WEAK MAXIMUM PRINCIPLE FOR ELLIPTIC OPERATORS ON A STRATIFIED SET

A. A. Gavrilov and O. M. Penkin UDC 517.951

At the beginning of the 1980th, in connection with the problem on small oscillation of a system of coupled strings, Yu. V. Pokornyi began to study differential equations on geometric graphs. Independently, in connection with the problem on diffusion on a spatial net, G. Lumer in Belgium and J. von Below in Germany arrived at the same equations. Equations on stratified sets are a natural continuation of this topic. We arrive at them when membranes are added to a string system. A stratified set is a connected subset of $\mathbb{R}^n$ composed of finitely many manifolds of different dimensions sufficiently regularly adjacent to each other (below, this will be made more precise). In the definition of an elliptic operator, we follow the approach suggested by Pokornyi in studying equations on graphs. The essence of this approach manifests itself even on the stage of modeling the mentioned mechanical systems. It is clear that, for example, small deformations of a system of coupled string are described by a tuple of second-order differential equations (whose number coincides with the number of strings) whose solutions must be in concordance with the continuity conditions and the so-called transmission conditions (tension balance) at places of joining separate strings. It turns out that if we extend the classical concept of divergence to the case of measures described below, then the equations mentioned and the transmission conditions can be interpreted as one second-order equation as a whole on a geometric graph. The idea to consider differential operations with respect to abstract measures is well known, but it almost never is applied to the class of problems considered. A recent work of Zhikov (see [8]) shows that the class of sets on which it is possible not to consider an elliptic operator but to obtain effective results can be enlarged.

This treatment of equations on graphs turned out to be very fruitful in studying qualitative properties of solutions of equations on graphs (see, e.g., [6, 7]). In this paper, we discuss the weak maximum principle for an elliptic equation on an arbitrary stratified set. Early, we have succeeded in doing this only on a two-dimensional stratified set (see [3]).

1. Main concepts. In this section, we briefly describe main concepts. A detailed presentation can be found in [4, 5].

A connected set in $\mathbb{R}^n$ is said to be stratified if it is represented as the union of finitely many smooth manifolds (strata) σ_{ki} of different dimension. The subscript k is the dimension and the subscript i is the number of a stratum of this dimension. Moreover, it is assumed that strata regularly are adjacent to each other; if a stratum σ_{ki} is adjacent to $\sigma_{k+1,j}$ (which symbolically is written in the form $\sigma_{ki} \prec \sigma_{k+1,j}$), then when a point $y \in \sigma_{k+1,j}$ approaches a point $x \in \sigma_{ki}$, the tangent spaces $T_y\sigma_{k+1,j}$ tend to a certain limit position containing $T_x\sigma_{ki}$. Finally, as a whole, the set Ω must have a cell complex structure.

On the set Ω, a measure μ is naturally defined. Precisely, s set $\omega \subset \Omega$ is said to be measurable if each intersection $\omega \cap \sigma_{ki}$ is measurable in the usual k-dimensional Lebesgue sense. The measure of the set ω is set equal to the sum of measures of the above intersections. This measure will be naturally called the Lebesgue–Stieltjes measure, since the density of this measure can have different dimension on different strata. The Lebesgue integral with respect to such a measure will be called the Lebesgue–Stieltjes integral. It should be noted that for functions u Riemann integrable on each of the strata, we have

$$\int_\Omega u\,d\mu = \sum_{\sigma_{ki}\subset\Omega}\int_{\sigma_{ki}} u\,d\mu,$$

and this formula can be taken as the definition of the integral of such functions.

Along with Ω, we distinguish and fix a certain open connected (in the topology on Ω induced from $\mathbb{R}^n$) subset Ω_0 composed of strata and generating Ω in the sense that $\overline{\Omega}_0 = \Omega$. Then the difference $\partial\Omega_0 = \Omega\setminus\Omega_0$ is the boundary of Ω_0.

2. Divergence and the Green formulas. A vector field $\boldsymbol{F}$ in $\mathbb{R}^n$ generates the corresponding vector field on Ω. It is said to be tangent to Ω if for any σ_{ki} and $x \in \sigma_{ki}$, we have $\boldsymbol{F}(x) \in T_x\sigma_{ki}$. The divergence $\nabla\boldsymbol{F}(x)$ of a tangent vector field at a point $x \in \sigma_{k-1,i} \subset \Omega_0$ is defined as follows:

$$(\nabla\boldsymbol{F})(x) = (\widetilde{\nabla}\boldsymbol{F})(x) + \sum_{\sigma_{kj}\succ\sigma_{k-1,i}} (\boldsymbol{F}\cdot\boldsymbol{\nu})_{\overline{kj}}(x), \tag{1}$$

where $\widetilde{\nabla}\boldsymbol{F}$ is the classical divergence on $\sigma_{k-1,i}$ and $\boldsymbol{\nu}$ is the unit normal at a point $x \in \sigma_{k-1,i}$ directed inward σ_{kj}; the connection of the vector $\boldsymbol{\nu}$ with the stratum σ_{kj} is not indicated in its notation; however, when considering the sum (1), one should keep in mind this connection. Everywhere in what follows, the writing of the type $f_{\overline{kj}}$ means an extension by continuity to $\overline{\sigma}_{kj}$ of the restriction of a function $f : \Omega \to \mathbb{R}$ to σ_{kj}. For the presented definition of divergence have a meaning, we will assume that the field $\boldsymbol{F}$ is uniformly continuous inside each of the strata and continuously differentiable in it; in this case, we will write $\boldsymbol{F} \in C^1_\sigma(\Omega_0)$. The defined divergence can be interpreted as the density of the flow of a vector field with respect to the measure μ mentioned above, i.e., similarly to the classical divergence.

Let $C^2_\sigma(\Omega_0)$ be the set of "stratawise two times differentiable" functions on Ω_0, which is similar to $C^1_\sigma(\Omega_0)$, and let $C^2(\Omega_0) = C^2_\sigma(\Omega_0)\cap C(\Omega_0)$, where $C(\Omega_0)$ is the set of continuous functions on Ω_0. Denoting by ∇u the gradient of u, it is easy to note that for $u \in C^2(\Omega_0)$, we have $\nabla u \in C^1_\sigma(\Omega_0)$ (componentwise inclusion) and, therefore, the operator $\Delta u = \nabla(\nabla u)$, an analogue of usual Laplace–Beltrami operator, is defined. We consider a slightly more general divergence-type operator

$$(\Delta_p u)(x) = (\nabla(p\nabla u))(x) + \sum_{\sigma_{kj}\succ\sigma_{k-1,i}} (p\nabla u\cdot\boldsymbol{\nu})_{\overline{kj}}(x).$$

The function $p \in C^1_\sigma(\Omega_0)$ is assumed to be strictly positive.

Let us pass to the deduction of the Green formulas. Note that such formulas are contained in [4, 5], but all the functions are assumed to be continuous there. We assume the continuity of functions only inside the strata. As a result, we have more complicated formulas. We are forced to deal with them, since we do not know an analogue of the Friedrichs–Sobolev smoothing operation on a stratified set. Therefore, we use "stratawise" smoothing which, refining functions inside strata, "spoils" them as a whole; the function smoothed in such a way can have discontinuity when passing from one stratum to another. Nevertheless, discontinuities are removed when the smoothing parameter tends to zero. This circumstance allows us to extend the well-known technique for proving the weak maximum principle (see, e.g., [2]) to stratified sets.

Recall that according to the accepted definition, the integral over Ω_0 is the sum of integrals over separate strata. Consider a certain stratum $\sigma_{li} \subset \Omega_0$ and the integral over it of $\varphi\Delta_p u$. If $\varphi \in C^1_\sigma(\Omega_0)$, $u \in \overline{C}^2_\sigma(\Omega_0)$ (the bar means the continuity of higher derivatives inside each stratum), then transforming the integral of the classical part of Δ_p by the Green formula and not changing the integral of the nonclassical part, we obtain

$$\int_{\sigma_{li}} \varphi\Delta_p u\,d\mu = -\int_{\sigma_{li}} p\nabla u\nabla\varphi\,d\mu - \sum_{\sigma_{l-1,k}\prec\sigma_{li}} \int_{\sigma_{l-1,k}} \varphi\Big|_{\overline{li}}(p\nabla u\cdot\boldsymbol{\nu})\Big|_{\overline{li}}\,d\mu + \sum_{\sigma_{l+1,j}\succ\sigma_{li}} \int_{\sigma_{li}} \varphi(p\nabla u\cdot\boldsymbol{\nu})\Big|_{\overline{l+1,j}}\,d\mu. \tag{2}$$

For strata of higher dimension, the latter sum vanishes since Δ_p does not contain the nonclassical part; for strata of dimension 0, there is no the classical part of the operator.

A part of the strata $\sigma_{l-1,k}$ belongs to the boundary; it is convenient for us to isolate them in a separate sum. As a result, summing equations of the form (2) over all strata of dimension l, we obtain

$$-\sum_{\sigma_{li}}\int\limits_{\sigma_{li}}\varphi\Delta_p u\,d\mu=-\sum_{\sigma_{li}}\int\limits_{\sigma_{li}}p\nabla u\nabla\varphi\,d\mu-\sum_{\sigma_{li}}\sum_{\substack{\sigma_{l-1,k}\prec\sigma_{li}\\ \sigma_{l-1,k}\subset\partial\Omega_0}}\int\limits_{\sigma_{l-1,k}}\varphi\Big|_{\overline{li}}(p\nabla u\cdot\boldsymbol{\nu})\Big|_{\overline{li}}\,d\mu-$$
$$-\sum_{\sigma_{li}}\sum_{\substack{\sigma_{l-1,k}\prec\sigma_{li}\\ \sigma_{l-1,k}\not\subset\partial\Omega_0}}\int\limits_{\sigma_{l-1,k}}\varphi\Big|_{\overline{li}}(p\nabla u\cdot\boldsymbol{\nu})\Big|_{\overline{li}}\,d\mu+\sum_{\sigma_{li}}\sum_{\substack{\sigma_{l+1,j}\prec\sigma_{li}\\ \sigma_{l+1,j}\not\subset\partial\Omega_0}}\int\limits_{\sigma_{li}}\varphi(p\nabla u\cdot\boldsymbol{\nu})\Big|_{\overline{l+1,j}}\,d\mu. \tag{3}$$

Now it is natural to take the sum over all the strata in general. Before doing this, it is convenient to introduce the notation

$$A_{l-1}=\sum_{\sigma_{li}}\sum_{\substack{\sigma_{l-1,k}\prec\sigma_{li}\\ \sigma_{l-1,k}\not\subset\partial\Omega_0}}\int\limits_{\sigma_{l-1,k}}\varphi\Big|_{\overline{li}}(p\nabla u\cdot\boldsymbol{\nu})\Big|_{\overline{li}}\,d\mu,$$
$$B_l=\sum_{\sigma_{li}}\sum_{\substack{\sigma_{l+1,j}\prec\sigma_{li}\\ \sigma_{l+1,j}\not\subset\partial\Omega_0}}\int\limits_{\sigma_{li}}\varphi(p\nabla u\cdot\boldsymbol{\nu})\Big|_{\overline{l+1,j}}\,d\mu.$$

In this notation, the result of summation has the form

$$\int\limits_{\Omega_0}\varphi\Delta_p u\,d\mu=\int\limits_{\Omega_0}p\nabla u\nabla\varphi\,d\mu-\sum_{\sigma_{li}\subset\Omega_0}\sum_{\substack{\sigma_{l-1,k}\prec\sigma_{li}\\ \sigma_{l-1,k}\subset\partial\Omega_0}}\int\limits_{\sigma_{l-1,k}}\varphi\Big|_{\overline{li}}(p\nabla u\cdot\boldsymbol{\nu})\Big|_{\overline{li}}\,d\mu-\sum_{l=0}^{k}(A_l+B_l). \tag{4}$$

Accepting the notation

$$j(\varphi)(x)=\varphi\Big|_{l-1,i}(x)-\varphi\Big|_{\overline{lj}}(x),\quad x\in\sigma_{l-1,i}\succ\sigma_{lj}$$

for the last sum, we obviously obtain

$$\sum_{l=0}^{k}(A_l+B_l)=-\sum_{\substack{\sigma_{li}\subset\Omega_0\\ l\neq k}}\sum_{\sigma_{l+1,j}\succ\sigma_{li}}\int\limits_{\sigma_{li}}j(\varphi)(p\nabla u\cdot\boldsymbol{\nu})\Big|_{\overline{li}}\,d\mu.$$

By the transformation

$$\varphi\Big|_{\overline{li}}=\varphi\Big|_{l-1,k}-j(\varphi),$$

the latter sum reduces to the form

$$\sum_{\sigma_{li}\subset\Omega_0}\sum_{\substack{\sigma_{l-1,k}\prec\sigma_{li}\\ \sigma_{l-1,k}\subset\partial\Omega_0}}\int\limits_{\sigma_{l-1,k}}\varphi\Big|_{\overline{li}}(p\nabla u\cdot\boldsymbol{\nu})\Big|_{\overline{li}}\,d\mu=$$
$$=-\sum_{\sigma_{li}\subset\Omega_0}\sum_{\substack{\sigma_{l-1,k}\prec\sigma_{li}\\ \sigma_{l-1,k}\subset\partial\Omega_0}}\int\limits_{\sigma_{l-1,k}}\varphi\Big|_{l-1,k}(p\nabla u\cdot\boldsymbol{\nu})\Big|_{\overline{li}}\,d\mu+\sum_{\sigma_{li}\subset\Omega_0}\sum_{\substack{\sigma_{l-1,k}\prec\sigma_{li}\\ \sigma_{l-1,k}\subset\partial\Omega_0}}\int\limits_{\sigma_{l-1,k}}j(\varphi)(p\nabla u\cdot\boldsymbol{\nu})\Big|_{\overline{li}}\,d\mu.$$

It is natural to combine the summands with “jumps” $j(\varphi)$; then introducing the notation

$$\sum_{\substack{\sigma_{l+1,j}\succ\sigma_{li}\\ \sigma_{l+1,j}\not\subset\partial\Omega_0}}j(\varphi)(p\nabla u\cdot\boldsymbol{\nu})\Big|_{\overline{li}}=\{j(\varphi),(p\nabla u)_\nu\},$$

we obtain

$$\int\limits_{\Omega_0} \varphi \Delta_p u \, d\mu = -\int\limits_{\Omega_0} p\nabla u \nabla \varphi \, d\mu - \int\limits_{\partial\Omega_0} \varphi (p\nabla u)_\nu \, d\mu + \int\limits_{\Omega\setminus\sigma_k} \{j(\varphi), (p\nabla u)_\nu\} \, d\mu.$$

Therefore, the following assertion holds.

Lemma. *The following formula holds for any functions* $u \in \overline{C}^2_\sigma(\Omega_0)$, $\varphi \in C^1_\sigma(\Omega_0)$:

$$\int\limits_{\Omega_0} \varphi \Delta_p u \, d\mu = -\int\limits_{\Omega_0} p\nabla u \nabla \varphi \, d\mu - \int\limits_{\partial\Omega_0} \varphi (p\nabla u)_\nu \, d\mu + \int\limits_{\Omega\setminus\sigma_k} \{j(\varphi), (p\nabla u)_\nu\} \, d\mu. \tag{5}$$

Remark. As a consequence from the obtained formula, we obtain the following analogue of the first Green formula:

$$\int\limits_{\Omega_0} \varphi \Delta_p u \, d\mu = -\int\limits_{\Omega_0} p\nabla \varphi \nabla u \, d\mu - \int\limits_{\partial\Omega_0} \varphi (p\nabla u)_\nu \, d\mu \quad \text{for} \quad u \in \overline{C}^2_\sigma(\Omega_0), \ \varphi \in \overline{C}^1(\Omega_0), \tag{6}$$

since $\overline{C}^1_\sigma(\Omega_0) \subset C^1_\sigma(\Omega_0)$, and for $\varphi \in C(\Omega)$, obviously, $j(\varphi)(x) = 0$; then (5) immediately implies (6).

3. Weak maximum principle.

Theorem. *Let $q \in C_\sigma(\Omega_0)$ be nonnegative and the set Ω be oriented. Then the following relation holds for a solution of the inequality $L_q u \geq 0$, $u \in C^2(\Omega_0) \cap C(\Omega)$:*

$$u(y) \leq \max_{x \in \partial\Omega_0} u^+(x), \quad u^+(x) = max\{0, u(x)\}$$

for any $y \in \Omega$.

Proof. First, we prove the relation of the formulation of the theorem for a solution of the inequality

$$\Delta_p u \geq 0. \tag{7}$$

Let G be the set of functions such that

(1) $\varphi \in C^1_\sigma(\Omega_0)$;
(2) $\varphi(x) \geq 0$, $x \in \Omega$;
(3) $\operatorname{supp} \varphi \subset \Omega_0$.

Then the following inequality holds for a solution of inequality (7) and any function $\varphi \in G$:

$$(\Delta_p u \cdot \varphi)(x) \geq 0 \quad \forall x \in \Omega_0.$$

Integrating this inequality over Ω_0, we obtain the inequality

$$\int\limits_{\Omega_0} \Delta_p u \cdot \varphi \, d\mu \geq 0.$$

Then according to Lemma 2,

$$-\int\limits_{\Omega_0} p\nabla u \nabla \varphi \, d\mu - \int\limits_{\partial\Omega_0} \varphi (p\nabla u)_\nu \, d\mu + \int\limits_{\Omega\setminus\sigma_k} \{j(\varphi), (p\nabla u)_\nu\} \, d\mu \geq 0.$$

The second summand on the left-hand side of this inequality is equal to 0 since $\varphi\Big|_{\partial\Omega_0} = 0$ for any function $\varphi \in G$. Therefore, we have

$$-\int\limits_{\Omega_0} p\nabla u \nabla \varphi \, d\mu + \int\limits_{\Omega\setminus\sigma_k} \{j(\varphi), (p\nabla u)_\nu\} \, d\mu \geq 0.$$

Further, we assume the contrary, i.e., there exists a point $x_0 \in \sigma_{li}$ such that

$$u(x_0) > \max_{x\in\partial\Omega_0} u^+(x).$$

Then let c be a constant such that

$$u(x_0) > c > \sup_{x\in\partial\Omega_0} u^+(x).$$

Let $\Omega' \subset \Omega_0$ be a certain connected component of the set on which $u(x) - c \geq 0$. We set

$$v(x) = \begin{cases} u(x) - c, & x \in \Omega', \\ 0, & x \notin \Omega'. \end{cases}$$

Now let us define the function $v_\varepsilon(x)$ as follows: on each stratum $\sigma_{li} \subset \Omega_0$, for any $x \in \sigma_{li}$,

$$v_\varepsilon(x) = \frac{1}{\varepsilon^n} \int\limits_{\sigma'_{li}} \rho\Big(\frac{x-y}{\varepsilon}\Big) v(y)\, dy,$$

under the assumption that the function $v(x)$ is defined in a certain neighborhood σ'_{li} of the stratum σ_{li} (such an extension of $v(x)$ by continuity to the neighborhood of σ_{li} is possible; see, e.g., [1]). The function ρ is the so=called averaging kernel; it is chosen so that

$$\int\limits_{\|z\|\leq 1} \rho(z)\, dz = 1.$$

Obviously, for a sufficiently small $\varepsilon > 0$,

$$\frac{1}{\varepsilon^n} \int\limits_{\sigma'_{li}} \rho\Big(\frac{x-y}{\varepsilon}\Big)\, dy = 1.$$

Then for a sufficiently small ε, which, obviously, can be taken uniform for all strata,

(1) $v_\varepsilon \in C^1_\sigma(\Omega_0)$;
(2) $v_\varepsilon(x) \geq 0,\ x \in \Omega$;
(3) $\operatorname{supp} v_\varepsilon \subset \Omega_0$.

Therefore,

$$-\int\limits_{\Omega_0} p\nabla u \nabla v_\varepsilon\, d\mu + \int\limits_{\Omega\setminus\sigma_k} \{j(v_\varepsilon), (p\nabla u)_\nu\}\, d\mu \geq 0. \tag{8}$$

As is well known (see, e.g., [2]), the function v_ε, together with its gradient, uniformly converges to v on each of the strata as $\varepsilon \to 0$; therefore, passing to the limit in (8) as $\varepsilon \to 0$, we obtain

$$-\int\limits_{\Omega_0} p\nabla u \nabla v\, d\mu + \int\limits_{\Omega\setminus\sigma_k} \{j(v), (p\nabla u)_\nu\}\, d\mu \geq 0.$$

Since the function v is continuous, $j(v) \equiv 0$ on Ω. Since

$$(\nabla v)(x) = \begin{cases} (\nabla u)(x), & x \in \Omega', \\ 0, & x \notin \Omega', \end{cases}$$

we have

$$-\int\limits_{\Omega'} p\nabla v \nabla v\, d\mu \geq 0,$$

but since $p \geq 0$, it follows that

$$\int\limits_{\Omega'} p(\nabla v)^2\, d\mu \geq 0.$$

The obtained contradiction proves the assertion for the case of the operator Δ_p.

In the proof for the operator L_q, the changes are not substantial. The formula for the operator L_q in Lemma 2 obviously becomes

$$\int\limits_{\Omega_0} \varphi L_q u \, d\mu = -\int\limits_{\Omega_0} (p\nabla u \nabla \varphi + qu) \, d\mu - \int\limits_{\partial\Omega_0} \varphi(p\nabla u)_\nu \, d\mu + \int\limits_{\Omega\setminus\sigma_k} \{j(\varphi), (p\nabla u)_\nu\} \, d\mu;$$

Therefore, using similar arguments on a definite stage, we obtain the following formula:

$$-\int\limits_{\Omega'} (p\nabla v \nabla v + qu) \, d\mu \geq 0,$$

but since $u \geq 0$ on Ω', it follows that

$$-\int\limits_{\Omega'} p\nabla v \nabla v \, d\mu \geq \int\limits_{\Omega'} qu \, d\mu \geq 0,$$

and a contradiction can be obtained in the same way. The theorem is proved. □

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A. A. Gavrilov
Voronezh State University
O. M. Penkin
Belgorod State University