\varphi|_{L^p}$.

In particular, it follows from Theorem 2 that $H^p(D)$ equipped with the norm

$$|\phi| = |\phi_y - J\phi_x|_{L^p(D)} + \sup_n |\phi|_{L^p(\Gamma_n)} \tag{17}$$

is a Banach space. By using Theorem 2, one can readily give the following equivalent description of the space H^p.

Theorem 3. *In the class of smooth functions in D, the norm* (17) *is equivalent to the norm*

$$|\phi| = |\phi_y - J\phi_x|_{L^p(D)} + |\phi|_{L^p(\Gamma)}. \tag{18}$$

Proof. By virtue of the expansion (15), it suffices to verify the equivalence of the norms (17) and (18) for J-analytic functions.

By $C^{+0}\left(\bar{D}\right)$ we denote the class of Hölder continuous functions. If ϕ is a J-analytic function in $C^{+0}\left(\bar{D}\right)$, then, by applying Theorem 2 to the Cauchy formula

$$2\phi(z) = \frac{1}{\pi i}\int_\Gamma (t-z)_J^{-1} dt_J \phi(t),$$

we obtain the estimate $|\phi|_{H^p(D)} \le C|\phi|_{L^p(\Gamma)}$. Conversely, let $\phi \in H^p(D) \cap C^{+0}\left(\bar{D}\right)$. Then, by (10), the sequence α_n uniformly converges to ϕ on Γ, whence we obtain $|\phi|_{L^p(\Gamma)} \le C_1|\phi|_{H^p(D)}$. This implies the desired equivalence of the norms.

Theorem 3, together with its proof, implies that the space $H^p(D)$ can be obtained as the closure of the class $C^{1,+0}\left(\bar{D}\right)$ in the norm (18). In particular, to each function $\phi \in H^p$, one can assign its limit value $\phi^+ \in L^p(\Gamma)$.

The expansion (16) can be complemented as follows.

Theorem 4. *Let D be the domain bounded by a simple piecewise Lyapunov contour without return points, and let J be a triangular matrix. Then each function $\phi \in H^p(D)$ can be uniquely represented in the form*

$$\phi = TF + I\varphi + i\xi, \tag{19}$$

where $\xi \in \mathbb{R}^l$, $F = \phi_y - J\phi_x$, and $\varphi \in L^p(\Gamma)$ is a vector function taking real values.

Proof. By replacing ϕ by $\phi - T\phi$, without loss of generality, one can assume that $F = 0$. In this case, it suffices to use the results in [4, 6].

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