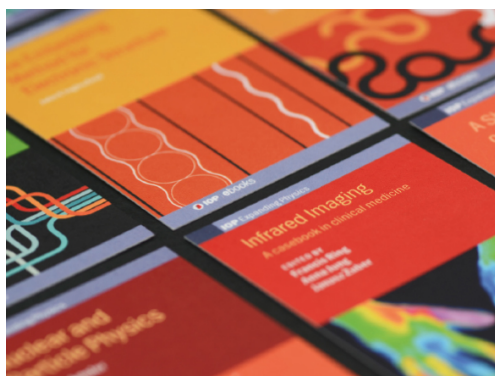


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On elliptic problems in domains with low dimensional boundaries

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Abstract. We study a solvability for some concrete boundary value problems in domains with a low dimensional boundary. Such boundaries are considered as a limit situation of a volume problem for which one or more parameters of a domain tend to their endpoint values. The solvability conditions in Sobolev–Slobodetskii spaces are given using the wave factorization concept for an elliptic symbol.

1. Introduction

Based on the theory of boundary value problems for elliptic pseudo-differential equations on manifolds with a smooth boundary [1], the theory of singular integral equations [2], multidimensional complex analysis [3] and our previous results [4, 5, 6] we develop this direction to more complicated specific situations. This report is a continuation of studies [7, 8, 9] related to the theory of pseudo-differential equations and boundary value problems in special canonical non-smooth domains of \mathbf{R}^m .

2. Equations in cones

2.1. Canonical domains and model operators

Let $D \subset \mathbf{R}^m$ be a domain. A model pseudo-differential operator in a domain D is defined in the following way

$$(Au)(x) = \int_D \int_{\mathbf{R}^m} A(\xi)u(y)e^{i(y-x)\cdot\xi}d\xi dy, \quad x \in D,$$

where the function $A(\xi)$ is called a symbol of the pseudo-differential operator A . We will consider the class of symbols satisfying the condition

$$c_1 \leq |A(\xi)(1 + |\xi|)^{-\alpha}| \leq c_2, \quad \xi \in \mathbf{R}^m.$$

The number $\alpha \in \mathbf{R}$ we call an order of a pseudo-differential operator A .

A domain $D \subset \mathbf{R}^m$ is called a canonical domain if it a cone $C \subset \mathbf{R}^m$ non-including a whole straight line in \mathbf{R}^m .

Moreover, we will consider such symbols $A(\xi)$ which admit the wave factorization [3, 4]

$$A(\xi) = A_{\neq}(\xi) \cdot A_{=}(\xi).$$



where $A_{\neq}(\xi)$ admits analytical continuation into $T(-C^*)$ and $A_{=}(\xi)$ into $T(C^*)$. It is necessary because we consider equations in outer of a cone.

2.2. Solvability for equations in cones

To describe our advances for low dimensional cones we consider the following equation

$$(Au)(x) = 0 \quad x \in \mathbf{R}^3 \setminus \overline{C_+^{ab}}. \quad (1)$$

in Sobolev–Slobodetskii space $H^s(\mathbf{R}^3 \setminus \overline{C_+^{ab}})$, where

$$C_+^{ab} = \{x \in \mathbf{R}^3 : x = (x_1, x_2, x_3), x_3 < a|x_1| + b|x_2|, a, b > 0\}.$$

To present a general solution of the equation (1) for the case $\alpha - s = 1 + \delta, |\delta| < 1/2$ we use some results from [7, 8, 9]. Let us introduce the following one-dimensional singular integral operators [2]

$$(S_1 u)(\xi_1, \xi_2, \xi_3) = v.p. \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{u(\tau, \xi_2, \xi_3) d\tau}{\xi_1 - \tau}, \quad (S_2 u)(\xi_1, \xi_2, \xi_3) = v.p. \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{u(\xi_1, \eta, \xi_3) d\eta}{\xi_2 - \eta}.$$

and write

$$\begin{aligned} A_{\neq}(\xi)\tilde{u}(\xi) &= \tilde{C}_1(\xi_1 - a\xi_3, \xi_2 - b\xi_3) + \tilde{C}_2(\xi_1 - a\xi_3, \xi_2 + b\xi_3) + \\ &\tilde{C}_3(\xi_1 + a\xi_3, \xi_2 - b\xi_3) + \tilde{C}_4(\xi_1 + a\xi_3, \xi_2 + b\xi_3), \end{aligned} \quad (2)$$

where

$$\begin{aligned} \tilde{C}_1(\xi_1 - a\xi_3, \xi_2 - b\xi_3) &= \frac{1}{4}\tilde{c}_0(\xi_1 - a\xi_3, \xi_2 - b\xi_3) - \frac{1}{2}(S_1\tilde{c}_0)(\xi_1 - a\xi_3, \xi_2 - b\xi_3) - \\ &-\frac{1}{2}(S_2\tilde{c}_0)(\xi_1 - a\xi_3, \xi_2 - b\xi_3) + (S_1S_2\tilde{c}_0)(\xi_1 - a\xi_3, \xi_2 - b\xi_3); \\ \tilde{C}_2(\xi_1 - a\xi_3, \xi_2 + b\xi_3) &= \frac{1}{4}\tilde{c}_0(\xi_1 - a\xi_3, \xi_2 + b\xi_3) - \frac{1}{2}(S_1\tilde{c}_0)(\xi_1 - a\xi_3, \xi_2 + b\xi_3) + \\ &+\frac{1}{2}(S_2\tilde{c}_0)(\xi_1 - a\xi_3, \xi_2 + b\xi_3) - (S_1S_2\tilde{c}_0)(\xi_1 - a\xi_3, \xi_2 + b\xi_3); \\ \tilde{C}_3(\xi_1 + a\xi_3, \xi_2 - b\xi_3) &= \frac{1}{4}\tilde{c}_0(\xi_1 + a\xi_3, \xi_2 - b\xi_3) + \frac{1}{2}(S_1\tilde{c}_0)(\xi_1 + a\xi_3, \xi_2 - b\xi_3) - \\ &-\frac{1}{2}(S_2\tilde{c}_0)(\xi_1 + a\xi_3, \xi_2 - b\xi_3) - (S_1S_2\tilde{c}_0)(\xi_1 + a\xi_3, \xi_2 - b\xi_3); \\ \tilde{C}_4(\xi_1 + a\xi_3, \xi_2 + b\xi_3) &= \frac{1}{4}\tilde{c}_0(\xi_1 + a\xi_3, \xi_2 + b\xi_3) + \frac{1}{2}(S_1\tilde{c}_0)(\xi_1 + a\xi_3, \xi_2 + b\xi_3) + \\ &+\frac{1}{2}(S_2\tilde{c}_0)(\xi_1 + a\xi_3, \xi_2 + b\xi_3) + (S_1S_2\tilde{c}_0)(\xi_1 + a\xi_3, \xi_2 + b\xi_3), \end{aligned}$$

where $c_0(x_1, x_2)$ is an arbitrary function from the space $H^{s-\alpha+1/2}(\mathbf{R}^2)$. In other words, a kernel of the operator A is a one-dimensional subspace.

To determine uniquely the arbitrary function $c_0(\xi_1, \xi_2)$ we require certain additional condition. for example, we assume that the restriction $\tilde{u}(\xi_1, \xi_2, 0)$ is given, i.e. the following integral

$$\int_{-\infty}^{+\infty} u(x_1, x_2, x_3) dx_3 \equiv g(x_1, x_2), \quad (3)$$

it gives the equality

$$\tilde{u}(\xi_1, \xi_2, 0) = \tilde{g}(\xi_1, \xi_2). \quad (4)$$

Taking into account $\xi_3 = 0$ in the formula (2) we find

$$\begin{aligned} \sum_{k=1}^4 \tilde{C}_k(\xi_1, \xi_2) = & \\ & \frac{1}{4}\tilde{c}_0(\xi_1, \xi_2) - \frac{1}{2}(S_1\tilde{c}_0)(\xi_1, \xi_2) - \frac{1}{2}(S_2\tilde{c}_0)(\xi_1, \xi_2) + (S_1S_2\tilde{c}_0)(\xi_1, \xi_2) + \\ & + \frac{1}{4}\tilde{c}_0(\xi_1, \xi_2) - \frac{1}{2}(S_1\tilde{c}_0)(\xi_1, \xi_2) + \frac{1}{2}(S_2\tilde{c}_0)(\xi_1, \xi_2) - (S_1S_2\tilde{c}_0)(\xi_1, \xi_2) + \\ & + \frac{1}{4}\tilde{c}_0(\xi_1, \xi_2) + \frac{1}{2}(S_1\tilde{c}_0)(\xi_1, \xi_2) - \frac{1}{2}(S_2\tilde{c}_0)(\xi_1, \xi_2) - (S_1S_2\tilde{c}_0)(\xi_1, \xi_2) + \\ & + \frac{1}{4}\tilde{c}_0(\xi_1, \xi_2) + \frac{1}{2}(S_1\tilde{c}_0)(\xi_1, \xi_2) + \frac{1}{2}(S_2\tilde{c}_0)(\xi_1, \xi_2) + (S_1S_2\tilde{c}_0)(\xi_1, \xi_2) = \tilde{c}_0(\xi_1, \xi_2). \end{aligned}$$

In view of the condition (4) we obtain

$$\tilde{c}_0(\xi') = \tilde{A}_{\neq}(\xi', 0)\tilde{g}(\xi'). \quad (5)$$

Theorem 1. *Let $\varkappa - s = 1 + \delta$, $|\delta| < 1/2$, $g \in H^{s+1/2}(\mathbf{R}^2)$. Then the unique solution of the problem (1),(3) is given by the formula (2), and $c_0(x_1, x_2)$ is determined by the formula (5).*

The proof of Theorem 1 in detail is given in [7, 8].

Further, our main goal is describing behavior of the unique solution of the problem (1),(3) when the parameters a, b tend to their endpoint values, 0 and ∞ . Of course, we will assume that the needed wave factorization exists for enough small (large) values. Let us note that the cases for some small values a, b were studied in [8].

In the next sections we'll describe some domains in Euclidean space \mathbf{R}^m which we can obtain under endpoint values of the parameters. Such domains we call domains with cuts. In the plane case one can obtain only one type of such a domain, but in 3-dimensional space there are a few types.

We will show below certain simple examples. Based on the formula (2) we'll find certain condition for the boundary function g .

2.2.1. A space without a half-plane In the above section the cone had two parameters a and b . It is very interesting case when one of parameters or both tend to 0 or $+\infty$. We would like to remind here that the case $a, b \rightarrow 0$ is considered in the book [1], it corresponds to a half-space. The cases $a \rightarrow 0, b = \text{const}$ and $a = \text{const}, b \rightarrow 0$ are described in [8]. Here we consider the case $a \rightarrow +\infty, b = \text{const}$, it is analogous to the case $a = \text{const}, b \rightarrow +\infty$.

A starting point is the equality (2). We apply change of variables $\xi_1 - a\xi_3 = t_1, \xi_1 + a\xi_3 = t_3$ from which we find $\xi_1 = \frac{t_3+t_1}{2}, \xi_3 = \frac{t_3-t_1}{2a}$. For new variables t_1, ξ_2, t_3 using the condition (4) we can determine the arbitrary function \tilde{c}_0 by the formula (5). We rewrite the formula (2) for new variables t_1, ξ_2, t_3 and get

$$\begin{aligned} A_{\neq} \left(\frac{t_2+t_1}{2}, \xi_2, \frac{t_3-t_1}{2a} \right) \tilde{u} \left(\frac{t_2+t_1}{2}, \xi_2, \frac{t_3-t_1}{2a} \right) = \tilde{C}_1 \left(t_1, \xi_2 - b \frac{t_3-t_1}{2a} \right) + \\ + \tilde{C}_2 \left(t_1, \xi_2 + b \frac{t_3-t_1}{2a} \right) + \tilde{C}_3 \left(t_3, \xi_2 - b \frac{t_3-t_1}{2a} \right) + \tilde{C}_4 \left(t_3, \xi_2 + b \frac{t_3-t_1}{2a} \right), \quad (6) \end{aligned}$$

Tending a to $+\infty$ we obtain the relation

$$A_{\neq} \left(\frac{t_2 + t_1}{2}, \xi_2, 0 \right) \tilde{u} \left(\frac{t_2 + t_1}{2}, \xi_2, 0 \right) = \tilde{C}_1(t_1, \xi_2) + \tilde{C}_2(t_1, \xi_2) + \tilde{C}_3(t_3, \xi_2) + \tilde{C}_4(t_3, \xi_2),$$

Further, we evaluate

$$\begin{aligned} & \tilde{C}_1(t_1, \xi_2) + \tilde{C}_2(t_1, \xi_2) + \tilde{C}_3(t_3, \xi_2) + \tilde{C}_4(t_3, \xi_2) = \\ & = \frac{\tilde{c}_0(t_1, \xi_2) + \tilde{c}_0(t_3, \xi_2)}{2} - (S_1 \tilde{c}_0)(t_1, \xi_2) + (S_1 \tilde{c}_0)(t_3, \xi_2). \end{aligned}$$

Taking into account the condition (4), the formula (5) in the equality (6) and using new notations

$$\tilde{A}_{\neq}(\xi_1, \xi_2, 0) \tilde{g}(\xi_1, \xi_2) \equiv h(\xi_1, \xi_2)$$

we find the following equality (equation) with the parameter ξ_2

$$h \left(\frac{t_2 + t_1}{2}, \xi_2 \right) = \frac{h(t_1, \xi_2) + h(t_3, \xi_2)}{2} - (S_1 h)(t_1, \xi_2) + (S_1 h)(t_3, \xi_2) \tag{7}$$

Therefore, we have the following result.

Theorem 2. *If the symbol $A(\xi)$ admits the wave factorization with respect to the cone C_+^{ab} with index \varkappa such that $\varkappa - s = 1 + \delta, |\delta| < 1/2$, for enough large a then the unique solution of boundary value problem (1), (3) has a limit under $a \rightarrow +\infty$ iff the boundary function $g \in H^{s+1/2}(\mathbf{R}^2)$ is a solution of the equation (7).*

2.3. A plane case

Let us consider here the 2-dimensional case. The equation (1) takes the form

$$(Au)(x) = 0 \quad x \in \mathbf{R}^2 \setminus \overline{C_+^a}. \tag{8}$$

and it will be studied in the space $H^s(\mathbf{R}^2 \setminus \overline{C_+^a})$. As before [4] the notation \varkappa means the index of wave factorization. We assume that condition $1/2 < \varkappa - s < 3/2$, holds, where s is order of the space H^s [4], [1] and a plane sector has the form

$$C_+^a = \{x \in \mathbf{R}^2 : x_2 > a|x_1|, a > 0\}.$$

If the symbol $A(\xi)$ admits the wave factorization [4] then one can show [7] that a general solution of the equation (1) in Sobolev–Slobodetskii space $H^s(\mathbf{R}^2 \setminus \overline{C_+^a})$ has the form

$$\begin{aligned} \tilde{u}(\xi) = & \frac{\tilde{c}_0(\xi_1 + a\xi_2) + \tilde{c}_0(\xi_1 - a\xi_2)}{2A_{\neq}(\xi_1, \xi_2)} + \\ & + A_{\neq}^{-1}(\xi_1, \xi_2) \left(v.p. \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{\tilde{c}_0(\eta) d\eta}{\xi_1 + a\xi_2 - \eta} - v.p. \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{\tilde{c}_0(\eta) d\eta}{\xi_1 - a\xi_2 - \eta} \right), \end{aligned}$$

where c_0 is an arbitrary function from the space $H^{s-\varkappa+1/2}(\mathbf{R})$.

Let us denote

$$v.p. \frac{i}{\pi} \int_{-\infty}^{+\infty} \frac{\tilde{c}_0(\eta) d\eta}{\xi_1 + a\xi_2 - \eta} \equiv \tilde{d}_0(\xi_1 + a\xi_2), \quad v.p. \frac{i}{\pi} \int_{-\infty}^{+\infty} \frac{\tilde{c}_0(\eta) d\eta}{\xi_1 - a\xi_2 - \eta} \equiv \tilde{d}_0(\xi_1 - a\xi_2).$$

Then we obtain

$$\begin{aligned}\tilde{u}(\xi_1, \xi_2) &= \frac{\tilde{c}_0(\xi_1 + a\xi_2) + \tilde{c}_0(\xi_1 - a\xi_2) + \tilde{d}_0(\xi_1 + a\xi_2) - \tilde{d}_0(\xi_1 - a\xi_2)}{2A_{\neq}(\xi_1, \xi_2)} \equiv \\ &\equiv \frac{\tilde{c}(\xi_1 + a\xi_2) + \tilde{d}(\xi_1 - a\xi_2)}{2A_{\neq}(\xi_1, \xi_2)},\end{aligned}\quad (9)$$

We will use the integral condition

$$\int_{-\infty}^{+\infty} u(x_1, x_2) dx_2 \equiv g(x_1). \quad (10)$$

which means $\tilde{u}(\xi_1, 0) = \tilde{g}(\xi)$ for the Fourier images. According to the formula (9) we conclude

$$\frac{\tilde{c}_0(\xi_1)}{A_{\neq}(\xi_1, 0)} = \tilde{g}(\xi_1).$$

Thus, at least formally we can find the function $\tilde{c}_0(\xi_1) = A_{\neq}(\xi_1, 0)\tilde{g}(\xi_1)$. Further, using the formula (9) we find $\tilde{d}_0(\xi_1)$. So, the formula (9) is the solution of the equation (8). Finally, the solution of the equation (8) under the condition (10) will take the form

$$\begin{aligned}\tilde{u}(\xi_1, \xi_2) &= \frac{A_{\neq}(\xi_1 + a\xi_2, 0)\tilde{g}(\xi_1 + a\xi_2) + A_{\neq}(\xi_1 - a\xi_2, 0)\tilde{g}(\xi_1 - a\xi_2)}{2A_{\neq}(\xi_1, \xi_2)} + \\ &+ \frac{1}{2A_{\neq}(\xi_1, \xi_2)} v.p. \frac{i}{\pi} \int_{-\infty}^{+\infty} \frac{A_{\neq}(\eta, 0)\tilde{g}(\eta)d\eta}{\xi_1 + a\xi_2 - \eta} - \frac{1}{2A_{\neq}(\xi_1, \xi_2)} v.p. \frac{i}{\pi} \int_{-\infty}^{+\infty} \frac{A_{\neq}(\eta, 0)\tilde{g}(\eta)d\eta}{\xi_1 - a\xi_2 - \eta}\end{aligned}$$

(one can find some details in [9]).

2.3.1. A plane without a ray Using the notation $a_{\neq}(t_1, t_2) \equiv A_{\neq}(\frac{t_1+t_2}{2}, \frac{t_1-t_2}{2a})$, $\tilde{U}(t_1, t_2) \equiv \tilde{u}(\frac{t_1+t_2}{2}, \frac{t_1-t_2}{2a})$, we can write

$$\begin{aligned}\tilde{U}(t_1, t_2) &= \frac{A_{\neq}(t_1, 0)\tilde{g}(t_1) + A_{\neq}(t_2, 0)\tilde{g}(t_2)}{2a_{\neq}(t_1, t_2)} + \\ &+ \frac{1}{2a_{\neq}(t_1, t_2)} v.p. \frac{i}{\pi} \int_{-\infty}^{+\infty} \frac{A_{\neq}(\eta, 0)\tilde{g}(\eta)d\eta}{t_1 - \eta} - \frac{1}{2a_{\neq}(t_1, t_2)} v.p. \frac{i}{\pi} \int_{-\infty}^{+\infty} \frac{A_{\neq}(\eta, 0)\tilde{g}(\eta)d\eta}{t_2 - \eta}.\end{aligned}$$

Tending $a \rightarrow +\infty$, denoting $A_{\neq}(t, 0)\tilde{g}(t) \equiv G(t)$ and $\lim_{a \rightarrow +\infty} a_{\neq}(t_1, t_2) \equiv h(t_1, t_2)$ we have the following relation

$$\begin{aligned}\tilde{u}\left(\frac{t_1 + t_2}{2}, 0\right) &= \tilde{U}(t_1, t_2) = \frac{A_{\neq}(t_1, 0)\tilde{g}(t_1) + A_{\neq}(t_2, 0)\tilde{g}(t_2)}{2h(t_1, t_2)} + \\ &\frac{1}{2h(t_1, t_2)} v.p. \frac{i}{\pi} \int_{-\infty}^{+\infty} \frac{A_{\neq}(\eta, 0)\tilde{g}(\eta)d\eta}{t_1 - \eta} - \frac{1}{2h(t_1, t_2)} v.p. \frac{i}{\pi} \int_{-\infty}^{+\infty} \frac{A_{\neq}(\eta, 0)\tilde{g}(\eta)d\eta}{t_2 - \eta}.\end{aligned}\quad (11)$$

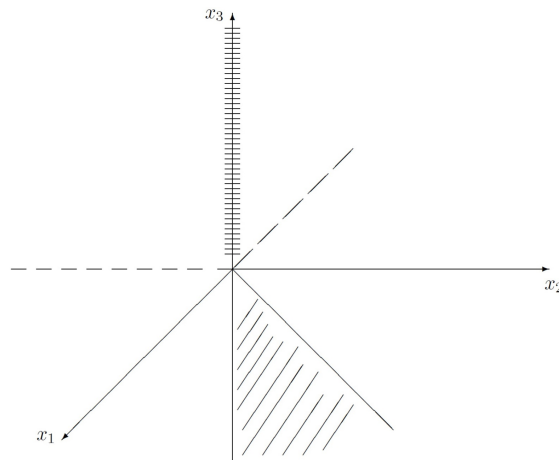


Figure 1. 3D cut in 5D space.

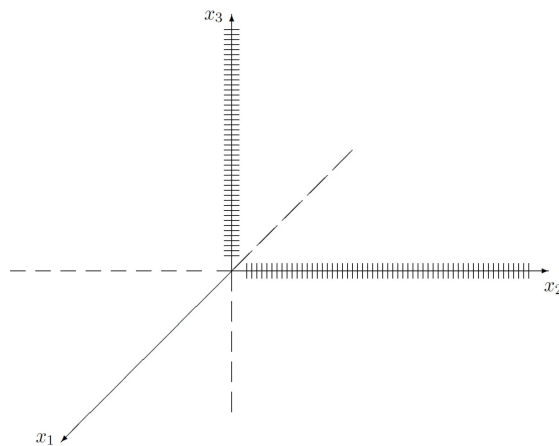


Figure 2. 2D cut in 4D space.

According to the condition (10) we obtain

$$G\left(\frac{t_1 + t_2}{2}\right) = \frac{G(t_1) + G(t_2)}{2} + (SG)(t_1) - (SG)(t_2), \tag{12}$$

where

$$(SG)(t) = v.p. \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{G(\eta)d\eta}{t - \eta}.$$

Final result for a plane case is the following.

Theorem 3. *If the symbol $A(\xi_1, \xi_2)$ admits the wave factorization with respect to the cone C_+^a for enough large a then a limit of the solution (9) exists under $a \rightarrow +\infty$ and the boundary value problem (8),(10) is solvable iff the condition (12) holds.*

3. Elliptic equations in domains with degenerated boundaries

This section is devoted to the boundary problems in which parameters of cones tend to their endpoint values, i.e. 0 and ∞ . Such boundaries we call degenerated or low dimensional

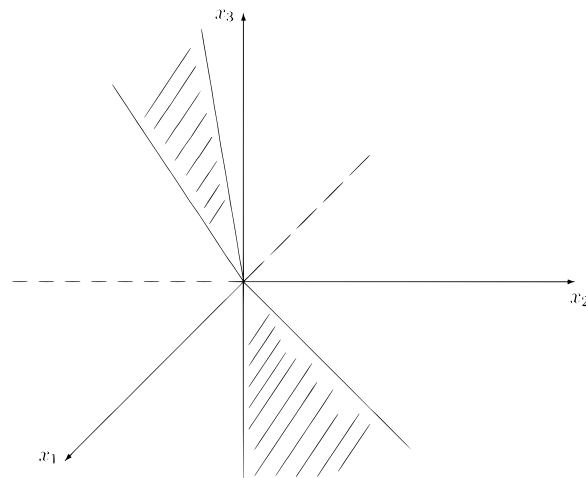


Figure 3. 4D cut in 6D space.

boundaries. Using Theorems 1,2,3 we will construct new domains in a multidimensional space with low dimensional boundaries and will describe their solvability conditions.

A general concept is the following. Let C_1 and C_2 be cones in \mathbf{R}^m and \mathbf{R}^n respectively non-including a whole straight line. Then the set $C_1 \times C_2$ will be a cone in \mathbf{R}^{m+n} non-including a whole straight line from \mathbf{R}^{m+n} . If so, we can consider boundary value problem like (1),(5) in the domain $\mathbf{R}^{m+n} \setminus (C_1 \times C_2)$, Further, if we will obtain a solution formula for this boundary value problem we can manipulate with parameters of cones and consider boundary value problems in domains with low dimensional boundaries. .

3.1. A ray and a plane sector in 5-dimensional space

We formulate here the following boundary value problem

$$\begin{cases} (Au)(x) = 0, & x \in \mathbf{R}^5 \setminus \overline{(C_+^a \times C_+^{bd})} \\ \int_{\mathbf{R}^2} u(x_1, x_2, x_3, x_4, x_5) dx_2 dx_5 = g(x_1, x_3, x_4) \end{cases} \quad (13)$$

where $C_+^a \subset \mathbf{R}^2, C_+^{bd} \subset \mathbf{R}^3$.

Figure 1 presents the corresponding cut in 5-dimensional space, it corresponds to situation in which $a \rightarrow \infty, b \rightarrow 0, d = const$.

3.2. Two rays in 4-dimensional space

This situation is related to the following boundary value problem

$$\begin{cases} (Au)(x) = 0, & x \in \mathbf{R}^4 \setminus \overline{(C_+^a \times C_+^b)} \\ \int_{\mathbf{R}^2} u(x_1, x_2, x_3, x_4) dx_2 dx_4 = g(x_1, x_3) \end{cases} \quad (14)$$

where $C_+^a \subset \mathbf{R}^2, C_+^b \subset \mathbf{R}^2$.

Figure 2 presents the corresponding cut in 4-dimensional space, it corresponds to situation in which $a \rightarrow \infty, b \rightarrow \infty$.

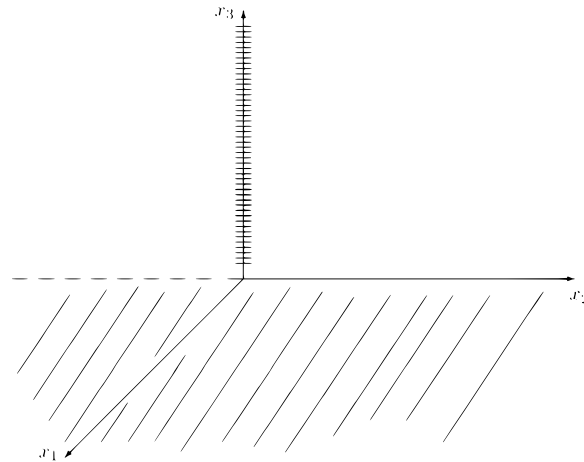


Figure 4. Another 3D cut in 5D space.

3.3. Two plane sectors in 6-dimensional space

We need to work here with two polyhedrons in \mathbf{R}^6 . This case corresponds to the following boundary value problem

$$\begin{cases} (Au)(x) = 0, & x \in \mathbf{R}^6 \setminus \overline{(C_+^{ab} \times C_+^{dl})} \\ \int_{\mathbf{R}^2} u(x_1, x_2, x_3, x_4, x_5, x_6) dx_3 dx_6 = g(x_1, x_2, x_4, x_5) \end{cases} \quad (15)$$

where $C_+^{ab} \subset \mathbf{R}^3$, $C_+^{dl} \subset \mathbf{R}^3$.

Figure 3 presents the corresponding cut in 6-dimensional space, it corresponds to situation in which $a \rightarrow 0, b = \text{const}, d \rightarrow 0, l = \text{const}$.

3.4. A ray and a half-plane in 5-dimensional space

We return here to the boundary value problem (13) and consider the same boundary value problems. Of course, the solution will be the same but for this case we use the following limit $a \rightarrow \infty, b \rightarrow \infty, d = \text{const}$. This cut is shown in figure 4.

All four cases and boundary value problems (13)–(15) can be studied by methods of Section 2 by combining limit situations which were described above. More details and solvability conditions will be given in forthcoming papers.

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