A Necessary Condition for an Extremum on a Stratified Set

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In this paper, on a connected set composed of finitely many convex polyhedra of different dimensions, we introduce a special (stratified) measure and define a divergence operator (as the flow density of the tangent vector field) and a Laplacian on smooth functions. We also give an exact analogue of the strong maximum principle for an elliptic inequality with a Laplacian on a stratified set.

1. INTRODUCTION

If u is a smooth function on a domain $\Omega_0 \subset \mathbb{R}^n$ and X is a point of nontrivial local maximum of u (i.e., a point of local maximum having no neighborhood in which the function u is constant), then it is easy to show that there exists an arbitrarily small $r > 0$ for which the integral of the outer normal derivative of u over the sphere of radius r centered at X is negative. This necessary condition for an extremum immediately implies the strong maximum principle for solutions to the inequality $\Delta u \geq 0$.

In this paper, we prove a similar assertion for a smooth function on a stratified set. The argument is far from being trivial, but nevertheless, the proof remains within the framework of elementary mathematical analysis. As an application, we prove the strong maximum principle for an elliptic inequality on a stratified set. The two-dimensional case of the last result was announced for long ago [1], but the general case has not been proved (partial results were published in [2, 3]).

2. BASIC DEFINITIONS

A general definition of a stratified set Ω suitable for setting boundary value problems can be found in [4, 5]. In this paper, we consider a narrower class. Namely, we assume that $\Omega \subset \mathbb{R}^n$ is a connected set consisting of finitely many open (in the ambient planes of minimum dimensions) convex polyhedra (strata) σ_{kj} (k is the dimension of the stratum and j is its number in an autonomous numeration of strata of each dimension) adjacent to each other as in a simplicial complex; i.e., we assume that the boundary $\partial\sigma_{kj}$ consists of strata of lower dimensions and any two strata either are disjoint or intersect in a common face.

In Ω, consider an open connected (in the topology induced on Ω from $\mathbb{R}^n$) subset Ω_0 consisting of strata of Ω and such that $\overline{\Omega}_0 = \Omega$. The difference $\Omega\backslash\Omega_0$ is the topological boundary of Ω_0; thus, we denote it by $\partial\Omega_0$. Possibly, $\partial\Omega_0 = \emptyset$.

For our purposes in this paper, it is convenient to introduce the so-called stratified measure on Ω. For $\omega \subset \Omega$, it is defined by

$$\mu(\omega) = \sum_{\sigma_{kj}} \mu_k(\omega \cap \sigma_{kj}), \qquad (1)$$

where μ_k is the standard k-dimensional Lebesgue measure on the stratum σ_{kj}. The sets measurable with respect to this measure are those whose intersections with all strata are Lebesgue measurable on these strata. The measure of the zero-dimensional stratum is defined to be 1. The Lebesgue integral of a measurable function $f: \Omega \to \mathbb{R}$ with respect to this measure is the sum

$$\int_\Omega f\, d\mu = \sum_{\sigma_{kj}} \int_{\sigma_{kj}} f\, d\mu_k$$

of the Lebesgue integrals over separate strata. In what follows, we largely omit $d\mu$ and $d\mu_k$ in the notation of integrals.

3. AN AUXILIARY FORMULA

Let $X \in \Omega$. If $r > 0$ is less than the distance to all strata not containing X in their closures (such a number r is said to be admissible), then the set

$$S_r(X) = \{Y \in \Omega: \|Y - X\| = r\}$$

is called a sphere. It coincides with the intersection of the sphere $\hat{S}_r(X)$ in the ambient space with Ω. We

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denote the corresponding open ball by $B_r(X)$. The union of the intersections of the sphere $S_r(X)$ with all $(m+1)$-dimensional strata is called the m-dimensional fragment of this sphere and denoted by $S_r^m(X)$. The sphere can be regarded as a stratified set as well; it is natural to define its m-strata to be the connected components of the set $S_r^m(X)$. Integrals over a sphere are calculated with respect to the stratified measure introduced above. For admissible r_1 and r_2, the spheres $S_{r_1}(X)$ and $S_{r_2}(X)$, as well as their fragments $S_{r_1}^m(X)$ and $S_{r_2}^m(X)$, are similar. This easily implies the formula

$$\frac{d}{dr}\left(\frac{1}{r^m}\int\limits_{S_r^m(X)} u\right) = \frac{1}{r^m}\int\limits_{S_r^m(X)} \frac{\partial u}{\partial \nu}, \qquad (2)$$

where ν is the outer normal to the sphere $\hat{S}_r(X)$ at a point of the sphere $S_r(X)$. Here, it is assumed that $X \in \Omega_0$ and the function u is globally continuous on Ω_0 (this condition is not used in this section, but we need it in what follows) and differentiable inside each stratum $\sigma_{kj} \subset \Omega_0$, and the integrals on the right-hand side of (2) converge for all m. We denote the set of all such functions by $C^1(\Omega_0)$.

Multiplying both sides of (2) by r^m and taking the sum over all m, we obtain

$$\int\limits_{S_r(X)} \frac{\partial u}{\partial \nu} = \sum_{m=0}^{d} r^m \frac{d}{dr}\left(\frac{1}{r^m}\int\limits_{S_r^m(X)} u\right). \qquad (3)$$

A stratified sphere not necessarily contains fragments of all dimensions, and formally, the right-hand side may include redundant terms. To avoid confusion, we assume that the integrals over empty $S_r^m(X)$ vanish.

4. A NECESSARY CONDITION FOR AN EXTREMUM

In this section, we give the main result.

Theorem 1. *Let $X \in \Omega_0$ be a point of nontrivial maximum of a function $u \in C^1(\Omega_0)$.*

Then, there exists an arbitrarily small admissible $r > 0$ such that

$$\int\limits_{S_r(X)} \frac{\partial u}{\partial \nu} < 0. \qquad (4)$$

Recall that we call X a point of nontrivial maximum of the function u if, for points Y close to it, $u(Y) \le u(X)$ and the function X is constant in no neighborhood of X.

The proof is based on the following two lemmas.

Lemma 1. *Let f_0 and f_1 be continuous functions on $[0; a]$ continuously differentiable on $(0; a]$ and such that $f_0(0) = f_1(0) = 0$. If the function f_0 is nonpositive and*

$$rf_1'(r) + f_0'(r) \ge 0, \qquad (5)$$

then the function f_1 is nonnegative.

Lemma 2. *Let $f_0, f_1, \ldots, f_n$ be continuous functions on $[0; a]$ continuously differentiable on $(0; a]$ and such that $f_i(0) = 0$ for $i = 0, 1, \ldots, n$. If the functions f_i are nonpositive and*

$$r^n f_n'(r) + r^{n-1} f_{n-1}'(r) + \ldots + f_0'(r) \ge 0, \qquad (6)$$

then $f_i(r) \equiv 0$ for all i.

These lemmas easily imply the above theorem, because assuming the opposite, we can find a positive number a such that the integral on the left-hand side of (4) is nonnegative for $r \in (0; a]$. By virtue of (3), this implies $r^d f_d'(r) + r^{d-1} f_{d-1}'(r) + \ldots + f_0'(r) \ge 0$ for $r \in (0; a]$, where

$$f_m(r) = \frac{1}{r^m}\int\limits_{S_r^m(X)} u$$

for $m = 0, 1, \ldots, d$. Without loss of generality, we can assume that $u(X) = 0$ at the point of maximum. Then, the functions f_m are nonpositive and, by continuity, can be set to zero at $r = 0$. Thus, they satisfy all conditions of Lemma 2. Therefore, $f_m(r) \equiv 0$ for all m. This immediately implies that $u \equiv 0$ on the ball $B_a(X)$, which contradicts the nontriviality of the maximum at the point X.

The above theorem plays the key role in the proof of the strong maximum principle for elliptic inequalities on stratified sets, which is discussed later on.

5. DIVERGENCE OPERATORS AND LAPLACIANS ON STRATIFIED SETS

We say that a vector field $\vec{F}$ on Ω_0 is tangent to Ω_0 if, for any stratum $\sigma_{kj} \subset \Omega_0$ and any point $X \in \sigma_{kj}$, the vector $\vec{F}(X)$ belongs to the space σ_{kj} tangent to $T_X\sigma_{kj}$ at the point X; it is natural to assume that $\vec{F} = 0$ on zero-dimensional strata.

The divergence of the field $\vec{F}$ at a point $X \in \Omega_0$ is defined by the relation

$$(\nabla\vec{F})(X) = \lim_{S \to X} \frac{\Phi_{\vec{F}}(S)}{\mu(B)}, \qquad (7)$$

where $\Phi_{\vec{F}}(S)$ denotes the flow of the vector field $\vec{F}$ through the "stratified" surface S obtained as the intersection of some smooth surface $\hat{S}$ in the ambient space $\mathbb{R}^n$ with Ω and B is the part of Ω contained inside $\hat{S}$. We assume that S is contained in some ball $B_r(X)$ of admissible radius and the normal to $\hat{S}$ at the point $X \in \sigma_{kj} \cap S$ belongs to the space tangent to σ_{kj} for all k and j; for example, $S = S_r(X)$ satisfies this condition. The flow $\Phi_{\vec{F}}(S)$ is composed of flows through the separate m-fragments S_m ($m = 1, 2, \ldots, d$) of the surface S. The summation of these separate flows yields

$$\Phi_{\vec{F}}(S) = \int_S \vec{F} \cdot \vec{\nu}\, d\mu,$$

where $\vec{\nu}$ is the outer normal to S with the properties specified above and μ is the stratified measure on the surface S treated as a stratified set; stratification is performed in the same way as in Section 3 (for spheres).

We denote the set of tangent vector fields continuous on each stratum and having a divergence on Ω_0 by $\vec{C}^1(\Omega_0)$. A field belonging to $\vec{C}^1(\Omega_0)$ may not be globally continuous on Ω_0; i.e., a vector field on Ω_0 is a set of independent fields on separate strata. It can be shown (see [6]) that if $X \in \sigma_{k-1i}$, then

$$(\nabla\vec{F})(X) = (\nabla_{k-1}\vec{F})(X) + \sum_{\sigma_{kj} > \sigma_{k-1i}} \vec{\nu} \cdot \vec{F}\big|_{\overline{kj}}(X), \quad (8)$$

where ∇_{k-1} is the classical divergence operator on σ_{k-1i} and an expression of the form $\sigma_{kj} > \sigma_{k-1i}$ means that the stratum σ_{k-1i} is adjacent to σ_{kj}. The vector $\vec{\nu}$ is the unit normal to σ_{k-1i} at the point X directed inward the stratum σ_{kj}. In what follows, for a function, say, u: $\Omega \to \mathbb{R}$, a notation of the form $u|_{\overline{kj}}(X)$ (where $X \in \sigma_{k-1i} > \sigma_{kj}$) is used for the continuous extension to the point X of the restriction $u|_{\sigma_{kj}}$ of the function u to σ_{kj}. Certainly, it is assumed that such an extension exists. This is so if, e.g., the restriction is uniformly continuous on σ_{kj}. If u is not globally continuous on Ω, then, generally, $u|_{\overline{kj}}(X) \neq u(X)$.

Suppose that a function u: $\Omega_0 \to \mathbb{R}$ is differentiable inside each stratum. Then, we can consider the vector field ∇u, whose restriction to each stratum σ_{ki} of Ω_0 coincides with the field of the gradient $\nabla_k u$. Note that we do not assume the existence of any relations between the restrictions of u to separate strata; thus, in fact, ∇u is a set of independent vector fields (one field on each stratum). If ∇u belongs to the class $C^1(\Omega_0)$, then we can consider the operator $\Delta u = \nabla(\nabla u)$, which is natural to call the Laplace operator. As usual, we use the same symbol ∇ for the gradient and the divergence, interpreting it as the divergence if it is applied to a vector field and as the gradient if it is applied to a scalar function. In the rest of the paper, u is assumed to be continuous on Ω_0. If, in addition, $\nabla u \in \vec{C}^1(\Omega_0)$, then we write $u \in C^2(\Omega_0)$.

Together with the operator Δ, we consider the operators Δ_p acting as $\Delta_p u = \nabla(p\nabla u)$, where p is the so-called stratified constant (a function constant on each separate stratum). In this paper, we assume that p equals 0 or 1 on each stratum. This is needed to simplify the statements of results; in the general case, proofs are similar. All such operators can be considered as analogues of the Laplace operator. In the simplest case, p is nonzero only on the so-called free strata (that is, the strata not contained in the boundaries of other strata); examples of free strata are the strata of highest dimension. We refer to the corresponding Laplacian as the soft Laplacian, as opposed to the rigid Laplacian, which corresponds to $p \equiv 1$ on the entire set Ω_0. The results obtained in this paper apply to these two extreme cases and all cases between them. It is possible to consider also the case where $p = 0$ on some free strata, but this case turns out to be fairly meaningless.

6. THE STRONG MAXIMUM PRINCIPLE

First, note that, unlike for the classical strong maximum principle, the inequality $\Delta u \geq 0$ on a stratified set may admit nonconstant solutions having local maxima. Nevertheless, the following exact analogue of the strong maximum principle is valid.

Theorem 2. *Let $u \in C^2(\Omega_0)$ be a solution to the inequality $\Delta u \geq 0$ on Ω_0. Then, u may have no points of nontrivial local maximum in Ω_0.*

This theorem is easily derived from the fact that, as well as in the classical case, the inequality $\Delta u \geq 0$ implies the nonnegativity of the integrals of the normal derivative over the admissible spheres. As mentioned above, the integral of the normal derivative over a sphere can be represented in the form $r^n f_n'(r) + r^{n-1} f_{n-1}'(r) + \ldots + f_0'(r)$. As a result, the inequality $\Delta u \geq 0$ implies (6), which contradicts (if X is a point of nontrivial maximum) Theorem 1. The proof that the inequality $\Delta u \geq 0$ implies the nonnegativity of the integrals of the normal derivative over the admissible spheres is based on the following analogue of Green's formula.

Theorem 3. *If u, $v \in C^2(\Omega_0)$, then, on the balls of admissible radius,*

$$\int_{B_r(X)} (u\Delta v - v\Delta u)\,d\mu = \int_{S_r(X)} (u(\nabla v)_\nu - v(\nabla u)_\nu)\,d\mu.$$

This theorem is a special case of the corresponding theorem from [6].

It should be mentioned that, if $\partial\Omega_0 = \emptyset$, then the inequality admits only constant solutions (see [6]), and Theorem 2 is trivial in this case. It is also clear that this theorem implies the absence of positive nontrivial maxima for the solutions to the inequality $\Delta u - qu \geq 0$ provided that the function q is nonnegative. This function is also assumed to be continuous on each stratum of Ω_0.

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