DOUBLE POROSITY MODELS FOR LIQUID FILTRATION IN INCOMPRESSIBLE POROELASTIC MEDIA

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Double porosity models for the liquid filtration in a naturally fractured reservoir is derived from the homogenization theory. The governing equations on the microscopic level consist of the stationary Stokes system for an incompressible viscous fluid, occupying a crack-pore space (liquid domain), and stationary Lame equations for an incompressible elastic solid skeleton, coupled with the corresponding boundary conditions on the common boundary "solid skeleton-liquid domain". We assume that the liquid domain is a union of two independent systems of cracks (fissures) and pores, and that the dimensionless size δ of pores depends on the dimensionless size ε of cracks: $\delta = \varepsilon^r$ with $r > 1$. The rigorous justification is fulfilled for homogenization procedure as the dimensionless size of the cracks tends to zero, while the solid body is geometrically periodic. As the result we derive the well-known Biot–Terzaghi system of liquid filtration in poroelastic media, which consists of the usual Darcy law for the liquid in cracks coupled with anisotropic Lame's equation for the common displacements in the solid skeleton and in the liquid in pores and a continuity equation for the velocity of a mixture. The proofs are based on the method of reiterated homogenization, suggested by Allaire and Briane. As a consequence of the main result we derive the double porosity model for the filtration of the incompressible liquid in an absolutely rigid body.

1. Introduction

The liquid motion in a naturally fractured reservoir is described by different mathematical models. These models take into account a geometry of a space, occupied by the liquid (liquid domain), and physical properties of the liquid and the solid skeleton. Among different models, the simplest one is Darcy equations

$$\mathbf{v} = -k\nabla q + \mathbf{F}, \quad \nabla \cdot \mathbf{v} = 0, \tag{1.1}$$

for the macroscopic velocity $\mathbf{v}$ and the pressure q of the liquid, when the solid skeleton is assumed to be an absolutely rigid body and the liquid domain is a pore space.

For more complicated geometry, when the liquid domain is a union of system of pores and cracks, there are different types of models (see, for example, Refs. 4, 11, 20 and 22). Note that pores differ from cracks by its characteristic size: if l_p is a characteristic size of pores and l_c is a characteristic size of cracks, then $l_p \ll l_c$. The well-known double-porosity model, suggested by Barenblatt, Zheltov and Kochina,[4] describes two-velocity continuum where macroscopic velocity $\mathbf{v}_p$ and pressure q_p in pores and macroscopic velocity $\mathbf{v}_c$ and pressure q_c in cracks satisfy two different Darcy laws

$$\mathbf{v}_p = -k_p \nabla q_p + \mathbf{F}, \quad \mathbf{v}_c = -k_c \nabla q_c + \mathbf{F}, \tag{1.2}$$

and two continuity equations

$$\nabla \cdot \mathbf{v}_p = J, \quad \nabla \cdot \mathbf{v}_c = -J. \tag{1.3}$$

The model is completed by postulating that the overflow J from pores to cracks linearly depends on the difference $(q_c - q_p)$.

Scientific and practical value of mathematical models describing such complicated processes is obvious. But their physical reliability is also very important. Namely, we say, that the given phenomenological model is physically correct, if it is one of basic model of continuum mechanics (as, for example, Stokes equations describing a slow motion of a viscous liquid, or Lame's equations describing a motion of an elastic solid body) or asymptotically closed to some physically correct phenomenological model on the microscopic level (that is a model, obtained by homogenization of some model on the microscopic level, depending on the small parameter). In view of the importance of models for the liquid motion in a naturally fractured reservoir, it is very natural to show that they are physically correct. That is, rigorously derive the governing equations for each model, starting with detailed microstructure of the liquid domain and the linearized equations of fluid and solid dynamics on the microscopic level. In their fundamental paper Burridge and Keller[8] have used this scheme to justify a well-known in contemporary acoustics and filtration phenomenological model of poroelasticity, suggested by Biot.[5] As a model of the porous medium on the microscopic level the authors have considered the mathematical model, consisting of Stokes equations describing liquid motion in pores and cracks, and Lame's equations, describing motion of a solid skeleton. The differential equations in the solid skeleton and in the liquid domain are completed by boundary conditions on the common boundary "liquid domain — solid skeleton", which express a continuity of displacements and normal tensions. The suggested microscopic model is a physically correct one, because it follows from basic laws of continuum mechanics (see also Sanchez-Palencia[19]). After scaling there appears a natural small parameter δ which is the pore characteristic size l_p divided by the characteristic size L of the entire porous body: $\delta = l_p/L$. The small parameter enters both into coefficients of the differential equations, and in the geometry of the domain in consideration. The homogenization (that is a finding of all limiting regimes as $\delta \searrow 0$) of this model is a model, asymptotically closed to the basic model. But even this approach is too difficult to be realized, and some additional simplifying assumptions are necessary. In terms of geometrical

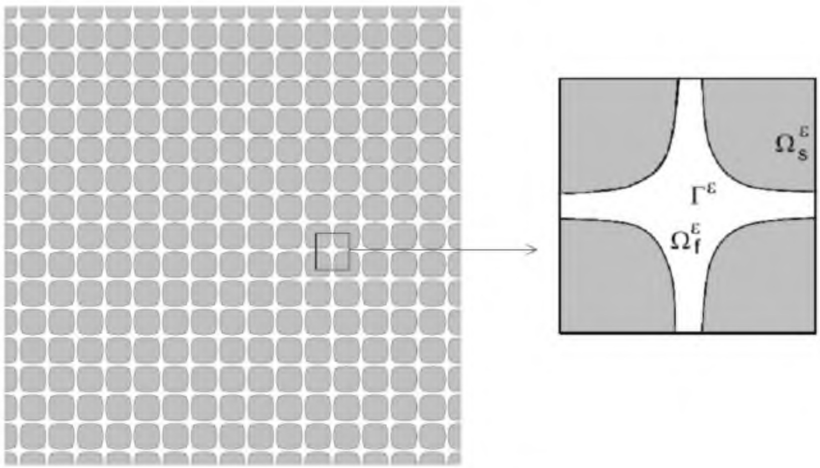

Fig. 1. Single porosity geometry.

properties of the medium, it is most expedient to simplify the problem by postulating that the porous structure is periodic with the period δ. Under this assumption Burridge and Keller, using a method of two-scale asymptotic expansion, have formally justified Biot's model. For the same geometry of the pore space (let us call such a model as a *single porosity model*, see Fig. 1) and for absolutely rigid solid skeleton when a liquid motion is described by the Stokes system, Tartar have rigorously justified the Darcy law of filtration (see Appendix in Ref. 19). Later a rigorous justification of Biot's models, under the same assumptions on the geometry of a pore space as in Ref. 8, has been rigorously proved in Refs. 13–16 and 18.

For more complicated geometry, when the liquid domain is a crack-pore space (let us call such a geometry as a *double porosity geometry* and corresponding

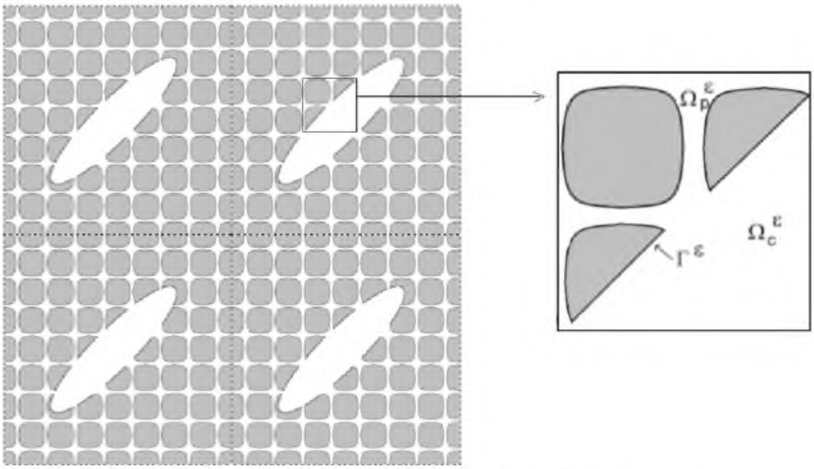

Fig. 2. Double porosity geometry: isolated cracks.

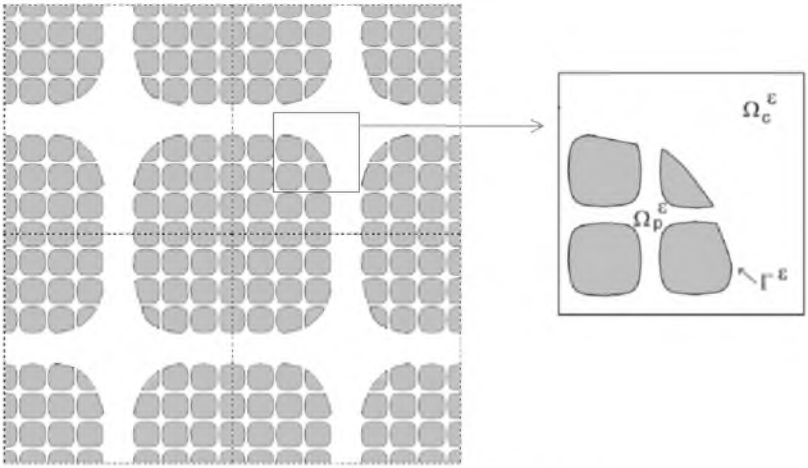

Fig. 3. Double porosity geometry: connected crack space.

mathematical model as a *double porosity model*, see Figs. 2 and 3), some attempts to derive macroscopic models, asymptotically closed to some phenomenological models on the microscopic level have been made by Arbogast *et al.*,[3] Bourgeat *et al.*[7] and Chen.[9] Because the last two papers repeat ideas of the first one, let us briefly discuss the main idea in Ref. 3. As a basic model on the microscopic level, the authors have considered a periodic structure, consisting of "solid" blocks of size ε surrounded by the fluid. The solid component is assumed to be already homogenized: there is no pore space and the motion of the fluid in blocks is governed by the usual Darcy equations of filtration. The motion of the fluid in crack space (the space between "solid" blocks) is described by some artificial system, similar to Darcy equations of filtration. There is no physical base, but from mathematical point of view such a choice of equations of fluid dynamics in cracks is very clear: it is impossible to find reasonable boundary conditions on the common boundary "solid" block-crack space, if the fluid dynamics is described by the Stokes equations. But there are reasonable boundary conditions, if the liquid motion is described by Darcy equations of filtration. Therefore, the final macroscopic models in Refs. 3, 7 and 9 are *physically incorrect* (see Ref. 17).

The physically correct double porosity model for the liquid filtration in an absolutely rigid body has been derived by Meirmanov.[17] Following the scheme, suggested by Burridge and Keller,[8] the author starts with a liquid domain, composed of a periodic system of pores with dimensionless size δ and a periodic system of cracks with dimensionless size ε, where $\delta = \varepsilon^r$, $r > 1$. The liquid motion is described by the Stokes system

$$\alpha_\tau \rho_f \frac{\partial \mathbf{v}}{\partial t} = \alpha_\mu \triangle \mathbf{v} - \nabla q + \rho_f \mathbf{F}, \quad \frac{\partial q}{\partial t} + \alpha_q \nabla \cdot \mathbf{v} = 0, \tag{1.4}$$

for dimensionless microscopic velocity $\mathbf{v}$ and pressure q of the liquid, where

$$\alpha_\tau = \frac{L}{g\tau^2}, \quad \alpha_\mu = \frac{2\mu}{\tau L g \rho_0}, \quad \alpha_q = \frac{c^2 \rho_f}{Lg},$$

L is a characteristic size of the domain in consideration, τ is a characteristic time of the process, ρ_f is the mean dimensionless density of the liquid, scaled with the mean density of water ρ_0, g is the value of acceleration of gravity, μ is the viscosity of fluid, c is a speed of sound in fluid, and the given function $\mathbf{F}(\mathbf{x}, t)$ is the dimensionless vector of distributed mass forces.

It is assumed that all dimensionless parameters depend on the small parameter ε and the (finite or infinite) limits exist:

$$\lim_{\varepsilon \searrow 0} \alpha_\tau(\varepsilon) = \tau_0, \quad \lim_{\varepsilon \searrow 0} \alpha_\mu(\varepsilon) = \mu_0, \quad \lim_{\varepsilon \searrow 0} \alpha_q(\varepsilon) = c_f^2, \quad \lim_{\varepsilon \searrow 0} \frac{\alpha_\mu}{\varepsilon^2} = \mu_1, \quad \lim_{\varepsilon \searrow 0} \frac{\alpha_\mu}{\delta^2} = \mu_2.$$

The aim of any homogenization procedure of some mathematical model, depending on the small parameter ε, is to find all possible limiting regimes in this model as $\varepsilon \searrow 0$. Of course, these regimes for the model (1.4) depend on criteria τ_0 and μ_1, which characterize different types of physical processes. We may roughly divide all these processes on two groups: long-time processes (filtration) and short-time processes (acoustics). It is well known that the characteristic time of the liquid filtration is about a month, while the characteristic size of the domain is about a thousand meters. Therefore, we may assume that for filtration $\tau_0 = 0$. The rest of processes we call acoustics and all these situations characterized by criterion $\tau_0 > 0$.

Under restrictions

$$\mu_0 = 0, \quad \tau_0 < \infty, \quad 0 < c_f < \infty,$$

the author has shown that the homogenization procedure for the liquid filtration ($\tau_0 = 0$) makes sense only if $\mu_1 > 0$. This criterion automatically implies the equality $\mu_2 = \infty$ and that the unique limiting regime for the liquid in pores is a rest state. For the case when the crack space is connected and $\mu_1 < \infty$, the author, using the method of reiterated homogenization suggested by Allaire and Briane,[2] has shown that the limiting velocity of the liquid in cracks and the limiting liquid pressure satisfy the usual Darcy equations of filtration. For disconnected crack space (isolated cracks), or for the case $\mu_1 = \infty$ the unique limiting regime is a rest state.

In the present publication we deal with the liquid filtration ($\tau_0 = 0$) and the same liquid domain as in Ref. 14, composed of a periodic system of pores with dimensionless size δ and a periodic system of cracks with dimensionless size ε, where $\delta = \varepsilon^r$, $r > 1$. We define the liquid domain Ω_f^ε, which is a subdomain of the unit cube Ω. Let $\Omega = Z_f \cup Z_s \cup \gamma_c$, where Z_f and Z_s are open sets, the common boundary $\gamma_c = \partial Z_f \cap \partial Z_s$ is a Lipschitz continuous surface, and a periodic repetition in $\mathbb{R}^3$ of the domain Z_s is a connected domain with a Lipschitz continuous boundary. The elementary cell Z_f models a crack space Ω_c^ε: the domain Ω_c^ε is an intersection of the cube Ω with a periodic repetition in $\mathbb{R}^3$ of the elementary cell εZ_f. In the same way we define the pore space Ω_p^δ: $\Omega = Y_f \cup Y_s \cup \gamma_p$, γ_c is a Lipschitz continuous surface, a periodic repetition in $\mathbb{R}^3$ of the domain Y_s is a connected domain with a Lipschitz continuous boundary, and Ω_p^δ is an intersection of $\Omega\backslash\Omega_c^\varepsilon$ with a periodic repetition in $\mathbb{R}^3$ of the elementary cell δY_f. Finally, we put $\Omega_f^\varepsilon = \Omega_p^\delta \cup \Omega_c^\varepsilon$, $\Omega_s^\varepsilon = \Omega\backslash\bar{\Omega}_f^\varepsilon$ is a solid skeleton, and $\Gamma^\varepsilon = \partial\Omega_s^\varepsilon \cap \partial\Omega_f^\varepsilon$ is a "solid skeleton—liquid domain" interface.

Following Burridge and Keller[8] and Sanchez-Palencia,[19] we describe the joint motion of the mixture of solid and liquid components on the microscopic level by well-known system, consisting of the Stokes and Lame's equations, coupled with corresponding boundary conditions on the common boundary "solid skeleton—liquid domain". For filtration processes ($\tau_0 = 0$) we may neglect the inertial terms and consider stationary equations. That is, the motion of the incompressible liquid in the liquid domain Ω_f^ε is governed by the stationary Stokes system

$$\alpha_\mu \triangle \frac{\partial \mathbf{w}_f}{\partial t} - \nabla q_f + \rho_f \mathbf{F} = 0, \quad \nabla \cdot \mathbf{w}_f = 0, \tag{1.5}$$

for dimensionless microscopic displacements $\mathbf{w}_f$ and pressure q_f, and the motion of the incompressible solid skeleton Ω_s^ε is governed by the stationary Lame's system

$$\alpha_\lambda \triangle \mathbf{w}_s - \nabla q_s + \rho_s \mathbf{F} = 0, \quad \nabla \cdot \mathbf{w}_s = 0, \tag{1.6}$$

for dimensionless microscopic displacements $\mathbf{w}_s$ and pressure q_s. On the common boundary Γ^ε "solid skeleton—liquid domain" the displacement vectors and pressures satisfy the usual continuity conditions

$$\mathbf{w}_f = \mathbf{w}_s, \tag{1.7}$$

and the momentum conservation law of the form

$$\left(\alpha_\mu \mathbb{D}\left(\frac{\partial \mathbf{w}_f}{\partial t}\right) - q_f \mathbb{I}\right) \cdot \mathbf{n} = (\alpha_\lambda \mathbb{D}(\mathbf{w}_s) - q_s \mathbb{I}) \cdot \mathbf{n}, \tag{1.8}$$

where $\mathbf{n}(\mathbf{x}_0)$ is the unit normal to the boundary at the point $\mathbf{x}_0 \in \Gamma^\varepsilon$.

In (1.5)–(1.8) $\mathbb{D}(\mathbf{u})$ is a symmetric part of the gradient $\nabla \mathbf{u}$, $\mathbb{I}$ is a unit tensor,

$$\alpha_\lambda = \frac{2\lambda}{Lg\rho_0},$$

ρ_s is the mean dimensionless density of the solid phase correlated with the mean density of water ρ_0 and λ is the elastic Lamé's constant.

The problem is endowed with the homogeneous initial and boundary conditions

$$\mathbf{w}(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in \Omega_f^\varepsilon \cup \Gamma^\varepsilon \cup \Omega_s^\varepsilon, \tag{1.9}$$

$$\mathbf{w}(\mathbf{x}, t) = 0, \quad \mathbf{x} \in S = \partial\Omega, \quad t \geq 0, \tag{1.10}$$

where $\mathbf{w} = \mathbf{w}_f$ in $\bar{\Omega}_f^\varepsilon$ and $\mathbf{w} = \mathbf{w}_s$ in $\bar{\Omega}_s^\varepsilon$.

Note that the assumption about incompressibility of the liquid is quite natural. It is well-known that the measure of incompressibility is a speed of sound of compressible waves. For filtration processes we assume that this value is equal to infinity. But the speed of a sound in a solid skeleton in two or three times is more than the speed of a sound in a liquid. Therefore, we may assume that *for filtration of incompressible liquid the solid skeleton is also an incompressible medium.*

The case $r = 1$ corresponds to already studied situation of a simple pore space, and the case $r > 1$ corresponds to a real double-porosity geometry. In what follows, we assume that

$$\mu_0 = 0 \quad \text{and} \quad 0 < \lambda_0 < \infty, \tag{1.11}$$

where

$$\lim_{\varepsilon \searrow 0} \alpha_\lambda(\varepsilon) = \lambda_0.$$

For the simple geometry ($r = 1$) the homogenization procedure makes sense only if $\mu_1 > 0$ (see Ref. 13). Moreover, if $\mu_1 = \infty$ (extremely viscous liquid), then the unique limiting regime is one velocity continuum, described by anisotropic Stokes system for the common velocity in the solid skeleton and in the liquid. This fact (that the velocity in the liquid coincides with the velocity in the solid skeleton) is a simple consequence of the Friedrichs–Poincaré inequality. The same situation is repeated for the case $r > 1$ of more complicated geometry. We show that, as before, the homogenization procedure makes sense if and only if $\mu_1 > 0$. But this criterion automatically implies the equality $\mu_2 = \infty$. Therefore, due to the same Friedrichs–Poincaré inequality the limiting velocity of the liquid in pores is proportional to the limiting velocity of the solid skeleton. If the crack space is connected and $\mu_1 < \infty$, then using the method of reiterated homogenization, suggested by Allaire and Briane,[2] we prove that the limiting displacements $\mathbf{u}$ of the solid skeleton and the limiting liquid pressure q_f satisfy some anisotropic Lame's equation

$$\lambda_0 \nabla \cdot (\mathbb{A}^{(s)} : \mathbb{D}(\mathbf{u})) - \frac{1}{m} \nabla q_f = \hat{\rho} \mathbf{F}, \tag{1.12}$$

coupled with Darcy law for the liquid velocity in cracks

$$\mathbf{v}_c = m_c \mathbf{v}_s + \frac{1}{\mu_1} \mathbb{B}^{(c)} \left(\rho_f \mathbf{F} - \frac{1}{m} \nabla q_f \right), \tag{1.13}$$

and common continuity equation:

$$\nabla \cdot (\mathbf{v}_c + (1 - m_c) \mathbf{v}_s) = 0, \tag{1.14}$$

where $\mathbf{v}_s = \partial \mathbf{u} / \partial t$ is a velocity of the solid component.

For the case $\mu_1 = \infty$, or for disconnected crack space $\mathbf{v}_c = m_c \mathbf{v}_s$ and the limiting displacements of the solid skeleton and the limiting liquid pressure satisfy the usual Stokes system

$$\lambda_0 \nabla \cdot (\mathbb{A}^{(s)} : \mathbb{D}(\mathbf{u})) - \frac{1}{m} \nabla q_f = \hat{\rho} \mathbf{F}, \quad \nabla \cdot \mathbf{u} = 0. \tag{1.15}$$

Here symmetric and strictly positively definite fourth-rank constant tensor $\mathbb{A}^{(s)}$ depends only on the geometry of the solid cells Y_s and Z_s and does not depend on criteria λ_0 and μ_1, strictly positively definite constant matrix $\mathbb{B}^{(c)}$ depends only on the geometry of the liquid cell Z_f and does not depend on criteria λ_0 and μ_1, $\hat{\rho} = m\rho_f + (1-m)\rho_s$, $m = \int_Y \int_Z \chi dydz$ is the porosity of the liquid domain, and $m_c = \int_Z \chi_c dz$ is the porosity of the crack space.

The system (1.12)–(1.14) is well-known as Biot's system of poroelasticity (Ref. 6), or Terzaghi system of filtration (Ref. 21). We call it as *Biot–Terzaghi system of liquid filtration in poroelastic media.*

Finally, for $\mu_1 < \infty$ we consider the family $\{\mathbf{v}_c^{\lambda_0}, \mathbf{u}^{\lambda_0}, q_f^{\lambda_0}\}$ of the solutions to the problem (1.12)–(1.14) and show that these solutions converge as $\lambda_0 \nearrow \infty$ to the solution of the problem

$$\mathbf{v}_c = \frac{1}{\mu_1}\mathbb{B}^{(c)}\left(\rho_f \mathbf{F} - \frac{1}{m}\nabla q_f\right), \quad \nabla \cdot \mathbf{v}_c = 0, \tag{1.16}$$

which is the usual Darcy system of filtration and, on the other hand, is *a physically correct double porosity model for filtration of an incompressible liquid in an absolutely rigid body.*

2. Main Results

To define the generalized solution to the problem (1.5)–(1.10), we characterize liquid and solid domains using indicator functions in Ω. Let $\eta(\mathbf{x})$ be the indicator function of the domain Ω in $\mathbb{R}^3$, that is $\eta(\mathbf{x}) = 1$ if $\mathbf{x} \in \Omega$ and $\eta(\mathbf{x}) = 0$ if $\mathbf{x} \in \mathbb{R}^3 \setminus \Omega$. Let also $\chi_p(\mathbf{y})$ be the one-periodic extension of the indicator function of the domain Y_f in Y and $\chi_c(\mathbf{z})$ be the one-periodic extension of the indicator function of the domain Z_f in Z. Then $\chi_c^\varepsilon(\mathbf{x}) = \eta(\mathbf{x})\chi_c(\mathbf{x}/\varepsilon)$ stands for the indicator function of the domain Ω_c^ε, $\chi_p^\varepsilon(\mathbf{x}) = \eta(\mathbf{x})(1-\chi_c(\mathbf{x}/\varepsilon))\chi_p(\mathbf{x}/\delta)$ stands for the indicator function of the domain Ω_p^δ and $\chi^\varepsilon(\mathbf{x}) = \chi_c^\varepsilon(\mathbf{x}) + \chi_p^\varepsilon(\mathbf{x})$ stands for the indicator function of the liquid domain Ω_f^ε.

We say, that functions $\{\mathbf{w}^\varepsilon, q^\varepsilon\}$, where

$$\mathbf{w}^\varepsilon = \mathbf{w}_f^\varepsilon \chi^\varepsilon + \mathbf{w}_s^\varepsilon(1-\chi^\varepsilon), \quad q^\varepsilon = q_f^\varepsilon \chi^\varepsilon + q_s^\varepsilon(1-\chi^\varepsilon),$$

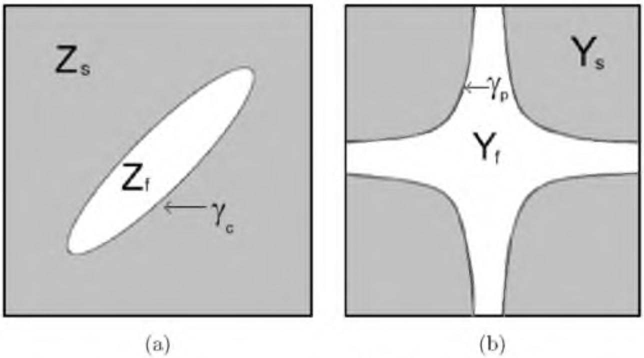

Fig. 4. Elementary (a) crack and (b) pore cells.

such that

$$\mathbf{w}^\varepsilon \in L^\infty((0,T);\overset{\circ}{W}{}^1_2(\Omega)), \quad \frac{\partial \mathbf{w}^\varepsilon}{\partial t} \in L^2((0,T);\overset{\circ}{W}{}^1_2(\Omega^\varepsilon_f)), \quad q^\varepsilon \in L^2(G_T)$$

is a generalized solution to the problem (1.5)–(1.10), if they satisfy normalization condition

$$\int_\Omega q^\varepsilon(\mathbf{x},t)dx = 0$$

almost everywhere in $(0,T)$, continuity equation

$$\nabla \cdot \mathbf{w} = 0 \tag{2.1}$$

in a usual sense almost everywhere in $G_T = \Omega \times (0,T)$, initial condition (1.9), and the integral identity

$$\int_0^T\int_\Omega \left(\left(\alpha_\mu \chi^\varepsilon \mathbb{D}\left(\frac{\partial \mathbf{w}^\varepsilon}{\partial t}\right) + \alpha_\lambda(1-\chi^\varepsilon)\mathbb{D}(\mathbf{w}^\varepsilon) - q^\varepsilon \mathbb{I}\right) : \mathbb{D}(\boldsymbol{\varphi}) + \rho^\varepsilon \mathbf{F}\cdot\boldsymbol{\varphi}\right)dxdt = 0 \tag{2.2}$$

for any vector functions $\boldsymbol{\varphi} \in L^2((0,T);\overset{\circ}{W}{}^1_2(\Omega))$. In (2.2)

$$\rho^\varepsilon = \rho_f \chi^\varepsilon + \rho_s(1-\chi^\varepsilon).$$

The homogeneous boundary condition (1.10) is already included into the corresponding functional space. Functions $\partial\mathbf{F}/\partial t$ and $\partial^2\mathbf{F}/\partial t^2$ are assumed to be L^2-integrable:

$$F_1 = \int_0^T\int_\Omega \left|\frac{\partial \mathbf{F}}{\partial t}\right|^2 dxdt < \infty, \quad F_2 = \int_0^T\int_\Omega \left|\frac{\partial^2 \mathbf{F}}{\partial t^2}\right|^2 dxdt < \infty.$$

In the same standard way, as in Ref. 13, one can show that for any $\varepsilon > 0$ there exists a unique generalized solution to the problem (1.5)–(1.10). To formulate basic *a priori* estimates we need to extend the function $\mathbf{w}^\varepsilon$ from Ω^ε_s to Ω^ε_s. To do that we use well-known results (see Conca[10] and Acerbi *et al.*[1]) in the following form: for any $\varepsilon > 0$ there exists an extension $\mathbf{u}^\varepsilon \in L^\infty((0,T);W^1_2(\Omega))$ such that $\mathbf{w}^\varepsilon = \mathbf{u}^\varepsilon$ in Ω^ε_s and

$$\int_\Omega |\mathbf{u}^\varepsilon|^2 dx \le C\int_{\Omega^\varepsilon_s} |\mathbf{w}^\varepsilon|^2 dx, \quad \int_\Omega |\mathbb{D}(\mathbf{u}^\varepsilon)|^2 dx \le C\int_{\Omega^\varepsilon_s} |\mathbb{D}(\mathbf{w}^\varepsilon)|^2 dx, \tag{2.3}$$

where C is independent of ε and t.

The following holds:

Lemma 2.1. *Let* $\mu_1 > 0$ *and* $r > 1$. *Then there exists sufficiently small* $\varepsilon_0 > 0$, *such that for any* $0 < \varepsilon < \varepsilon_0$ *and for any* $0 < t < T$

$$\int_\Omega |\mathbf{w}^\varepsilon(\mathbf{x},t)|^2 dx + \alpha_\mu \int_{\Omega^\varepsilon_f} |\mathbb{D}(\mathbf{w}^\varepsilon(\mathbf{x},t))|^2 dx + \alpha_\lambda \int_{\Omega^\varepsilon_s} |\mathbb{D}(\mathbf{w}^\varepsilon(\mathbf{x},t))|^2 dx \le CF_1, \tag{2.4}$$

$$\int_\Omega |\mathbf{v}^\varepsilon(\mathbf{x},t)|^2 dx + \alpha_\mu \int_{\Omega^\varepsilon_f} |\mathbb{D}(\mathbf{v}^\varepsilon(\mathbf{x},t))|^2 dx + \alpha_\lambda \int_{\Omega^\varepsilon_s} |\mathbb{D}(\mathbf{v}^\varepsilon(\mathbf{x},t))|^2 dx \le CF_2, \tag{2.5}$$

$$\int_\Omega |q^\varepsilon(\mathbf{x},t)|^2 dx = \int_\Omega (|q^\varepsilon_f(\mathbf{x},t)|^2 + |q^\varepsilon_s(\mathbf{x},t)|^2) dx \leq C(F_1 + F_2) = CF, \tag{2.6}$$

$$\frac{\alpha_\mu}{\delta^2}\int_{\Omega^\delta_p} |(\mathbf{w}^\varepsilon - \mathbf{u}^\varepsilon)(\mathbf{x},t)|^2 dx + \frac{\alpha_\mu}{\varepsilon^2}\int_{\Omega^\varepsilon_c} |(\mathbf{w}^\varepsilon - \mathbf{u}^\varepsilon)(\mathbf{x},t)|^2 dx \leq CF, \tag{2.7}$$

$$\frac{\alpha_\mu}{\delta^2}\int_{\Omega^\delta_p} \left|\left(\mathbf{v}^\varepsilon - \frac{\partial \mathbf{u}^\varepsilon}{\partial t}\right)(\mathbf{x},t)\right|^2 dx + \frac{\alpha_\mu}{\varepsilon^2}\int_{\Omega^\varepsilon_c} \left|\left(\mathbf{v}^\varepsilon - \frac{\partial \mathbf{u}^\varepsilon}{\partial t}\right)(\mathbf{x},t)\right|^2 dx \leq CF, \tag{2.8}$$

where $\mathbf{v}^\varepsilon = \partial \mathbf{w}^\varepsilon/\partial t$ *and* C *is independent of* ε *and* T.

Theorem 2.1. *Under conditions* (1.11) *and conditions of Lemma* 2.1 *there exist functions* $\mathbf{u}^\varepsilon \in L^\infty((0,T); W^1_2(\Omega))$, *such that* $\mathbf{u}^\varepsilon = \mathbf{w}^\varepsilon$ *in* Ω^ε_s, *a subsequence of small parameters* $\{\varepsilon > 0\}$, *and functions* $\mathbf{v}_p \in L^\infty((0,T); L^2(\Omega))$ — *the limiting velocity of the liquid in pores,* $\mathbf{v}_c \in L^\infty((0,T); L^2(\Omega))$ — *the limiting velocity of the liquid in cracks,* $\mathbf{u} \in L^\infty((0,T); \overset{\circ}{W}{}^1_2(\Omega))$ — *the limiting displacements of the solid skeleton, and* $q_f \in L^\infty((0,T); L^2(\Omega))$ — *the limiting pressure in the liquid, such that the sequences* $\{\chi^\delta_p \partial \mathbf{w}^\varepsilon/\partial t\}$, $\{\chi^\varepsilon_c \partial \mathbf{w}^\varepsilon/\partial t\}$, *and* $\{q^\varepsilon_f\}$ *converge as* $\varepsilon \searrow 0$ *weakly in* $L^2((0,T); L^2(\Omega))$ *to the functions* $\mathbf{v}_p$, $\mathbf{v}_c$ *and* q_f, *respectively. At the same time the sequence* $\{\mathbf{u}^\varepsilon\}$ *converges as* $\varepsilon \searrow 0$ *weakly in* $L^2((0,T); \overset{\circ}{W}{}^1_2(\Omega))$ *to the function* $\mathbf{u}$.

(I) If $\mu_1 = \infty$, *or the crack space is disconnected (isolated cracks), then*

$$\mathbf{v}_p = (1-m_c)m_p\frac{\partial \mathbf{u}}{\partial t}, \quad \mathbf{v}_c = m_c\frac{\partial \mathbf{u}}{\partial t}, \quad \mathbf{v} \equiv \mathbf{v}_c + \mathbf{v}_p + (1-m)\frac{\partial \mathbf{u}}{\partial t} = \frac{\partial \mathbf{u}}{\partial t},$$

and functions $\mathbf{u}$ *and* q_f *satisfy in* G_T *the anisotropic Stokes system*

$$\lambda_0 \nabla\cdot(\mathbb{A}^{(s)} : \mathbb{D}(\mathbf{u})) - \frac{1}{m}\nabla q_f = \hat{\rho}\mathbf{F}, \quad \nabla\cdot\mathbf{u} = 0, \tag{2.9}$$

with homogeneous initial and boundary conditions

$$q_f(\mathbf{x},0) = 0, \quad \mathbf{x}\in\Omega, \quad \mathbf{u}(\mathbf{x},t) = 0, \quad \mathbf{x}\in S, \quad t \geq 0, \tag{2.10}$$

where fourth-rank constant tensor $\mathbb{A}^{(s)}$ *is defined below by formula* (4.36), $\hat{\rho} = m\rho_f + (1-m)\rho_s$, $m = \int_Y\int_Z \chi dydz$ — *the porosity of the liquid domain,* $m_p = \int_Y \chi_p dy$ — *the porosity of the pore space, and* $m_c = \int_Z \chi_c dz$ — *the porosity of the crack space. The tensor* $\mathbb{A}^{(s)}$ *is symmetric, strictly positively definite, and depends only on the geometry of the solid cells* Y_s *and* Z_s.

(II) If $\mu_1 < \infty$, *and the crack space is connected, then*

$$\mathbf{v}_p = (1-m_c)m_p\frac{\partial \mathbf{u}}{\partial t}, \quad \mathbf{v} = \mathbf{v}_c + (1-m_c)\frac{\partial \mathbf{u}}{\partial t},$$

functions $\mathbf{u}$, $\mathbf{v}_c$ *and* q_f *satisfy in* G_T *Eqs.* (2.9), *Darcy law in the form*

$$\mathbf{v}_c = m_c\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{\mu_1}\mathbb{B}^{(c)}\left(\rho_f\mathbf{F} - \frac{1}{m}\nabla q_f\right), \quad \mathbf{x}\in\Omega, \tag{2.11}$$

initial and boundary conditions (2.10), *and boundary condition*

$$\mathbf{v} \cdot \mathbf{n} = 0, \quad \mathbf{x} \in S, \tag{2.12}$$

where $\mathbf{n}$ *is a unit normal vector to the boundary* S *at* $\mathbf{x} \in S$. *In* (2.11) *the strictly positively definite constant matrix* $\mathbb{B}^{(c)}$, *is defined below by formula* (4.18) *and depends only on the geometry of the liquid cell* Z_f.

Remark 2.1. Without loss of generality we may assume that

$$\int_\Omega q_f(\mathbf{x}, t) dx = 0.$$

Theorem 2.2. *Under conditions of Theorem* 2.1 *let* $\mu_1 < \infty$ *and* $\mathbf{u}^{(\lambda_0)}$, $\mathbf{v}_c^{(\lambda_0)}$ *and* $q_f^{(\lambda_0)}$ *be a solution to the problem* (2.9)–(2.12). *Then there exists a subsequence of parameters* $\{\lambda_0\}$, *such that the sequence* $\{\mathbf{u}^{(\lambda_0)}\}$ *converges as* $\lambda_0 \nearrow \infty$ *strongly in* $L^\infty((0,T); \overset{\circ}{W}{}^1_2(\Omega))$ *to zero, and sequences* $\{\mathbf{v}_c^{(\lambda_0)}\}$ *and* $\{q_f^{(\lambda_0)}\}$ *converge as* $\lambda_0 \nearrow \infty$ *weakly in* $L^2(G_T)$ *to functions* $\mathbf{v}_c$, *and* q_f *respectively, which are a solution to the problem*

$$\mathbf{v}_c = \frac{1}{\mu_1} \mathbb{B}^{(c)} \left(\rho_f \mathbf{F} - \frac{1}{m} \nabla q_f \right), \quad \mathbf{x} \in \Omega, \tag{2.13}$$

$$\nabla \cdot \mathbf{v}_c = 0, \quad \mathbf{x} \in \Omega, \quad \mathbf{v}_c \cdot \mathbf{n} = 0, \quad \mathbf{x} \in S. \tag{2.14}$$

3. Proof of Lemma 2.1

To prove (2.4) we choose as a test function in (2.2) the function $h(t)\partial \mathbf{w}^\varepsilon / \partial t$, where $h(t) = 1, t \in (0,t)$ and $h(t) = 0, \tau \in [t, T)$:

$$\alpha_\mu \int_0^t \int_\Omega \chi^\varepsilon \left| \mathbb{D}\left(\frac{\partial \mathbf{w}^\varepsilon}{\partial t}(\mathbf{x}, \tau) \right) \right|^2 dx d\tau + \frac{1}{2} \alpha_\lambda \int_\Omega (1 - \chi^\varepsilon) |\mathbb{D}(\mathbf{w}^\varepsilon(\mathbf{x}, t))|^2 dx$$
$$= \int_0^t \int_{\Omega^\varepsilon} \mathbf{F} \cdot \frac{\partial \mathbf{w}^\varepsilon}{\partial t} dx d\tau.$$

Passing the time derivative from $\partial \mathbf{w}^\varepsilon / \partial t$ to $\mathbf{F}$ on the integral on the right, applying after that to this integral Hölder inequality and the evident estimate

$$\int_\Omega \chi^\varepsilon |\mathbb{D}(\mathbf{w}^\varepsilon(\mathbf{x}, t))|^2 dx \le C \int_0^t \int_\Omega \chi^\varepsilon \left| \mathbb{D}\left(\frac{\partial \mathbf{w}^\varepsilon}{\partial t}(\mathbf{x}, \tau) \right) \right|^2 dx d\tau,$$

we arrive at

$$J(t) \equiv \alpha_\mu \int_\Omega \chi^\varepsilon |\mathbb{D}(\mathbf{w}^\varepsilon(\mathbf{x}, t))|^2 dx + \alpha_\lambda \int_\Omega (1 - \chi^\varepsilon) |\mathbb{D}(\mathbf{w}^\varepsilon(\mathbf{x}, t))|^2 dx$$
$$\le C F_1 + \int_0^t \int_\Omega |\mathbf{w}^\varepsilon(\mathbf{x}, \tau)|^2 dx d\tau. \tag{3.1}$$

Next we put $\mathbf{w}_0^\varepsilon = \mathbf{w}^\varepsilon - \mathbf{u}^\varepsilon$. By construction $\mathbf{w}_0^\varepsilon \in \overset{\circ}{W}{}_2^1(\Omega_f^\varepsilon)$. To estimate the integral

$$I_f^\varepsilon = \int_{\Omega_f^\varepsilon} |\mathbf{w}_0^\varepsilon|^2 dx$$

we divide it by two parts:

$$I_f^\varepsilon = I_p^\delta + I_c^\varepsilon, \quad I_p^\delta = \int_{\Omega_p^\delta} |\mathbf{w}_0^\varepsilon|^2 dx, \quad I_c^\varepsilon = \int_{\Omega_c^\varepsilon} |\mathbf{w}_0^\varepsilon|^2 dx.$$

Let $G_p^{(\mathbf{k})}$, where $\mathbf{k} = (k_1, k_2, k_3) \in \mathbb{Z}^3$, be the intersection of Ω_p^δ with a set $\{\mathbf{x} : \mathbf{x} = \varepsilon(\mathbf{y} + \mathbf{k}), \mathbf{y} \in Y\}$. Then $\Omega_p^\delta = \cup_{\mathbf{k}\in\mathbb{Z}^3} G_p^{(\mathbf{k})}$ and

$$I_p^\delta = \sum_{\mathbf{k}\in\mathbb{Z}^3} I_p^\delta(\mathbf{k}), \quad I_p^\delta(\mathbf{k}) = \int_{G_p^{(\mathbf{k})}} |\mathbf{w}_0^\varepsilon|^2 dx.$$

In each integral I_p^δ we change variable by $\mathbf{x} = \delta \mathbf{y}$, then apply the Friedrichs–Poincaré inequality and finally return to original variables:

$$\begin{aligned}
\int_{G_p^{(\mathbf{k})}} |\mathbf{w}_0^\varepsilon|^2 dx &= \delta^3 \int_{Y^{(\mathbf{k})}} |\bar{\mathbf{w}}_0^\varepsilon|^2 dy \\
&\leq \delta^3 C^{(\mathbf{k})} \int_{Y^{(\mathbf{k})}} |\mathbb{D}_y(\bar{\mathbf{w}}_0^\varepsilon)|^2 dy \\
&= \delta^2 C^{(\mathbf{k})} \int_{G_p^{(\mathbf{k})}} |\mathbb{D}_x(\mathbf{w}_0^\varepsilon)|^2 dx.
\end{aligned}$$

Here $\bar{\mathbf{w}}_0^\varepsilon(\mathbf{y}, t) = \mathbf{w}_0^\varepsilon(\mathbf{x}, t)$, $Y^{(\mathbf{k})} \subset Y$ is an appropriate translation to origin of the set $(1/\delta) G_p^{(\mathbf{k})}$, and $C^{(\mathbf{k})}$ is a constant in the Friedrichs–Poincaré inequality for the domain $Y^{(\mathbf{k})}$. To estimate these constants uniformly with respect to δ (or ε) let us clarify the structure of the domain $Y^{(\mathbf{k})}$. If the closure of $G_p^{(\mathbf{k})}$ has no intersection with the boundary between pore and crack spaces, then $Y^{(\mathbf{k})} = Y_f$ and $C^{(\mathbf{k})}$ coincides with a fixed constant C. Otherwise, $Y^{(\mathbf{k})}$ is one of the two domains, obtained after splitting Y_f by some smooth surface, asymptotically closed to the plane as $\varepsilon \searrow 0$. Due to supposition on the structure of the solid part Y_f, constants $C^{(\mathbf{k})}$ uniformly bounded for all possible planes, splitting Y_f. Therefore, $\sup C^{(\mathbf{k})} \leq C$ (for simplicity we denote all constants independent of ε as C) and

$$I_p^\delta \leq \delta^2 C \sum_{\mathbf{k}\in\mathbb{Z}^3} \int_{G_p^{(\mathbf{k})}} |\mathbb{D}_x(\mathbf{w}_0^\varepsilon)|^2 dx \leq \delta^2 C \int_{\Omega_f^\varepsilon} |\mathbb{D}_x(\mathbf{w}_0^\varepsilon)|^2 dx. \tag{3.2}$$

To explain ideas we consider the easiest geometry, when the liquid part Y_f is "surrounded" by the solid part Y_s. That is, for each facet $S \subset \partial Y$ of Y the liquid part $S \cap \partial Y_f$ is completely surrounded by the solid part $S \cap \partial Y_s$. Due to construction ($\mathbf{w}_0^\varepsilon = 0$ in Y_s) the constant in the Friedrichs–Poincaré inequality for $Y^{(\mathbf{k})}$ depends only on the ratio $\sigma = V_f/V_s$ between the volume V_f of the liquid part $Y^{(\mathbf{k})} \cap Y_f$ of $Y^{(\mathbf{k})}$ and the volume V_s of the solid part $Y^{(\mathbf{k})} \cap Y_s$ of $Y^{(\mathbf{k})}$: $C^{(\mathbf{k})} \leq C\sigma$. It is easy to see that for chosen geometry of Y_f and for any type of splitting of Y by planes, this ratio σ is uniformly bounded.

In the same way we show that

$$I_c^\varepsilon \leq \varepsilon^2 C \int_{\Omega_f^\varepsilon} |\mathbb{D}_x(\mathbf{w}_0^\varepsilon)|^2 dx. \tag{3.3}$$

In fact, as before we again divide the integral I_c^ε into the sum of integrals over domains $G_c^{(\mathbf{k})}$ and make change of variables:

$$\mathbf{x} = \varepsilon \mathbf{z}, \quad \mathbf{w}_0^\varepsilon(\mathbf{x},t) = \tilde{\mathbf{w}}_0^\varepsilon(\mathbf{z},t), \quad \int_{G_c^{(\mathbf{k})}} |\mathbf{w}_0^\varepsilon|^2 dx = \varepsilon^3 \int_{Z^{(\mathbf{k})}} |\tilde{\mathbf{w}}_0^\varepsilon|^2 dz.$$

For integrals over domains $G_c^{(\mathbf{k})}$ we use the Friedrichs–Poincaré inequality, based on the fact that the function $\tilde{\mathbf{w}}_0^\varepsilon$ vanishes on the some periodic (with period δ/ε) part of the boundary $\partial G_c^{(\mathbf{k})}$ with strictly positive measure, which bounded from below independently of ε.

Thus,

$$I_f^\varepsilon \leq C(\delta^2 + \varepsilon^2) \int_{\Omega_f^\varepsilon} |\mathbb{D}(\mathbf{w}_0^\varepsilon)|^2 dx \leq C\left(\frac{\delta^2}{\alpha_\mu} + \frac{\varepsilon^2}{\alpha_\mu}\right)\alpha_\mu \int_{\Omega_f^\varepsilon} |\mathbb{D}(\mathbf{w}^\varepsilon)|^2 dx$$

$$+ C(\delta^2 + \varepsilon^2) \int_{\Omega_f^\varepsilon} |\mathbb{D}(\mathbf{u}^\varepsilon)|^2 dx \leq CJ(t),$$

$$J(t) = \alpha_\mu \int_{\Omega_f^\varepsilon} |\mathbb{D}(\mathbf{w}^\varepsilon)|^2 dx + \alpha_\lambda \int_{\Omega_s^\varepsilon} |\mathbb{D}(\mathbf{w}^\varepsilon)|^2 dx$$

and

$$\int_{\Omega_f^\varepsilon} |\mathbf{w}^\varepsilon|^2 dx \leq \int_{\Omega_f^\varepsilon} |\mathbf{w}_0^\varepsilon|^2 dx + \int_{\Omega_f^\varepsilon} |\mathbf{u}^\varepsilon|^2 dx \leq C\left(J(t) + \int_{\Omega_s^\varepsilon} |\mathbf{w}^\varepsilon|^2 dx\right).$$

To estimate the integral

$$I_s^\varepsilon = \int_{\Omega_s^\varepsilon} |\mathbf{w}^\varepsilon|^2 dx$$

we use the Friedrichs–Poincaré inequality, estimate (2.3) and supposition $\lambda_0 > 0$:

$$I_s^\varepsilon \leq \int_\Omega |\mathbf{u}^\varepsilon|^2 dx \leq C \int_\Omega |\mathbb{D}(\mathbf{u}^\varepsilon)|^2 dx \leq C\alpha_\lambda \int_{\Omega_s^\varepsilon} |\mathbb{D}(\mathbf{w}^\varepsilon)|^2 dx \leq CJ(t).$$

Gathering all together one has

$$\int_\Omega |\mathbf{w}^\varepsilon|^2 dx \leq CJ(t).$$

Estimate (2.4) follows now from (3.1) and Gronwall's inequality. The same estimate (2.4) together with (3.2) and (3.3) result (2.7).

To prove estimates (2.5) and (2.8) we just repeat all over again for the "time derivative" of identity (2.2) and $\partial^2 \mathbf{w}^\varepsilon / \partial t^2$.

Estimate (2.6) is a simple consequence of (2.4) and (2.5) (see, for example, Ref. 13).

4. Proof of Theorem 2.1

4.1. *Weak and tree-scale limits of sequences of displacements, velocities and pressure*

First, we define the velocity of the liquid in pores as $\mathbf{v}_p^\delta = \chi_p^\delta \partial \mathbf{w}^\varepsilon/\partial t$, the velocity of the liquid in cracks as $\mathbf{v}_c^\varepsilon = \chi_c^\varepsilon \partial \mathbf{w}^\varepsilon/\partial t$ and the velocity of the solid skeleton as $\mathbf{v}_s^\varepsilon = \partial \mathbf{u}^\varepsilon/\partial t$. By definition

$$\mathbf{v}^\varepsilon = \mathbf{v}_p^\delta + \mathbf{v}_c^\varepsilon + (1 - \chi^\varepsilon)\mathbf{v}_s^\varepsilon. \tag{4.1}$$

On the strength of Lemma 1, the sequences $\{q_f^\varepsilon\}$, $\{q_s^\varepsilon\}$, $\{\mathbf{v}^\varepsilon\}$, $\{\mathbf{v}_p^\delta\}$, $\{\mathbf{v}_c^\varepsilon\}$, $\{\mathbf{u}^\varepsilon\}$, $\{\mathbf{v}_s^\varepsilon\}$, and $\{\nabla \mathbf{u}^\varepsilon\}$ are bounded in $L^2(\Omega_T)$. Hence there exists a subsequence of small parameters $\{\varepsilon > 0\}$ and functions q_f, q_s, $\mathbf{v}$, $\mathbf{v}_p$, $\mathbf{v}_c$, $\mathbf{v}_s \in L^2(G_T)$ and $\mathbf{u} \in L^\infty((0,T); \overset{\circ}{W}{}_2^1(\Omega))$ such that

$$\left.\begin{array}{l} q_f^\varepsilon \rightharpoonup q_f, \quad q_s^\varepsilon \rightharpoonup q_s, \quad \mathbf{v}^\varepsilon \rightharpoonup \mathbf{v}, \quad \mathbf{v}_p^\delta \rightharpoonup \mathbf{v}_p, \quad \mathbf{v}_c^\varepsilon \rightharpoonup \mathbf{v}_c, \\ \mathbf{v}_s^\varepsilon \rightharpoonup \mathbf{v}_s, \quad \mathbf{u}^\varepsilon \rightharpoonup \mathbf{u}, \quad \nabla \mathbf{u}^\varepsilon \rightharpoonup \nabla \mathbf{u} \end{array}\right\} \tag{4.2}$$

weakly in $L^2(\Omega_T)$ as $\varepsilon \searrow 0$.

Note also that

$$\chi^\varepsilon \alpha_\mu \mathbb{D}(\mathbf{v}^\varepsilon) \to 0 \tag{4.3}$$

strongly in $L^2(\Omega_T)$ as $\varepsilon \searrow 0$.

Next we apply the method of reiterated homogenization (see Allaire and Briane[2]): there exist functions $Q_f(\mathbf{x},t,\mathbf{y},\mathbf{z})$, $Q_s(\mathbf{x},t,\mathbf{y},\mathbf{z})$, $\mathbf{V}(\mathbf{x},t,\mathbf{y},\mathbf{z})$, $\mathbf{V}_c(\mathbf{x},t,\mathbf{y},\mathbf{z})$, $\mathbf{U}_c(\mathbf{x},t,\mathbf{z})$ and $\mathbf{U}_p(\mathbf{x},t,\mathbf{y},\mathbf{z})$ that are one-periodic in $\mathbf{y}$ and $\mathbf{z}$ and satisfy the condition that the sequences $\{q_f^\varepsilon\}$, $\{q_s^\varepsilon\}$, $\{\mathbf{v}^\varepsilon\}$, $\{\mathbf{v}_c^\varepsilon\}$ and $\{\nabla \mathbf{u}^\varepsilon\}$ tree-scale converge (up to some subsequences) to $Q_f(\mathbf{x},t,\mathbf{y},\mathbf{z})$, $Q_s(\mathbf{x},t,\mathbf{y},\mathbf{z})$, $\mathbf{V}(\mathbf{x},t,\mathbf{y},\mathbf{z})$, $\mathbf{V}_c(\mathbf{x},t,\mathbf{y},\mathbf{z})$ and $\nabla \mathbf{u} + \nabla_z \mathbf{U}_c(\mathbf{x},t,\mathbf{z}) + \nabla_y \mathbf{U}_p(\mathbf{x},t,\mathbf{y},\mathbf{z})$, respectively. The sequence $\{\mathbf{u}^\varepsilon\}$ three-scale converges to the function $\mathbf{u}(\mathbf{x},t)$.

Relabeling if necessary, we assume that the sequences themselves converge.

Note that *three-scale convergence* of the sequence $\{\pi^\varepsilon\}$ to the function $\Pi(\mathbf{x},t,\mathbf{y},\mathbf{z})$ means the convergence of integrals

$$\int_0^T \int_\Omega \pi^\varepsilon(\mathbf{x},t)\varphi\Big(\mathbf{x},t,\frac{\mathbf{x}}{\varepsilon},\frac{\mathbf{x}}{\delta}\Big)dxdt \to \int_0^T \int_\Omega \int_Y \int_Z \Pi(\mathbf{x},t,\mathbf{y},\mathbf{z})\varphi(\mathbf{x},t,\mathbf{y},\mathbf{z})dzdydxdt,$$

for any smooth one-periodic in $\mathbf{y}$ and $\mathbf{z}$ function $\varphi(\mathbf{x},t,\mathbf{y},\mathbf{z})$. By definition the function

$$\pi(\mathbf{x},t) = \langle\langle \Pi\rangle_Y\rangle_Z,$$

where

$$\langle \Pi\rangle_Y = \int_Y \Pi dy, \quad \langle \Pi\rangle_Z = \int_Z \Pi dz,$$

is a weak limit in $L^2(G_T)$ of the sequence $\{\pi^\varepsilon\}$.

4.2. *Macro- and microscopic equations*

We start the proof of the theorem from the macro- and microscopic equations related to the liquid motion and to the continuity equation.

Lemma 4.1. *For almost all* $(\mathbf{x},t) \in G_T$, $\mathbf{y} \in Y$ *and* $\mathbf{z} \in Z$, *the weak and three-scale limits of the sequences* $\{q_f^\varepsilon\}$, $\{q_s^\varepsilon\}$, $\{\mathbf{v}^\varepsilon\}$, $\{\mathbf{v}_c^\varepsilon\}$, $\{\mathbf{v}_p^\varepsilon\}$ *and* $\{\mathbf{u}^\varepsilon\}$ *satisfy the relations*

$$Q_f = \frac{1}{m} q_f(\mathbf{x},t)\chi(\mathbf{y},\mathbf{z}), \quad Q_s = Q_s(1-\chi), \quad \chi = \chi_c(\mathbf{z}) + (1-\chi_c(\mathbf{z}))\chi_p(\mathbf{y}), \tag{4.4}$$

$$\mathbf{v}_p = (1-m_c)m_p\mathbf{v}_s, \quad \mathbf{v} = \mathbf{v}_c + (1-m_c)\mathbf{v}_s, \tag{4.5}$$

$$\nabla\cdot\mathbf{v} = 0, \quad (1-\chi)(\nabla\cdot\mathbf{u} + \nabla_z\cdot\mathbf{U_c} + \nabla_y\cdot\mathbf{U_p}) = 0, \tag{4.6}$$

$$(1-m)\nabla\cdot\mathbf{u} + \langle(1-\chi)\nabla_z\cdot\mathbf{U_c}\rangle_Z + \langle\langle(1-\chi)\nabla_y\cdot\mathbf{U_p}\rangle_Y\rangle_Z = 0, \tag{4.7}$$

where $m = \langle\langle\chi\rangle_Y\rangle_Z$ — *the porosity of the liquid domain,* $m_p = \langle\chi_p\rangle_Y$ — *the porosity of the pore space, and* $m_c = \langle\chi_c\rangle_Z$ — *the porosity of the crack space.*

Proof. By definition of q_f^ε and q_s^ε, and properties of three-scale convergence one has equalities $Q_f = \chi Q_f$, $Q_s = (1-\chi)Q_s$. Choosing in (2.2) test function in the form $\boldsymbol{\varphi} = \delta h(t)\boldsymbol{\psi}^\varepsilon = \delta h(t)\boldsymbol{\psi}(\mathbf{x},\mathbf{x}/\varepsilon,\mathbf{x}/\delta)$, where $\boldsymbol{\psi}^\varepsilon$ is finite in Ω_f^ε, and passing to the limit as $\varepsilon \searrow 0$ we arrive at

$$\chi(\mathbf{y},\mathbf{z})\nabla_y Q_f = 0 \quad \text{or} \quad Q_f = \chi(\mathbf{y},\mathbf{z})Q_f(\mathbf{x},t,\mathbf{z}).$$

Now we repeat all over again with $\boldsymbol{\varphi} = \varepsilon h(t)\boldsymbol{\psi}^\varepsilon = \varepsilon h(t)\boldsymbol{\psi}(\mathbf{x},\mathbf{x}/\varepsilon)$, where $\boldsymbol{\psi}^\varepsilon$ is finite in Ω_f^ε, and get

$$\chi(\mathbf{y},\mathbf{z})\nabla_z Q_f = 0 \quad \text{or} \quad Q_f = \chi(\mathbf{y},\mathbf{z})Q_f(\mathbf{x},t),$$

which results in (4.4).

(4.5) is a simple consequence of (4.1), (2.8) and properties of three-scale convergence.

The first continuity equation in (4.6) follows from the continuity equation (2.1) in the form

$$\int_\Omega \mathbf{v}^\varepsilon\cdot\nabla\psi dx = 0, \tag{4.8}$$

which holds true for any smooth functions ψ, after passing there to the limit as $\varepsilon \searrow 0$.

Three-scale limit in continuity equation (2.1) in the form

$$(1-\chi^\varepsilon)\nabla\cdot\mathbf{v}_s^\varepsilon = 0$$

results the second continuity equation in (4.6). Finally, (4.7) is just an average of the first equation in (4.6). □

Remark 4.1. The first continuity equation in (4.6) is understood in the sense of distributions as integral identity

$$\int_\Omega \mathbf{v}\cdot\nabla\psi dx = 0,$$

which holds true for any smooth functions ψ.

Lemma 4.2. *Let $\tilde{\mathbf{V}} = \langle \mathbf{V}_c \rangle_Y$. If $\mu_1 = \infty$, then*

$$\tilde{\mathbf{V}} = \mathbf{V}_c = \mathbf{v_s}(\mathbf{x}, t)\chi_c(\mathbf{z}), \quad \mathbf{v_c} = m_c \mathbf{v_s}. \tag{4.9}$$

If $\mu_1 < \infty$, then for almost every $(\mathbf{x}, t) \in G_T$ the function $\tilde{\mathbf{V}}$ is a one-periodic in $\mathbf{z}$ solution to the Stokes system

$$-\mu_1 \triangle_z \tilde{\mathbf{V}} = -\nabla_z \tilde{\Pi} - \frac{1}{m}\nabla q_f + \rho_f \mathbf{F}, \tag{4.10}$$

$$\nabla_z \cdot \tilde{\mathbf{V}} = 0, \tag{4.11}$$

in the domain Z_f, such that

$$\tilde{\mathbf{V}}(\mathbf{x}, t, \mathbf{z}) = \mathbf{v_s}(\mathbf{x}, t), \quad \mathbf{z} \in \gamma_c. \tag{4.12}$$

Proof. First of all we derive the continuity equation (4.11). To do that we put $\psi = \varepsilon\psi_0(\mathbf{x}, \mathbf{x}/\varepsilon)$ in the integral identity (4.8), pass to the limit as $\varepsilon \searrow 0$, and get identity

$$\int_\Omega \int_{Z_f} \tilde{\mathbf{V}} \cdot \nabla_z \psi_0(\mathbf{x}, \mathbf{z}) dx dz = 0,$$

which is obviously equivalent to (4.11).

If $\mu_1 = \infty$, then (4.9) follows from estimate (2.8). Now let $\mu_1 < \infty$. If we choose in the integral identity (2.2) a test function $\boldsymbol{\varphi}$ in the form $\boldsymbol{\varphi} = h_0(t)h_1(\mathbf{x})\boldsymbol{\psi}(\mathbf{x}/\varepsilon)$, where $\operatorname{supp} h_1 \subset \Omega$, $\operatorname{supp} \boldsymbol{\psi}(\mathbf{z}) \subset Z_f$, $\nabla_z \cdot \boldsymbol{\psi} = 0$, and pass to the limit as $\varepsilon \searrow 0$, we arrive at

$$\int_\Omega \int_{Z_f} (h_1 \mu_1 \tilde{\mathbf{V}} \cdot (\nabla_z \cdot \mathbb{D}_z(\boldsymbol{\psi})) + \frac{1}{m} q_f (\nabla h_1 \cdot \boldsymbol{\psi}) + \rho_f (\mathbf{F} \cdot \boldsymbol{\psi}) h_1) dx dz = 0.$$

The desired Eq. (4.10) follows from the last identity, if we pass derivatives from the test function to $\tilde{\mathbf{V}}$ and take into account (4.11). The term $\nabla_z \tilde{\Pi}$ appears due to condition $\nabla_z \cdot \boldsymbol{\psi} = 0$.

Finally, the boundary condition (4.12) follows from the representation

$$\langle \mathbf{V} \rangle_Y = \tilde{\mathbf{V}} + (1 - \chi_c(\mathbf{z}))\mathbf{v_s}(\mathbf{x}, t),$$

and inclusion $\langle \mathbf{V} \rangle_Y \in W_2^1(Z)$ for almost every $(\mathbf{x}, t) \in \Omega_T$ (see Ref. 13). □

Now we derive macro- and microscopic equations for the solid motion. Let

$$\tilde{q}_f = \frac{1}{m\lambda_0} q_f, \quad \tilde{Q}_s = \left(\frac{1}{\lambda_0} Q_s - \tilde{q}_f\right)(1 - \chi), \quad \tilde{q}_s = \langle\langle \tilde{Q}_s \rangle_{Z_s} \rangle_{Y_s}.$$

Then

$$\frac{1}{\lambda_0}(q_f + q_s) = \frac{1}{\lambda_0}\langle\langle Q_f + Q_s \rangle_{Z_s}\rangle_{Y_s} = \langle\langle \tilde{q}_f + \tilde{Q}_s \rangle_Z \rangle_Y = \tilde{q}_f + \tilde{q}_s$$

Lemma 4.3. *Functions $\mathbf{u}$, $\mathbf{U}_c$, $\mathbf{U}_p$, $\tilde{q}_f$ and $\tilde{q}_s$ satisfy in G_T the macroscopic equation*

$$\nabla_x \cdot ((1 - m)\mathbb{D}(\mathbf{u}) + (1 - m_p)\langle \mathbb{D}_z(\mathbf{U}_c) \rangle_{Z_s} + \langle\langle \mathbb{D}_y(\mathbf{U}_p) \rangle_{Z_s} \rangle_{Y_s} - \tilde{q}\mathbb{I}) = \tilde{\mathbf{F}}, \tag{4.13}$$

where $\hat{\rho} = m\rho_f + (1 - m)\rho_s$, $\tilde{q} = \tilde{q}_f + \tilde{q}_s$, $\tilde{\mathbf{F}} = (\hat{\rho}/\lambda_0)\mathbf{F}$.

To prove this lemma we put in (2.2) $\boldsymbol{\varphi} = h_0(t)\mathbf{h}_1(\mathbf{x})$, where $\mathbf{h}$ is finite in Ω, and pass to the limit as $\varepsilon \searrow 0$, taking into account (4.3).

Lemma 4.4. *Functions* $\mathbf{u}$, $\mathbf{U}_c$, $\mathbf{U}_p$ *and* $\tilde{Q}_s$ *satisfy in* Z_s *and almost everywhere in* G_T *the microscopic equation*

$$\nabla_z \cdot ((1-\chi_c)((1-m_p)(\mathbb{D}(\mathbf{u}) + \mathbb{D}_z(\mathbf{U}_c)) + \langle \mathbb{D}_y(\mathbf{U}_p) - \tilde{Q}_s\mathbb{I}\rangle_{Y_s})) = 0. \tag{4.14}$$

To prove the lemma we put in (2.2) $\boldsymbol{\varphi} = \varepsilon h_0(t)h_1(\mathbf{x})\boldsymbol{\varphi}_0(\mathbf{x}/\varepsilon)$, where h_1 is finite in Ω, pass to the limit as $\varepsilon \searrow 0$, and use the equality $(1-\chi) = (1-\chi_p)(1-\chi_c)$.

Lemma 4.5. *Functions* $\mathbf{u}$, $\mathbf{U}_c$, $\mathbf{U}_p$ *and* $\tilde{Q}_s$ *satisfy in* Y_s *and almost everywhere in* $G_T \times Z_s$ *the microscopic equation*

$$\nabla_y \cdot ((1-\chi_p)(\mathbb{D}(\mathbf{u}) + \mathbb{D}_z(\mathbf{U}_c) + \mathbb{D}_y(\mathbf{U}_p) - \tilde{Q}_s\mathbb{I}) = 0. \tag{4.15}$$

To prove the lemma we put in (2.2) $\boldsymbol{\varphi} = \delta h_0(t)h_1(\mathbf{x})\varphi_0(\mathbf{x}/\varepsilon)\boldsymbol{\varphi}_1(\mathbf{x}/\delta)$, where h_1 is finite in Ω, and pass to the limit as $\varepsilon \searrow 0$.

4.3. *Homogenized equations*

The derivation of homogenized equations is quite standard (see Ref. 13). For the liquid motion we solve the microscopic system (4.9)–(4.12), find $\tilde{\mathbf{V}}$ as an operator on ∇q_f and $\partial\mathbf{u}/\partial t$, and then use the relation $\mathbf{v}_c = \langle\tilde{\mathbf{V}}\rangle_{Z_f}$. Namely, the following holds true

Lemma 4.6. *Let* $\mu_1 < \infty$. *Then functions* $\mathbf{v}_c$, $\mathbf{v}_s$, $\mathbf{v} = \mathbf{v}_c + (1-m_c)\mathbf{v}_s$, *and* q_f *satisfy in the domain* Ω *the usual Darcy system of filtration*

$$\mathbf{v}_c = m_c\mathbf{v}_s + \frac{1}{\mu_1}\mathbb{B}^{(c)}\left(\rho_f\mathbf{F} - \frac{1}{m}\nabla q_f\right), \quad \mathbf{x} \in \Omega, \tag{4.16}$$

$$\nabla \cdot \mathbf{v} = 0, \quad \mathbf{x} \in \Omega, \quad \mathbf{v} \cdot \mathbf{n} = 0, \quad \mathbf{x} \in S, \tag{4.17}$$

where $\mathbf{n}$ *is a unit normal vector to the boundary* S *at* $\mathbf{x} \in S$.

If the crack space is connected, then the strictly positively definite constant matrix $\mathbb{B}^{(c)}$, *is defined by formula*

$$\mathbb{B}^{(c)} = \frac{1}{\mu_1}\sum_{i=1}^{3}\langle\mathbf{V}^i\rangle_{Z_f} \otimes \mathbf{e}_i. \tag{4.18}$$

In (4.18) *functions* $\mathbf{V}^i(\mathbf{z})$, $i = 1, 2, 3$, *are solutions to the periodic boundary value problems*

$$\left.\begin{array}{l} -\triangle_z\mathbf{V}^i + \nabla\Pi^i = \mathbf{e}_i, \quad \nabla_y \cdot \mathbf{V}^i = 0, \quad \mathbf{z} \in Z_f, \\ \mathbf{V}^i = 0, \quad \mathbf{z} \in \gamma_c, \end{array}\right\} \tag{4.19}$$

where $\mathbf{e}_i$, $i = 1, 2, 3$, *are the standard Cartesian basis vectors and for any vectors* $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$ *the matrix* $\mathbf{a} \otimes \mathbf{b}$ *is defined as* $(\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c})$.

If the crack space is disconnected (isolated cracks), then the unique solution to the problem (4.19) *is* $\mathbf{V}^i = 0, i = 1, 2, 3$, $\mathbb{B}^{(c)} = 0$, *and*

$$\mathbf{v}_c = m_c \mathbf{v}_s.$$

The same procedure is applied for the solid motion. First, we solve the microscopic equation (4.15) coupled with the second equation in (4.6), find $\mathbf{U}_p$ as an operator on $\mathbb{D}_z(\mathbf{U}_c)$ and $\mathbb{D}(\mathbf{u})$, and substitute the result into Eq. (4.14). Next, we solve the obtained microscopic equation and find $\mathbf{U}_c$ as an operator on $\mathbb{D}(\mathbf{u})$. Finally, we substitute expressions $\mathbf{U}_p$ and $\mathbf{U}_c$ as operators on $\mathbb{D}(\mathbf{u})$ into macroscopic equation (4.13) and arrive at desired homogenized equation for the function $\mathbf{u}$.

Lemma 4.7. *For almost every* $(\mathbf{x}, t) \in G_T$ *functions* $\mathbf{u}$ *and* $\mathbf{U}_c$ *satisfy in* Z_s *the microscopic equation*

$$\nabla_z \cdot ((1 - \chi_c)\mathbb{A}^{(c)} : (\mathbb{D}(\mathbf{u}) + \mathbb{D}_z(\mathbf{U}_c))) = 0, \tag{4.20}$$

where fourth-rank constant tensor $\mathbb{A}^c$ *is defined below by formula* (4.23).

Proof. Let

$$D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad d = \nabla \cdot \mathbf{u}, \quad \mathbf{u} = (u_1, u_2, u_3),$$

$$D_{ij}^{(c)} = \frac{1}{2}\left(\frac{\partial U_{c,i}}{\partial z_j} + \frac{\partial U_{c,j}}{\partial z_i}\right), \quad d^{(c)} = \nabla_z \cdot \mathbf{U}_c, \quad \mathbf{U}_c = (U_{c,1}, U_{c,2}, U_{c,3}),$$

$$D_{ij}^{(p)} = D_{ij} + D_{ij}^{(c)}, \quad d^{(p)} = d + d^{(c)}.$$

As usual, Eq. (4.20) follows from the microscopic equations (4.14), after we insert in the expression

$$\langle \mathbb{D}_y(\mathbf{U}_p)\rangle_{Y_s} - \langle \tilde{Q}_s\rangle_{Y_s}\mathbb{I} = \mathbb{C}^{(p)} : (\mathbb{D}(\mathbf{u}) + \mathbb{D}_z(\mathbf{U}_c)).$$

To find it we look for the solution $\mathbf{U}_p$ to the system of microscopic equations (4.15) and (4.6) in the form

$$\mathbf{U}_p = \sum_{i,j=1}^{3} \mathbf{U}_p^{ij}(\mathbf{y}) D_{ij}^{(p)} + \mathbf{U}_p^0(\mathbf{y}) d^{(p)}, \quad \tilde{Q}_s = \sum_{i,j=1}^{3} Q_p^{ij}(\mathbf{y}) D_{ij}^{(p)} + Q_p^0(\mathbf{y}) d^{(p)}$$

and arrive at the following periodic boundary value problems in Y_s:

$$\left.\begin{array}{l} \nabla_y \cdot ((1 - \chi_p)((\mathbb{D}_y(\mathbf{U}_p^{ij}) + \mathbb{J}^{ij}) - Q_p^{ij}\mathbb{I})) = 0, \quad \mathbf{y} \in Y, \\ \nabla_y \cdot \mathbf{U}_p^{ij} = 0, \quad \langle \mathbf{U}_p^{ij}\rangle_{Y_s} = 0, \quad \mathbf{y} \in Y_s, \end{array}\right\} \tag{4.21}$$

$$\left.\begin{array}{l} \nabla_y \cdot ((1 - \chi_p)(\mathbb{D}_y(\mathbf{U}_p^0) - P_0\mathbb{I})) = 0, \quad \mathbf{y} \in Y, \\ \nabla_y \cdot \mathbf{U}_p^0 + 1 = 0, \quad \langle \mathbf{U}_p^0\rangle_{Y_s} = 0, \quad \mathbf{y} \in Y_s. \end{array}\right\} \tag{4.22}$$

In (4.21)

$$\mathbb{J}^{ij} = \frac{1}{2}(\mathbb{I}^{ij} + \mathbb{I}^{ji}) = \frac{1}{2}(\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_i \otimes \mathbf{e}_i).$$

Problems (4.21) and (4.22) are understood in the sense of distributions. For example, the first equation in (4.21) is equivalent to the integral identity

$$\int_Y (1-\chi_p)((\mathbb{D}_y(\mathbf{U}_p^{ij})+\mathbb{J}^{ij})-Q_p^{ij}\mathbb{I}):\mathbb{D}_y(\boldsymbol{\varphi})dy=0$$

for any smooth and periodic in $\mathbf{y}$ function $\boldsymbol{\varphi}(\mathbf{y})$.

The solvability of the problem (4.21) directly follows from the *a priori* estimate

$$\int_{Y_s} |\nabla \mathbf{U}_p^{ij}|^2 dy \leq C,$$

and the latter is a consequence of the energy identity

$$\int_{Y_s} (\mathbb{D}_y(\mathbf{U}_p^{ij}):\mathbb{D}_y(\mathbf{U}_p^{ij})+\mathbb{J}^{ij}:\mathbb{D}_y(\mathbf{U}_p^{ij}))dy=0.$$

To solve problem (4.22) we first find a one-periodic function $\mathbf{V}_0 \in W_2^1(Y_s)$ such that

$$\nabla_y \cdot \mathbf{V}_0 + 1 = 0, \quad \mathbf{y} \in Y_s.$$

There are a lot of ways to construct such a function. In Ref. 12, for example, one may find non-periodic case. The periodic case is quite similar.

After that, the solvability of the problem (4.22) follows from the energy equality

$$\int_{Y_s} (\mathbb{D}_y(\mathbf{U}_p^0):(\mathbb{D}_y(\mathbf{U}_p^0)-\mathbb{D}_y(\mathbf{V}_0)))dy=0,$$

which is a result of a substitution into the corresponding to the first equation in (4.22) integral identity of the test function $(\mathbf{U}_0-\mathbf{V}_0)$.

Thus,

$$\begin{aligned}
&\langle \mathbb{D}_y(\mathbf{U}_p)\rangle_{Y_s} - \langle \tilde{Q}_s\rangle_{Y_s}\mathbb{I} \\
&\quad= \sum_{i,j=1}^{3} \langle \mathbb{D}_y(\mathbf{U}_p^{ij})\rangle_{Y_s} D_{ij}^{(p)} + \langle \mathbb{D}_y(\mathbf{U}_p^0)\rangle_{Y_s} d^{(p)} \\
&\qquad- \left(\sum_{i,j=1}^{3} \langle Q_p^{ij}\rangle_{Y_s} D_{ij}^{(p)}\right)\mathbb{I} - (\langle Q_p^0\rangle_{Y_s} d^{(p)})\mathbb{I} \\
&\quad= \sum_{i,j=1}^{3} ((\langle(\mathbb{D}_y \mathbf{U}_p^{ij})\rangle_{Y_s} - \langle Q_p^{ij}\rangle_{Y_s}\mathbb{I}) D_{ij}^{(p)} + (\langle \mathbb{D}_y(\mathbf{U}_p^0)\rangle_{Y_s} - \langle Q_p^0\rangle_{Y_s}\mathbb{I}) d^{(p)} \\
&\quad= \sum_{i,j=1}^{3} (\langle \mathbb{D}_y(\mathbf{U}_p^{ij})\rangle_{Y_s} \otimes \mathbb{J}^{ij} - \langle Q_p^{ij}\rangle_{Y_s}\mathbb{I}\otimes \mathbb{J}^{ij}) : (\mathbb{D}(\mathbf{u}) + \mathbb{D}_z(\mathbf{U}_c)) \\
&\qquad+ (\langle \mathbb{D}_y(\mathbf{U}_p^0)\rangle_{Y_s} \otimes \mathbb{I} - \langle Q_p^0\rangle_{Y_s}\mathbb{I}\otimes\mathbb{I}) : (\mathbb{D}(\mathbf{u}) + \mathbb{D}_z(\mathbf{U}_c)) \\
&\quad= (\mathbb{C}_1^{(p)} + \mathbb{C}_2^{(p)} + \mathbb{C}_3^{(p)} + \mathbb{C}_4^{(p)}) : (\mathbb{D}(\mathbf{u}) + \mathbb{D}_z(\mathbf{U}_c)) = \mathbb{C}^{(p)} : (\mathbb{D}(\mathbf{u}) + \mathbb{D}_z(\mathbf{U}_c)),
\end{aligned}$$

where $\mathbb{B} \otimes \mathbb{C}$ is a fourth-rank tensor such that its convolution with any matrix $\mathbb{A}$ is defined by the formula

$$(\mathbb{B} \otimes \mathbb{C}) : \mathbb{A} = \mathbb{B}(\mathbb{C} : \mathbb{A})$$

and

$$\mathbb{A}^{(c)} = (1 - m_p) \sum_{i,j=1}^{3} \mathbb{J}^{ij} \otimes \mathbb{J}^{ij} + \mathbb{C}^{(p)} = (1 - m_p)\mathbb{J} + \mathbb{C}^{(p)}, \tag{4.23}$$

where

$$\mathbb{J} = \sum_{i,j=1}^{3} \mathbb{J}^{ij} \otimes \mathbb{J}^{ij}, \quad \mathbb{C}^{(p)} = \mathbb{C}_1^{(p)} + \mathbb{C}_2^{(p)} + \mathbb{C}_3^{(p)} + \mathbb{C}_4^{(p)},$$

$$\mathbb{C}_1^{(p)} = \sum_{i,j=1}^{3} \langle \mathbb{D}_y(\mathbf{U}_p^{ij}) \rangle_{Y_s} \otimes \mathbb{J}^{ij}, \quad \mathbb{C}_2^{(p)} = \langle \mathbb{D}_y(\mathbf{U}_p^{0}) \rangle_{Y_s} \otimes \mathbb{I},$$

$$\mathbb{C}_3^{(p)} = -\sum_{i,j=1}^{3} \langle Q_p^{ij} \rangle_{Y_s} \mathbb{I} \otimes \mathbb{J}^{ij}, \quad \mathbb{C}_4^{(p)} = -\langle Q_p^{0} \rangle_{Y_s} \mathbb{I} \otimes \mathbb{I}. \qquad \square$$

Lemma 4.8. *Tensors* $\mathbb{A}^{(c)}$ *and* $\mathbb{C}^{(p)}$ *are symmetric and the tensor* $\mathbb{A}^{(c)}$ *is strictly positively definite, that is for any arbitrary symmetric matrices* $\zeta = (\zeta_{ij})$ *and* $\eta = (\eta_{ij})$

$$(\mathbb{A}^{(c)} : \zeta) : \eta = (\mathbb{A}^{(c)} : \eta) : \zeta, \quad \text{and} \quad (\mathbb{A}^{(c)} : \zeta) : \zeta \geq \beta(\zeta : \zeta),$$

where positive constant β *is independent of* ζ.

Proof. To prove the lemma we need some properties of the tensor $\mathbb{A}^{(c)}$, which follow from equalities

$$-\langle Q_p^0 \rangle_{Y_s} = \langle \mathbb{D}_y(\mathbf{U}_p^0) : \mathbb{D}_y(\mathbf{U}_p^0) \rangle_{Y_s}, \tag{4.24}$$

$$\langle \mathbb{D}_y(\mathbf{U}_p^{ij}) : \mathbb{D}_y(\mathbf{U}_p^0) \rangle_{Y_s} = 0, \tag{4.25}$$

$$\langle Q_p^{ij} \rangle_{Y_s} = -\langle \mathbb{D}_y(\mathbf{U}_p^0) : \mathbb{J}^{ij} \rangle_{Y_s}, \tag{4.26}$$

$$\langle \mathbb{D}_y(\mathbf{U}_p^{ij}) : \mathbb{D}_y(\mathbf{U}_p^{kl}) \rangle_{Y_s} + \langle \mathbb{J}^{ij} : \mathbb{D}_y(\mathbf{U}_p^{kl}) \rangle_{Y_s} = 0, \tag{4.27}$$

for all $i, j, k, l = 1, 2, 3$.

Equation (4.24) is a corresponding to the first equation in (4.22) integral identity with the test function $\mathbf{U}_p^0$. Equation (4.25) is the corresponding to the first equation in (4.22) integral identity with the test function $\mathbf{U}_p^{ij}$. Equation (4.26) is the corresponding to the first equation in (4.21) integral identity with the test function $\mathbf{U}_p^0$. Here we additionally took into account relations (4.25). Finally, Eq. (4.27) is the corresponding to the first equation in (4.21) integral identity with the test function $\mathbf{U}_p^{kl}$.

Next we put

$$\mathbf{Y}_\zeta = \sum_{i,j=1}^{3} \mathbf{U}_p^{ij} \zeta_{ij}, \quad \mathbf{Y}_\eta = \sum_{i,j=1}^{3} \mathbf{U}_p^{ij} \eta_{ij}, \quad \mathbf{Y}_\zeta^0 = \mathbf{U}_p^0 \operatorname{tr} \zeta, \quad \mathbf{Y}_\eta^0 = \mathbf{U}_p^0 \operatorname{tr} \eta.$$

Then

$$C_1^{(p)} : \zeta = \langle \mathbb{D}_y(\mathbf{Y}_\zeta)\rangle_{Y_s}, \quad C_2^{(p)} : \zeta = \langle \mathbb{D}_y(\mathbf{Y}^0_\zeta)\rangle_{Y_s},$$

and Eqs. (4.24)–(4.27) take a form

$$(C_4^{(p)} : \zeta) : \eta = \langle \mathbb{D}_y(\mathbf{Y}^0_\zeta) : \mathbb{D}_y(\mathbf{Y}^0_\eta)\rangle_{Y_s}, \tag{4.28}$$

$$\langle \mathbb{D}_y(\mathbf{Y}_\eta) : \mathbb{D}_y(\mathbf{Y}^0_\zeta)\rangle_{Y_s} = 0, \tag{4.29}$$

$$(\mathbb{C}_3^{(p)} : \zeta) : \eta = (\mathbb{C}_2^{(p)} : \eta) : \zeta, \tag{4.30}$$

$$(\mathbb{C}_1^{(p)} : \eta) : \zeta + \langle \mathbb{D}_y(\mathbf{Y}_\zeta) : \mathbb{D}_y(\mathbf{Y}_\eta)\rangle_{Y_s} = 0. \tag{4.31}$$

Therefore,

$$(\mathbb{A}^{(c)} : \zeta) : \eta = (1 - m_p)\zeta : \eta + (\mathbb{C}^{(p)} : \zeta) : \eta = \langle \mathbb{D}_y(\mathbf{Y}^0_\eta)\rangle_{Y_s} : \zeta$$
$$+ \langle \mathbb{D}_y(\mathbf{Y}^0_\zeta)\rangle_{Y_s} : \eta + \eta : \langle \mathbb{D}_y(\mathbf{Y}_\zeta)\rangle_{Y_s} + \langle \mathbb{D}_y(\mathbf{Y}^0_\zeta) : \mathbb{D}_y(\mathbf{Y}^0_\eta)\rangle_{Y_s} + (1 - m_p)\zeta : \eta.$$

Taking into account (4.29) and (4.31) we finally get

$$\begin{aligned}(\mathbb{A}^{(c)} : \zeta) : \eta &\\ = (1 - m_p)\zeta : \eta &+ \langle \mathbb{D}_y(\mathbf{Y}^0_\zeta) : \mathbb{D}_y(\mathbf{Y}^0_\eta)\rangle_{Y_s} + \langle \mathbb{D}_y(\mathbf{Y}^0_\eta)\rangle_{Y_s} : \zeta + \langle \mathbb{D}_y(\mathbf{Y}^0_\zeta)\rangle_{Y_s} : \eta \\ &+ \langle \mathbb{D}_y(\mathbf{Y}_\zeta) : \mathbb{D}_y(\mathbf{Y}_\eta)\rangle_{Y_s} + \zeta : \langle \mathbb{D}_y(\mathbf{Y}_\eta)\rangle_{Y_s} + \eta : \langle \mathbb{D}_y(\mathbf{Y}_\zeta)\rangle_{Y_s} \\ = \langle (\mathbb{D}_y(\mathbf{Y}_\zeta + \mathbf{Y}^0_\zeta) &+ \zeta) : (\mathbb{D}_y(\mathbf{Y}_\eta + \mathbf{Y}^0_\eta) + \eta)\rangle_{Y_s}.\end{aligned} \tag{4.32}$$

Equations (4.32) and (4.23) show that tensors $\mathbb{A}^{(c)}$ and $\mathbb{C}^{(p)}$ are symmetric:

$$(\mathbb{A}^{(c)} : \zeta) : \eta = (\mathbb{A}^{(c)} : \eta) : \zeta, \quad (\mathbb{C}^{(p)} : \zeta) : \eta = -(1 - m_p)\zeta : \zeta + (\mathbb{A}^{(c)} : \zeta) : \eta.$$

In particular,

$$(\mathbb{A}^{(c)} : \zeta) : \zeta = \langle (\mathbb{D}_y(\mathbf{Y}_\zeta + \mathbf{Y}^0_\zeta) + \zeta) : (\mathbb{D}_y(\mathbf{Y}_\zeta + \mathbf{Y}^0_\zeta) + \zeta)\rangle_{Y_s} > 0$$

and $\mathbb{A}^{(c)}$ is strictly positively definite. In fact, if $(\mathbb{A}^{(c)} : \zeta^0) : \zeta^0 = 0$ for some ζ^0, such that $\zeta^0 : \zeta^0 = 1$, then

$$\mathbb{D}_y(\mathbf{Y}_{\zeta^0} + \mathbf{X}_{\zeta^0}) + \zeta^0 = 0.$$

The last equality is possible if and only if the periodic function $\mathbf{Y}_{\zeta^0} + \mathbf{Y}^0_{\zeta^0}$ is a linear one. But due to geometry of the solid cell Y_s it is possible only if $\mathbf{Y}_{\zeta^0} + \mathbf{Y}^0_{\zeta^0} = \text{const}$. Therefore $\zeta^0 = 0$, which contradict to supposition. □

Lemma 4.9. *Functions* $\mathbf{u}$ *and* $\tilde{q}_f$ *satisfy a.e. in* G_T *the homogenized equation*

$$\nabla_x \cdot (\mathbb{A}^{(s)} : \mathbb{D}(\mathbf{u}) - \tilde{q}_f \mathbb{I}) = \frac{\hat{\rho}}{\lambda_0}\mathbf{F}, \tag{4.33}$$

where fourth-rank constant tensor $\mathbb{A}^{(s)}$ *is defined below by formula* (4.36).

Proof. Following the standard scheme, we look for the solution to the microscopic equation (4.20) in the form

$$\mathbf{U}_c(\mathbf{x},t,\mathbf{z}) = \sum_{i,j=1}^{3} \mathbf{U}_c^{ij}(\mathbf{z}) D_{ij}(\mathbf{x},t),$$

where functions $\mathbf{U}_c^{ij}$ satisfy in Z the periodic boundary value problem

$$\nabla_z \cdot ((1-\chi_c)\mathbb{A}^{(c)} : (\mathbb{D}_z(\mathbf{U}_c^{ij}) + J^{ij})) = 0, \quad \langle \mathbf{U}_c^{ij} \rangle_{Z_s} = 0, \tag{4.34}$$

which is understood in the sense of distributions. Thus

$$\begin{gathered} \langle D_z(\mathbf{U}_c)\rangle_{Z_s} = \left(\sum_{i,j=1}^{3} \langle D_z(\mathbf{U}_c^{ij})\rangle_{Z_s} \otimes \mathbb{J}^{ij} \right) : D(\mathbf{u}) = \mathbb{C}^{(c)} : D(\mathbf{u}), \\ \mathbb{C}^{(c)} = \sum_{i,j=1}^{3} \langle D_z(\mathbf{U}_c^{ij})\rangle_{Z_s} \otimes \mathbb{J}^{ij}, \end{gathered} \tag{4.35}$$

and

$$\begin{aligned} \langle\langle(\mathbb{D}_y(\mathbf{U}_p) - \tilde{Q}_s\mathbb{I})\rangle_{Y_s}\rangle_{Z_s} &= \mathbb{C}^{(p)} : ((1-m_c)\mathbb{D}(\mathbf{u}) + \langle \mathbb{D}_z(\mathbf{U}_c)\rangle_{Z_s}) \\ &= \mathbb{C}^{(p)} : ((1-m_c)\mathbb{D}(\mathbf{u}) + \mathbb{C}^{(c)} : D(\mathbf{u})) \\ &= \mathbb{C}^{(p)} : (((1-m_c)\mathbb{J} + \mathbb{C}^{(c)}) : D(\mathbf{u})) \\ &= ((1-m_c)\mathbb{C}^{(p)} + \mathbb{C}^{(p)} : \mathbb{C}^{(c)}) : D(\mathbf{u}), \\ \mathbb{A}^{(s)} &= (1-m)\mathbb{J} + (1-m_p)\mathbb{C}^{(c)} + (1-m_c)\mathbb{C}^{(p)} + \mathbb{C}^{(p)} : \mathbb{C}^{(c)} \\ &= (1-m)\mathbb{J} + ((1-m_p)\mathbb{J} + \mathbb{C}^{(p)}) : \mathbb{C}^{(c)} + (1-m_c)\mathbb{C}^{(p)} \\ &= (1-m)\mathbb{J} + \mathbb{A}^{(c)} : \mathbb{C}^{(c)} + (1-m_c)\mathbb{C}^{(p)} \\ &= (1-m_c)((1-m_p)\mathbb{J} + \mathbb{C}^{(p)}) + \mathbb{A}^{(c)} : \mathbb{C}^{(c)} \\ &= (1-m_c)\mathbb{A}^{(c)} + \mathbb{A}^{(c)} : \mathbb{C}^{(c)} = \mathbb{A}^{(c)} : ((1-m_c)\mathbb{J} + \mathbb{C}^{(c)}), \end{aligned}$$

where we have used equalities $(1-m) = (1-m_p)(1-m_c)$ and $\mathbb{J} : \mathbb{A} = \mathbb{A} : \mathbb{J} = \mathbb{A}$ for any fourth-rank tensor $\mathbb{A}$.

Finally

$$\mathbb{A}^{(s)} = \mathbb{A}^{(c)} : ((1-m_c)\mathbb{J} + \mathbb{C}^{(c)}), \tag{4.36}$$

where $\mathbb{C}^{(c)}$ is defined by (4.35). □

Lemma 4.10. *The tensor $\mathbb{A}^{(s)}$ is symmetric and strictly positively definite.*

Proof. To prove the second statement of the lemma we use the equality

$$\int_{Z_s} (\mathbb{A}^{(c)} : \mathbb{D}_z(\mathbf{U}_c^{ij})) : \mathbb{D}_z(\mathbf{U}_c^{kl})dz + \int_{Z_s} (\mathbb{A}^{(c)} : \mathbb{D}_z(\mathbf{J}^{ij})) : \mathbb{D}_z(\mathbf{U}_c^{kl})dz = 0, \tag{4.37}$$

which is just corresponding to Eq. (4.34) integral identity with the test function $\mathbf{U}_c^{kl}$.

Let

$$\mathbf{Z}_\zeta = \sum_{i,j=1}^{3} \mathbf{U}_c^{ij}\zeta_{ij}, \quad \mathbf{Z}_\eta = \sum_{i,j=1}^{3} \mathbf{U}_c^{ij}\eta_{ij}.$$

Then (4.37) take a form

$$\langle(\mathbb{A}^{(c)}:\mathbb{D}_z(\mathbf{Z}_\zeta)):\mathbb{D}_z(\mathbf{Z}_\eta)\rangle_{Z_s}+\langle(\mathbb{A}^{(c)}:\mathbb{D}_z(\mathbf{Z}_\eta)):\zeta\rangle_{Z_s}=0. \tag{4.38}$$

Note also that by definition

$$\mathbb{C}^{(c)}:\zeta=\langle\mathbb{D}_z(\mathbf{Z}_\zeta)\rangle_{Z_s}. \tag{4.39}$$

Relations (4.38) and (4.39) result

$$\begin{aligned}(\mathbb{A}^{(s)}:\zeta):\eta&=(1-m_c)(\mathbb{A}^{(c)}:\zeta):\eta+((\mathbb{A}^{(c)}:\mathbb{C}^{(c)}):\zeta):\eta\\&=(1-m_c)(\mathbb{A}^{(c)}:\zeta):\eta+(\mathbb{A}^{(c)}:\langle\mathbb{D}_z(\mathbf{Z}_\zeta)\rangle_{Z_s}):\eta\\&=(1-m_c)(\mathbb{A}^{(c)}:\zeta):\eta+\langle(\mathbb{A}^{(c)}:\mathbb{D}_z(\mathbf{Z}_\zeta)):\mathbb{D}_z(\mathbf{Z}_\eta)\rangle_{Z_s}\\&\quad+\langle(\mathbb{A}^{(c)}:\mathbb{D}_z(\mathbf{Z}_\eta)):\zeta\rangle_{Z_s}+(\mathbb{A}^{(c)}:\langle\mathbb{D}_z(\mathbf{Z}_\zeta)\rangle_{Z_s}):\eta\\&=\langle(\mathbb{A}^{(c)}:(\mathbb{D}_z(\mathbf{Z}_\zeta)+\zeta)):(\mathbb{D}_z(\mathbf{Z}_\eta)+\eta)\rangle_{Z_s},\end{aligned}$$

which proves the symmetry of $\mathbb{A}^{(s)}$. In particular,

$$(\mathbb{A}^{(s)}:\eta):\eta=\langle(\mathbb{A}^{(c)}:(\mathbb{D}_z(\mathbf{Z}_\eta)+\eta)):(\mathbb{D}_z(\mathbf{Z}_\eta)+\eta)\rangle_{Z_s}>\beta(\eta:\eta).$$ □

5. Proof of Theorem 2

First of all we rewrite the continuity equation in (2.9) and Darcy law (2.11) in the form

$$\nabla\cdot\mathbf{v}_s^{(\lambda_0)}-\frac{1}{m\mu_1}\nabla\cdot(\mathbb{B}^{(c)}\nabla q_f^{(\lambda_0)})=-\rho_f\nabla\cdot(\mathbb{B}^{(c)}\mathbf{F}). \tag{5.1}$$

The correctness (uniqueness and existence of the solution) of the problem (2.9)–(2.12) follows from the basic *a priori* estimate

$$\lambda_0\int_0^t\int_\Omega|\nabla\mathbf{v}_s^{(\lambda_0)}(\mathbf{x},\tau)|^2dxd\tau+\frac{1}{\mu_1}\int_\Omega|\nabla q_f^{(\lambda_0)}(\mathbf{x},t)|^2dx\leq C. \tag{5.2}$$

To derive (5.2) we just multiply (5.1) by $\partial q_f^{(\lambda_0)}/\partial t$, and the first equation in (2.9) by $m\mathbf{v}_s^{(\lambda_0)}$, sum results, integrate by parts over domain Ω. Integral over the boundary $S=\partial\Omega$ vanishes due to boundary condition (2.12). Estimate (5.2) follows now from Hölder, Gronwall and Korn's inequalities. Next we apply the standard compactness results to choose the convergent subsequences of $\{\mathbf{v}_c^{(\lambda_0)}\}$ and $\{q_f^{(\lambda_0)}\}$, and pass to the limit as $\lambda_0\nearrow\infty$ in (2.11) and in the integral identity, corresponding to the continuity equation in (2.9). Estimate (5.2) also guarantees the strong convergence of $\{\mathbf{v}_s^{(\lambda_0)}\}$ to zero as $\lambda_0\nearrow\infty$.

6. Conclusions

We have shown how the new rigorous homogenization methods can be used to clarify the structure of mathematical models for liquid filtration in natural reservoirs with very complicated geometry. Obvious advantage of suggested models are:

(1) their solid physical and mathematical bases — the models are asymptotically closed to trustable mathematical model on the microscopic level;
(2) their clear physical meaning — the choice of the model depends on ratios between physical parameters of a process in consideration;
(3) for most often met situation of disconnected crack space the suggested model is so simple as well as usual Darcy system of filtration, but, in contrast to the last one, its solutions are more regular, that is very important in applications to various nonlinear problems. For example, at the description of replacement of oil by water.

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