WAVE SCATTERING BY DEFECTS IN MEDIA WITH SPATIAL DISPERSION AND NONRADIATIVE DYNAMIC SOLITONS

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Specific features of dynamic solitons in a nonlinear system described by a differential equation with a fourth-order spatial derivative are discussed. Solutions of the linearized equation and the problem of scattering of a double-partial wave by a point defect are analyzed. Conditions of existence of a nonradiative soliton are formulated and demonstrated in the case in which the natural soliton frequency falls within the continuous spectrum of harmonic oscillations of the examined system. These conditions are determined by the dispersion law of linear oscillations.

1. We discuss the properties of dynamic solitons in systems from the viewpoint of their linear dynamics. By the dynamic soliton is meant a nonlinear spatially localized disturbance of the examined system whose stability is ensured by the presence of simple additive integrals of motion. Examples of such solitons are provided by the soliton of a nonlinear Schrödinger equation (NSE) or by the magnetic soliton [1]. The dynamic soliton is commonly characterized by its natural frequency whose value is one of the main soliton parameters. Traditionally, the dynamic solitons in the physics of condensed matter were studied based on nonlinear differential equations with second-order spatial derivatives. For a scalar field, the dynamic functional has frequencies of local states that necessarily fall beyond the continuous spectrum. Therefore, the natural frequency of the dynamic soliton of this physical field, by analogy with the frequency of any localized excitation, must lie outside the spectrum (for the NSE or the magnetic soliton, it must lie below the continuous spectrum).

The situation changes in the study of the dynamics of a discrete or continuous system described by differential equations with higher-order (compared to the second order) spatial derivatives. In this case, the soliton frequency may fall within the continuous spectrum of harmonic oscillations, and problems of soliton existence arise in the presence of linear wave emission and radiative soliton interaction (for example, see [2, 3]).

Here, we want to demonstrate that the majority of accompanying effects are determined by the properties of the linearized harmonic oscillation equations for the examined system. These equations describe field asymptotes at large distances from the soliton and contain abundant information on soliton solutions of nonlinear equations. In this connection, we will analyze solutions of the linearized equation expressed in terms of the corresponding Green's functions.

We restrict our consideration to a simple nonlinear model:

$$i\frac{\partial\psi}{\partial t} = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^4\psi}{\partial x^4} + F(\psi), \qquad (1)$$

$$F(\psi) = \psi|\psi|^2 + \gamma_1\psi\left|\frac{\partial\psi}{\partial x}\right| + \gamma_2\psi\frac{\partial^2\psi}{\partial x^2} - \gamma_3\psi|\psi|^4$$

that considers the highest-order dispersion and allows us to describe various situations arising in the soliton dynamics. We are interested only in stationary states $\psi(x,t) = \psi(x)\exp(-i\omega t)$, that is, in the properties of the nonlinear equation

$$\omega\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^4\psi}{\partial x^4} + F(\psi), \qquad (2)$$

where ω is the excitation frequency.

2. We first analyze solutions of linearized equation (2) without the term $F(\psi)$. The law of dispersion (the dependence of the frequency ω on the wave number *k*) of stationary states $\psi(x) = \psi_0 \exp(ikx)$ of linearized equation (2) has the form

$$\omega(k) = -k^2 + k^4 . \qquad (3)$$

At the point where $k^2 = k_m^2 = 1/2$, the frequency ω reaches its minimum $\omega_m = -1/4$. Therefore, the continuous spectrum of natural frequencies of harmonic oscillations lies in the frequency range $\omega_m < \omega < \infty$.

The Green's function of linearized equation (2) can be easily calculated

1) for frequencies $\omega > 0$:

$$G_\omega(x) = \frac{1}{2(k^2 + \kappa^2)} \left\{ \frac{ie^{ik|x|}}{k} + \frac{e^{-\kappa|x|}}{\kappa} \right\}, \qquad (4)$$

where

$$k^2 = \frac{1}{2}\left\{\sqrt{1+4\omega} + 1\right\}, \; \kappa^2 = \frac{1}{2}\left\{\sqrt{1+4\omega} - 1\right\}; \qquad (5)$$

2) for frequencies $\omega_m < \omega < 0$:

$$G_\omega(x) = \frac{1}{2i(k_1^2 - k_2^2)} \left\{ \frac{e^{ik_1|x|}}{k_1} - \frac{e^{ik_2|x|}}{k_2} \right\}, \qquad (6)$$

where

$$k_{1,2}^2 = \frac{1}{2}\left\{1 \pm \sqrt{1+4\omega}\right\}; \qquad (7)$$

3) for frequencies $\omega < \omega_m$:

$$G_\omega(x) = \frac{e^{-\gamma|x|}\sin(q|x| - \varphi)}{4q\gamma\sqrt{q^2+\gamma^2}}, \qquad (8)$$

where

$$q^2 = \frac{1}{2}\left\{\sqrt{|\omega|} + \frac{1}{2}\right\}, \; \gamma^2 = \frac{1}{2}\left\{\sqrt{|\omega|} - \frac{1}{2}\right\}, \; \tan\varphi = \gamma/q . \qquad (9)$$

An analysis of the Green's functions given by Eqs. (4), (6), and (8) irrespective of the structure of the nonlinear term entering into Eq. (2) allows us to conclude the following on soliton solutions of equations of this type.

1) The dynamic solitons with frequencies $\omega < \omega_m$ have exponentially decaying oscillating tails, and if they are exact solutions of Eq. (1), they cannot emit linear waves. The presence of nonmonotonically decaying tails is essential to the study of soliton interaction. This was first discussed in [4].

2) In principle, soliton solutions cannot be obtained for frequencies $\omega_m < \omega < 0$, because the corresponding stationary states have no decaying asymptotes.

3) A typical linear stationary state for frequencies $\omega > 0$ is a quasi-localized oscillation whose first component is localized in space, whereas the second component represents a standing wave with a constant amplitude [5, 6]. The presence of these asymptotes means that the soliton solutions of Eq. (1) with frequencies $\omega > 0$, as a rule, are accompanied by emissions of harmonic waves. There are various explanations of the physical reasons for and the conditions of existence of nonradiative solitons in this frequency range.

3. Solutions of the inhomogeneous equation

$$\omega\psi - \frac{\partial^2\psi}{\partial x^2} - \frac{\partial^4\psi}{\partial x^4} = f(x), \tag{10}$$

where $f(x)$ is a distributed force, can be written in terms of the Green's functions. A solution of Eq. (10) has the form [7]

$$\psi(x) = \int G_\omega(x - x')\, f(x')\, dx'. \tag{11}$$

Let a short-range force (simulating a point defect)

$$f(x) = U_0\delta(x)\psi_0 \tag{12}$$

be considered at first. For frequencies $\omega < \omega_m$ lying below the continuous spectrum, there exists a state localized near the defect whose wave function

$$\psi(x) = \frac{\psi_0 U_0 e^{-\gamma|x|}}{4q\gamma\sqrt{q^2+\gamma^2}} \sin(q|x| - \varphi) \tag{13}$$

is obtained from Eqs. (11) and (12) and whose Green's function is given by Eq. (8). Solution (13) describing a generalized local state was discussed in [8, 9]. The discrete frequency of local state (13) is specified by the relation

$$U_0 = -4\gamma(\gamma^2 + q^2) \tag{14}$$

and exists only for $U_0 < 0$.

4. Let us now analyze the problem of scattering of waves with frequencies $\omega > 0$ by point defect (12). In this case, the wave function can be represented in the form (see [7])

$$\psi(x) = \psi_0 e^{ikx} + \frac{U_0}{D(\omega)} G_\omega(x), \tag{15}$$

where $D(\omega) = 1 - U_0 G_\omega(0)$. Taking advantage of the explicit form of the Green's function given by Eq. (4), we can write a solution of the scattering problem in the form

$$\psi(x) = \begin{cases} \psi_0 e^{ikx} + R(\omega)e^{-ikx} + M(\omega)e^{\kappa x}, & x < 0, \\ T(\omega)e^{ikx} + M(\omega)e^{-\kappa x}, & x > 0, \end{cases} \tag{16}$$

where k and κ are specified by Eqs. (5). Then the amplitudes of solution (16) are

$$R(\omega) = \frac{i\kappa U_0\psi_0}{\Delta(\omega)}, \tag{17}$$

$$T(\omega) = \frac{k\{2\kappa(\kappa^2 + k^2) - U_0\}\psi_0}{\Delta(\omega)}, \tag{18}$$

$$M(\omega) = \frac{kU_0\psi_0}{\Delta(\omega)},\ \Delta(\omega) = k\{2\kappa(\kappa^2 + k^2) - U_0\} - i\kappa U_0, \tag{19}$$

from which we can derive the reflection and transmission coefficients:

$$r(\omega) = \frac{|R(\omega)|^2}{\psi_0^2} = \frac{\kappa^2 U_0^2}{\kappa^2 U_0^2 + k^2\{2\kappa(\kappa^2 + k^2) - U_0\}^2}, \tag{20}$$

$$t(\omega) = \frac{|T(\omega)|^2}{\psi_0^2} = \frac{k^2\{2\kappa(\kappa^2 + k^2) - U_0\}^2}{\kappa^2 U_0^2 + k^2\{2\kappa(\kappa^2 + k^2) - U_0\}^2}. \tag{21}$$

Curiously, the resonant wave reflection may occur when

$$U_0 = 2\kappa(k^2 + \kappa^2), \tag{22}$$

given that $r(\omega) = 1$ and $t(\omega) = 0$. In this case, the asymmetric stationary state $\psi(x) = \psi_S(x) + \psi_L(x)$ appears, which represents superposition of the standing wave

$$\psi_S(x) = \begin{cases} 2i\psi_0 \sin kx, & x < 0, \\ 0, & x > 0, \end{cases} \tag{23}$$

existing only in a half-space, and the state

$$\psi_L(x) = M(\omega)\exp(-\kappa|x|) \tag{24}$$

localized near the defect. Frequencies of this asymmetric state are specified by Eq. (22) and make sense only for $U_0 > 0$. Analogous peculiarities of wave scattering by defects were discussed recently in [10–13].

5. To identify the conditions at which emission at frequencies $\omega > 0$ is absent, we now consider a solution of linearized equation (10) with the driving distributed force $f(x)$ localized in a small interval x around $x = 0$. Let $f(x)$ decrease with distance faster than $\exp(-\kappa|x|)$. Then, according to Eq. (11), the solution beyond the region of force application is

$$\psi(x) = \frac{1}{2(k^2 + \kappa^2)}\left\{iB(\pm k)e^{ik|x|} + L(\pm\kappa)e^{-\kappa|x|}\right\}, \tag{25}$$

where the upper sign is for $x > 0$ and the lower sign is for $x < 0$,

$$B(k) = \frac{1}{k}\int f(x)e^{-ikx}dx, \quad L(\kappa) = \frac{1}{\kappa}\int f(x)e^{\kappa x}dx. \tag{26}$$

If for a certain value $k = p$ the Fourier component of the distributed force vanishes, that is, $B(p) = 0$, the forced oscillation with frequency $\omega = \omega(p)$ becomes nonradiative at infinity. The state then becomes localized:

$$\psi(x) = \frac{L(\pm\kappa)e^{-\kappa|x|}}{2(k^2 + \kappa^2)}. \tag{27}$$

Linear oscillations at large distances from the source are completely quenched at the expense of wave interference [2, 3]. For the symmetric force distribution $f(x) = f(-x)$, this condition has the form

$$\int f(x)\cos kx dx = 0, \tag{28}$$

whereas for the antisymmetric distribution $f(-x) = -f(x)$

$$\int f(x)\sin kx dx = 0. \tag{29}$$

However, it must be remembered that since the frequencies of this state fall within the quasi-continuous spectrum, in the linear dynamics its weight is very small, of the order of $1/\sqrt{N}$, where N is the number of atoms in the examined 1D chain. However, it is well known that the weight of these isolated stationary states may be radically different in the nonlinear dynamics (see [1, 7]).

6. Obvious mathematical conditions (28) or (29) of solvability of the inhomogeneous differential equation allow us to obtain the parameters of the stationary dynamic nonradiative solitons.

Analyzing the structure of Eq. (2), we expect that it admits a soliton solution of the form

$$\psi(x) = \frac{A}{\cosh \kappa x}, \tag{30}$$

where A = const and κ is related to the frequency ω by Eq. (5). Substituting solution (30) into the nonlinear term $F(\psi(x))$ of Eq. (2), we can consider it as an external force $f(x) = f(-x)$. According to Eq. (28), soliton (30) will be nonradiative when

$$\int F(\psi(x)) \cos kx dx = 0 . \tag{31}$$

Not to crowd our calculations with unwieldy formulas, we perform them separately for the following cases: 1) $\gamma_1 \neq 0$ and $\gamma_2 = \gamma_3 = 0$, 2) $\gamma_2 \neq 0$ and $\gamma_1 = \gamma_3 = 0$, and 3) $\gamma_3 \neq 0$ and $\gamma_1 = \gamma_2 = 0$.

In the first case, we derive the parameters of soliton (30) from Eqs. (31) and (5) in the form

$$k^2 = \frac{3(\gamma_1 - 4)}{2\gamma_1}, \quad \kappa^2 = \frac{\gamma_1 - 12}{2\gamma_1}, \quad \omega = \left(\frac{6 - \gamma_1}{\gamma_1} \right)^2 - \frac{1}{4} . \tag{32}$$

We see that there always exists soliton (30) nonradiating linear waves for $\gamma_1 > 12$. All the parameters of this soliton, except the amplitude A, can be derived from an analysis of the linearized equation. It is easy to verify that the exact solution of Eq. (2) corresponds to

$$A^2 = \frac{24}{\gamma_1} \kappa^2 = \frac{12(\gamma_1 - 12)}{\gamma_1^2} . \tag{33}$$

In the second case ($\gamma_2 \neq 0$ and $\gamma_1 = \gamma_3 = 0$), we have

$$k^2 = \frac{3(\gamma_2 - 4)}{2\gamma_2}, \quad \kappa^2 = \frac{\gamma_2 - 12}{2\gamma_2}, \quad \omega = \frac{3(\gamma_2 - 4)(\gamma_2 - 12)}{4\gamma_2^2} . \tag{34}$$

From Eq. (2) we obtain the amplitude:

$$A^2 = \frac{24}{\gamma_2} \kappa^2 = \frac{12(\gamma_2 - 12)}{\gamma_2^2} . \tag{35}$$

Thus, soliton (30) with parameters (34) and (35) exists for a system with $\gamma_2 > 12$.

Finally, in the case in which $\gamma_3 \neq 0$ and $\gamma_1 = \gamma_2 = 0$, from Eqs. (31) and (5) it follows that

$$k^2 = \frac{3(4 - 3\gamma_3)}{2(6 - 5\gamma_3)}, \quad \kappa^2 = \frac{\gamma_3}{2(6 - 5\gamma_3)}, \quad \omega = \frac{3\gamma_3(4 - 3\gamma_3)}{4(6 - 5\gamma_3)^2} . \tag{36}$$

The soliton amplitude

$$A^2 = \frac{24}{\gamma_3} \kappa^2 = \frac{12}{6 - 5\gamma_3} \tag{37}$$

is derived from Eq. (2). It is interesting that the soliton exists only when $0 < \gamma_3 < 6/5$.

We now apply the above considerations to an analysis of a more complex soliton solution of simplified equation (2) for $\gamma_1 = \gamma_2 = \gamma_3 = 0$. In this case, we expect the presence of the soliton

$$\psi(x) = A\frac{\sinh \kappa x}{\cosh^2 \kappa x}. \quad (38)$$

Because in this case $\psi(-x) = -\psi(x)$, condition (29) must be used:

$$\int \left(\frac{\sinh \kappa x}{\cosh^2 \kappa x} \right)^3 \sin kx dx = 0. \quad (39)$$

From condition (39) we obtain $k^2 = 11\kappa^2$ and, taking advantage of Eq. (5), we find

$$\kappa^2 = 0.1, \ \ \omega = 0.11. \quad (40)$$

These parameters correspond to unique nonradiative solution (38). For the exact solution obtained in [14], the amplitude was $A = \sqrt{6/5}$.

It is clear that the above-formulated conditions of existence of a nonradiative dynamic soliton and their demonstration for some particular cases provide no recomendations for a search for an analytical soliton solution. However, they demonstrate that the structure of the dynamic soliton is determined by the dispersion law of the linearized equation no less than by the form of nonlinear terms entering into the dynamic equations.

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