MAXIMUM PRINCIPLE FOR SUBHARMONIC FUNCTIONS ON A STRATIFIED SET

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Abstract. The strong maximum principle is proved for solutions of the inequality $\Delta u \geq 0$ on a stratified set.

If u is a smooth function in a domain $\Omega_0 \subset \mathbb{R}^n$ and X is a nontrivial maximum point of u (i.e, a local maximum point such that in any its neighborhood, the function u is not an identical constant), then it is easy to show that there exists an arbitrarily small $r > 0$ such that the integral of the exterior normal derivative of the function u calculated over the sphere of radius r centered at X is negative.

In this work, we give the proof of an analogous assertion for the case of a smooth function on a stratified set, and based on it, we prove the strong maximum principle for solutions of the inequality $\Delta u \geq 0$ with Laplacian on a stratified set. The proof of this fact, which was announced in [2], was previously obtained only for elliptic inequalities on two-dimensional stratified sets [1] by using a principally different approach.

1. Main Definitions

The general definition of a stratified set Ω can be found in [3]. The definition, which essentially narrows the class of stratified sets, but which is more convenient for the statement of boundary-value problems on them, can be found in [4]. Here, we restrict ourselves to a narrower class. Precisely, we assume that $\Omega \subset \mathbb{R}^n$ is a connected set composed of finitely many open (in planes of minimal dimensions containing them) convex polyhedra (strata) σ_{kj} (k is the stratum dimension, and j is its serial number under independent enumeration of strata of every dimension) that are adjacent to each other as in a simplicial complex (see,e.g., [5]), i.e., it is assumed that the boundary $\partial\sigma_{kj}$ consists of strata of lesser dimensions, and the closures of two arbitrary strata either have empty intersection or intersect along a common face.

In Ω, let us isolate an open and connected (in the topology on Ω induced from $\mathbb{R}^n$) subset Ω_0 composed of strata of the set Ω such that $\overline{\Omega}_0 = \Omega$. The difference $\Omega \setminus \Omega_0$ is the topological boundary of Ω_0 in this case, and, therefore, it is denoted by $\partial\Omega_0$; the case $\partial\Omega_0 = \emptyset$ is not excluded.

For the problems considered, on the set Ω, it is convenient to introduce the so-called stratified measure. For $\omega \subset \Omega$, it is defined by the formula

$$\mu(\omega) = \sum_{\sigma_{kj}} \mu_k(\omega \cap \sigma_{kj}), \tag{1}$$

where μ_k is the standard k-dimensional Lebesgue measure on the stratum σ_{kj}. The sets whose intersections with all strata are Lebesgue measurable turn out to be measurable in such a measure. By definition, the measure of zero-dimensional stratum is equal to 1. The Lebesgue integral of a measurable function $f : \Omega \to \mathbb{R}$ with respect to such a measure is the sum

$$\int_{\Omega} f\, d\mu = \sum_{\sigma_{kj}} \int_{\sigma_{kj}} f\, d\mu_k$$

of Lebesgue integrals over separate strata. In what follows, we omit $d\mu$ and $d\mu_k$ in writing the integrals.

2. Auxiliary Formula

Let $X \in \Omega$. If $r > 0$ is less than the distance to all strata that do not contain X in their closures (such r are said to be admissible), then the set

$$S_r(X) = \{Y \in \Omega : \ \|Y - X\| = r\}$$

is called the sphere; it is the intersection of sphere $\hat{S}_r(X)$ of the ambient space with Ω. The corresponding open ball is denoted by $B_r(X)$. By definition, the m-dimensional part of the sphere $S_r(X)$, which is denoted by $S_r^m(X)$, is the union of its intersections with all $(m+1)$-dimensional strata. The sphere can also be considered as a stratified set; as its m-dimensional strata, it is natural to consider the connected components of the set $S_r^m(X)$. The integral over the sphere is calculated with respect to the stratified measure mentioned above. For admissible r_1 and r_2, the spheres $S_{r_1}(X)$ and $S_{r_2}(X)$ and also their parts $S_{r_1}^m(X)$ and $S_{r_2}^m(X)$ are similar. Owing to this, we easily obtain the formula

$$\frac{d}{dr}\left(\frac{1}{r^m} \int_{S_r^m(X)} u\right) = \frac{1}{r^m} \int_{S_r^m(X)} \frac{\partial u}{\partial \nu}, \tag{2}$$

where ν is the exterior normal to the sphere $\hat{S}_r(X)$ at the points of the sphere $S_r(X)$. Here, it is assumed that $X \in \Omega_0$, and the function u is assumed to be continuous on Ω_0 as a whole (this property is not needed in this section, but we will need it in what follows), differentiable inside each strata $\sigma_{kj} \subset \Omega_0$ and such that the integrals in the right-hand side of (2) converge for all m. The set of such functions is denoted by $C^1(\Omega_0)$.

Multiplying both sides of the formula (2) by r^m and summing in all m, we obtain

$$\int_{S_r(X)} \frac{\partial u}{\partial \nu} = \sum_{m=0}^{d} r^m \frac{d}{dr}\left(\frac{1}{r^m} \int_{S_r^m(X)} u\right). \tag{3}$$

In fact, the stratified sphere need not contain parts of all dimensions. Therefore, to the right, there formally can arise extra summands. To avoid confusions, we consider the integrals over empty $S_r^m(X)$ to be equal to zero.

3. Necessary Extremum Condition

We present here the result, which we consider the basis of our proof of the strong maximum principle.

Theorem 1. *Let $X \in \Omega_0$ be a nontrivial local maximum point of a function $u \in C^1(\Omega_0)$. Then there exist arbitrarily small admissible $r > 0$ such that*

$$\int\limits_{S_r(X)} \frac{\partial u}{\partial \nu} < 0. \tag{4}$$

We call X a nontrivial local maximum point of the function u if for all points Y close to it, the inequality $u(Y) \leq u(X)$ holds, and in any neighborhood of X, the function u is not constant.

For the proof, we need the following two assertions.

Lemma 1. *Let f_0 and f_1 be functions continuous on $[0; a]$ and continuously differentiable on $(0; a]$ such that $f_0(0) = f_1(0) = 0$. If the function f_0 is nonpositive and*

$$r f_1'(r) + f_0'(r) \geq 0, \tag{5}$$

then the function f_1 is nonnegative.

Proof. Integrating (5) over the closed interval $[\epsilon; r]$ $(0 < \epsilon < r \leq a)$, after obvious transformations, we obtain

$$r f_1(r) - \epsilon f_1(\epsilon) - \int\limits_{\epsilon}^{r} f_1(\rho) d\rho + f_0(r) - f_0(\epsilon) \geq 0.$$

Then, as $\epsilon \to 0$, we obtain

$$r f_1(r) - \int\limits_{0}^{r} f_1(\rho) d\rho + f_0(r) \geq 0$$

and, finally, taking into account that f_0 is nonpositive, we arrive at the inequality

$$\frac{1}{r} \int\limits_{0}^{r} f_1(\rho)\, d\rho \leq f_1(r).$$

By the mean value theorem, for a certain $\xi \in [0; r)$, the latter inequality is rewritten as $f_1(\xi) \leq f_1(r)$. Let us show that the greatest lower bound (denoted by $\xi^\star$) of those ξ for which the latter inequality holds coincides with the point 0. On the contrary, replacing r by $\xi^\star$, we obtain

$$\frac{1}{\xi^\star} \int\limits_{0}^{\xi^\star} f_1(\rho)\, d\rho \leq f_1(\xi^\star),$$

which implies the existence of $\xi^0 \in [0; \xi^\star)$ such that $f_1(\xi^0) \leq f_1(\xi^\star) \leq f_1(r)$. The latter contradicts the definition of $\xi^\star$. Therefore, $\xi^\star = 0$, and hence $0 = f_1(0) \leq f_1(r)$ for all $r \in [0; a]$. □

Lemma 2. *Let $f_0, \ldots, f_n$ be functions continuous on $[0; a]$ and continuously differentiable on $(0; a]$ such that $f_i(0) = 0$, $i = 0, \ldots, n$. If the functions f_i are nonpositive and*

$$r^n f_n'(r) + r^{n-1} f_{n-1}'(r) + \cdots + f_0'(r) \geq 0, \tag{6}$$

then $f_i(r) \equiv 0$ for all i.

Proof. For $n = 0$, the assertion is trivial. For $n = 1$, it easily follows from the previous lemma. Let us show how we can perform the proof by induction.

Let us define the set of functions $\phi_0, \ldots, \phi_{n-1}$ in a recursive way setting

$$\phi_k = r\phi_{k-1} + f_{n-k} - \int\limits_{0}^{r} \phi_{k-1}(\rho) d\rho \tag{7}$$

for $k \geq 1$ and $\phi_0 = f_n$. Then inequality (6) can be transformed into the form $r\phi'_{n-1}(r) + f'_0(r) \geq 0$. Indeed,

$$\begin{gathered} r^n f'_n + r^{n-1} f'_{n-1} + \cdots + f'_0 = r(r(\ldots r(rf'_n + f'_{n-1}) + f'_{n-2}) + \cdots + f'_1) + f'_0 \\ = r(r(\ldots(r\phi'_1 + f'_{n-2}) + \cdots + f'_1) + f'_0 = r(r(\ldots(r\phi'_2 + f'_{n-3}) + \cdots + f'_1) + f'_0 \\ = \cdots = r\phi'_{n-1} + f'_0. \end{gathered}$$

Here, we repeatedly used formula (7). The same formula implies $\phi_i(0) = 0$ for all i. Using the fact that f_0 is nonpositive and Lemma 1, we obtain $\phi_{n-1}(r) \geq 0$ or

$$r\phi_{n-2}(r) + f_1(r) - \int\limits_0^r \phi_{n-2}(\rho) d\rho \geq 0$$

and, the more so,

$$\phi_{n-2}(r) \geq \frac{1}{r} \int\limits_0^r \phi_{n-2}(\rho)\, d\rho.$$

Arguing as in Lemma 1, we obtain from this that $\phi_{n-2}(r) \geq 0$. Continuing these constructions further, we conclude that $\phi_0(r) = f_n(r) \geq 0$. But, by condition, $f_n(r) \leq 0$. Hence $f_n(r) \equiv 0$, and inequality (6) reduces to

$$r^{n-1} f'_{n-1}(r) + \cdots + f'_0(r) \geq 0.$$

The induction step is made, and the further arguments are obvious. □

Proof of Theorem 1. On the contrary, there exists a positive number a such that the integral in the left-hand side of (4) is nonnegative for $r \in (0; a]$. But, by (3), this implies the inequality $r^d f'_d(r) + r^{d-1} f'_{d-1}(r) + \cdots + f'_0(r) \geq 0$ for $r \in (0; a]$, where

$$f_m(r) = \frac{1}{r^m} \int\limits_{S_r^m(X)} u$$

For $m = 0, 1, \ldots, d$. Without loss of generality, we can assume that $u(X) = 0$ at the maximum point. Then the functions f_m are nonpositive and are defined by zero for $r = 0$ by continuity. Therefore, all the conditions of Lemma 2 hold for them. Hence $f_m(r) \equiv 0$ for all m. This immediately implies $u \equiv 0$ in the ball $B_a(X)$, which contradicts the nontriviality of the maximum at the point X. □

Note that inessential modifications in our arguments lead to the following generalization of Theorem 1.

Theorem 2. *Let p be a nonnegative function constant on each stratum ("stratified" constant). Let $X \in \Omega_0$ be a nontrivial localmaximum point of a function $u \in C^1(\Omega_0)$. Then there exist arbitrarily small admissible $r > 0$ such that*

$$\int\limits_{S_r(X)} p \frac{\partial u}{\partial \nu} < 0 \tag{8}$$

under the assumption that p is positive on the so-called free strata (not adjacent to other strata by their interior points).

In the next section, this theorem is applied to the proof of the strong maximum principle for elliptic inequalities.

4. Divergence and Laplacian on Stratified Set

A vector field $\vec{F}$ on Ω_0 is said to be tangent to Ω_0 if for any stratum $\sigma_{kj} \subset \Omega_0$ and any point $X \in \sigma_{kj}$, the vector $\vec{F}(X)$ belong to the tangent space $T_X\sigma_{kj}$ of σ_{kj} at the point X; at zero-dimensional strata, it is natural to assume that $\vec{F} = 0$.

The divergence of the field $\vec{F}$ at an arbitrary point $X \in \Omega_0$ is defined by the relation

$$(\nabla\vec{F})(X) = \lim_{r\to 0} \frac{\Phi_{\vec{F}}(S_r(X))}{\mu(B_r(X))}, \tag{9}$$

where $\Phi_{\vec{F}}(S_r(X))$ denote the flow of the vector field $\vec{F}$ through the sphere of admissible radius. This flow is composed of flows through separate m-dimensional parts of the sphere ($m = 1, \dots, d$). The summation of these separate flows yields

$$\Phi_{\vec{F}}(S_r(X)) = \int\limits_{S_r(X)} \vec{F} \cdot \vec{\nu}\, d\mu,$$

where $\vec{\nu}$ is the exterior normal to $\hat{S}_r(X)$ and μ is the stratified measure on the sphere $S_r(X)$, which is stratified by the method shown in Sec. 2.

The set of differentiable vector fields (in the limits of each stratum) having divergence on Ω_0 is denoted by $\vec{C}^1(\Omega_0)$. The belonging of a field to $\vec{C}^1(\Omega_0)$ does not imply its continuity as a whole on Ω_0; i.e., a vector field on Ω_0 is a tuple of independent fields on separate strata. One can show that if $X \in \sigma_{k-1i}$, then

$$(\nabla\vec{F})(X) = (\nabla_{k-1}\vec{F})(X) + \sum_{\sigma_{kj}\succ\sigma_{k-1i}} \vec{\nu}\cdot\vec{F}\Big|_{\overline{kj}}(X), \tag{10}$$

where ∇_{k-1} is the classical divergence operator on σ_{k-1i} and the writing of the form $\sigma_{kj} \succ \sigma_{k-1i}$ means the adjustment of the stratum σ_{k-1i} to σ_{kj}. The vector $\vec{\nu}$ is the unit normal to σ_{k-1i} at the point X directed inside the stratum σ_{kj}. In what follows, for example, for a function $u : \Omega \to \mathbb{R}$, the notation $u\Big|_{\overline{kj}}(X)$ ($X \in \sigma_{k-1i} \succ \sigma_{kj}$) means the extension by continuity to the point X of the restriction $u\Big|_{\sigma_{kj}}$ of the function u to σ_{kj}. Of course, it is assumed that such an extension exists; this is the case where the above restriction is a function uniformly continuous on σ_{kj}. If u is not continuous as a whole on Ω, then, in general, $u\Big|_{\overline{kj}}(X) \neq u(X)$.

Let $u : \Omega_0 \to \mathbb{R}$ be continuously differentiable inside each stratum. Then we can consider the vector field ∇u whose restriction to each stratum σ_{ki} from Ω_0 coincides with the gradient field $\nabla_k u$. Note that we do not assume any connection between the restrictions of the function u to separate strata, and, therefore, ∇u is in fact a tuple of independent vector fields (one field on each of the strata). If ∇u belongs to the class $C^1(\Omega_0)$, then we can consider the operator $\Delta u = \nabla(\nabla u)$, which is naturally called the Laplace operator. As usual, we use the same symbol ∇ for the gradient and the divergence having agreed to treat it as the divergence when it is applied to a vector field and as the gradient when it is applied to a scalar-valued function. Everywhere in what follows u is assumed to be continuous on Ω_0. In this case, if $\nabla u \in C^1(\Omega_0)$, then we write $u \in C^2(\Omega_0)$.

Along with the operator Δ, we also consider the operators of the form Δ_p acting by the formula $\Delta_p u = \nabla(p\nabla u)$, where p is the stratified constant. If p assumes only zero and unit values, then the corresponding operator Δ_p can be considered as an analog of the Laplace operator. The simplest case is obtained when p is different from zero only in the so-called free strata (see the end of Sec. 3); these are, e.g., the strata of higher dimension. Such a Laplacian is said to be "soft" in contract to the "rigid" one corresponding to $p \equiv 1$ on the whole Ω_0. For the soft Laplacian, the proof of the strong maximum principle is not very difficult and was obtained long ago (see, e.g., [4] and the bibliography therein). The results of this works can be applied to all Δ_p. As in the presented theorem, we assume that p is positive on all free strata. On the other strata, the vanishing of p is admissible.

5. Strong Maximum Principle

On the basis of Theorem 2, we obtain a very simple proof of the strong maximum principle for solutions(subharmonic functions) of the inequality

$$\Delta_p u \geq 0 \tag{11}$$

on the stratified set. Even being applied to the classical case of the Laplacian in the domain, our proof turns out to be simpler than the well-known standard proofs.

Note that sometimes by the strong maximum principle one usually understands the assertion that a nonconstant solution of inequality (11) cannot attain the global maximum inside the domain. However, as is known, for solutions of the inequality $\Delta u \geq 0$, the strengthened maximum principle holds, which asserts that a solution of this inequality different from a constant cannot have local maxima inside the domain. In this work, by the strong maximum principle we mean an analog of the latter (strengthened) maximum principle. However, it does not hold in the presented formulation, since a solution of inequality (11) can be constant on the whole stratum (thus, it entirely consists of local maximum points if it is not adjacent to other strata by its interior points) and not constant everywhere on Ω_0. However, the following assertion holds.

Theorem 3. *A solution of inequality* (11) *cannot have nontrivial local maximum points in* Ω_0 *if* p *satisfies the conditions presented at the end of the previous section.*

Proof. For the operator Δ_p on the stratified set, an analog of the Green formula holds (see [4]), which implies that inequality (11) implies

$$\int\limits_{S_r(X)} p\frac{\partial u}{\partial \nu} \geq 0.$$

However, by Theorem 1, the latter inequality cannot hold at a nontrivial local maximum point of the function u for all sufficiently small $r > 0$. □

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REFERENCES

1. A. A. Gavrilov and O. M. Penkin, "Analog of lemma on normal derivative for an elliptic equation on a stratified set," *Differents. Uravn.*, **36**, No. 2, 226–232 (2000).
2. S. N. Oshchepkova and O. M. Penkin, "On a certain necessary extremum condition on a stratified set," *Dokl. Ross. Akad. Nauk*, **416**, No. 1, 22–25 (2007).
3. F. Pham, *Introduction a l'étude topologique des singularités de Landau*, Gauthier-Villars éditeur, Paris (1967).
4. Yu. V. Pokornyi, O. M. Penkin, V. L. Pryadiev et all, *Differential Equations on Geometric Graphs* [in Russian], Fizmatlit, Moscow (2004).
5. H. Seifert and W. Threlfall, *Topology* [Russian translation], GONTI, Moscow (1938).

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