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SOME PROBLEMS OF THE OPTIMAL FAST-ACTIONING OVER SHORE SYSTEM: CLIFF-BEACH

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SYMMARY

The problem of the fast-actioning for the shore system: cliff-beach, which is described with the help of control dynamic system of the second order to is presented in this paper. This task is relevant when artificial stable shore systems on seas and water reservoirs are created and operated.

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Similar problems were discussed in the papers [1, 2, 3] about the optimal control over shore systems of the type: cliff-beach on the basis of artificial regulation of the beach volumes.

In this papers based on the beach-forming material of equations we consider some tasks of the optimal fast-actioning on the transition dynamical system

$$\frac{dV}{dt} = sH\frac{df(W)}{dW}V - \frac{d\varphi(W)}{dW}V + \xi(t)$$

$$\frac{dW}{dt} = V, \quad |\xi(t)| \le \beta \tag{1}$$

from some initial state (W_0, V_0) to the stationary (W_{st}, 0) at the minimal time, where W_{st} is determined from solution of equation

$$aHf(W) - \varphi(W) = 0.$$

Here the first equation of the system (1) is a differential equation of the beach-forming material balance, where W is the beach-forming material volume per unit length of the shore-line, m^2; a = const is a portion of the beach-forming material in the shore-forming bedrock ($0 < a < 1$); H – the cliff height, m; $f(W)$ is the rate of cliff retreat, m/yr; $\varphi(W)$ is the intensity of the beach-forming material attrition due to the wave action, m^2/yr; $\xi(t) = \frac{du(t)}{dt}$; $u(t)$ is the intensity of artificial delivery of material ($u > 0$) or material removal ($u < 0$), m^2/yr; t is time, yr.

In the above mentioned papers we considered system of equations (1) with linear functions $f(W)$, $\varphi(W)$. As according to the principle of Pontrjagin [4] the synthesis of optimal controls is constructed based on solutions systems of equations (1) at $\xi(t) = \pm\beta$, so according to the dependence $\xi(t)$ to u(t) the researching controls $u(t)$ we are obtained in the class of a linear time functions (in the class of the equal-acceleratial or equal-delay deliveries or removals).

On the other hand a class optimal fast-actioning task directly for the equation of the beach-forming material balance exists in the class of the constants on an intensity control operations on transition shore system from a given state $(W, S) = (W_0, 0)$ to the stationary with a given value the cliff retreat $(W, S) = (W_{st}\ S_1)$. A practical approach of the considering this task is given. Let us take as the function $f(W)$ the most universal approximation function

$$f(W) = \frac{B(W+\varepsilon)}{(W+r)^2},$$

where B, ε, r = const > 0 [5] and as the function $\varphi(W)$ a linear law of attrition $\varphi(W) = KW$ (most wide-spread), then system (1) can be written in the form

$$\frac{dW}{dt} = \frac{aHB\,(W+\varepsilon)}{(W+r)^2}KW + u(t), \quad \frac{dS}{dt} = \frac{B(W+\varepsilon)}{(W+r)^2}, \tag{2}$$

$$|u(t)| \le \beta = const.$$

This system of the equations, as shown above, is transferred from the state $(W_0, 0)$ to the state (W_{st}, S_1) at the minimal time. The restrictions are on the phasal variables: $0 \le S \le S_1$, $W \ge 0$.

By using Pontrjagin's maximum principle for system of equations (2) (the Hamiltonian of the system) the presence of at most one switching point and bit-constancy of the control: $u = \pm\beta$ is shown.

Thus synthesis of the optimal controls for system of equations (2) is constructed based on its solutions in the phasal plane (W, S) when $u(t) = \beta$ and $u(t) = -\beta$. Dividing the first equation by the second of the system (2)

$$\frac{dW}{dS} = aH - \frac{KW(W+r)^2}{B(W+r)} \pm \frac{\beta(W+r)^2}{B(W+\varepsilon)}. \tag{3}$$

The integral of this equation has the form

$$\int \frac{B(W+\varepsilon)\,dW}{KW^3 + (2rK \pm\beta)W^2 + (Kr^2 - aHB \pm 2r\beta)W - aHB\varepsilon \pm \beta r^2} = -S + C, \tag{4}$$

where C – integration constant, sign plus corresponds of material removal ($u(t) = -\beta$ in the first equation of the system (2)).

The application of Pontrjagin's maximum principle reduces to analogous situation and in the case of nonlinear law of attrition

$$\varphi(W) = \frac{CW}{\gamma_1 + W} \qquad \text{where} \quad C, \gamma_1 = const > 0.$$

The solutions (4) essentially depends from the presence and lies on the axis W of the roots of cubic equation (the denominator of the integrand). Let us consider the case of existence of local function maximum $aHf(W)$ when $W > 0$, it is equal to $\frac{aHb}{4(r-\varepsilon)}$ and it is achieved when $W = r - 2\varepsilon > 0$. The roots of the cubic equation correspond to the crossing curve $Z(W) = aHf(W)$ and the right line $Z(W) = KW - u(t)$. Consider the case when crossing this lines (at material removal $u(t) = -\beta$) in two points of positive domain: $\overset{(1)}{W_\beta} > \overset{(2)}{W_{-\beta}} > 0$, $\overset{(3)}{W_{-\beta}} < 0$. Numeration of the roots from the right to the left, index „–β" corresponds to the control $u(t) = -\beta$. We shall have a unique positive root of the cubic equation (the rest roots are negative) under positive control, moreover $\overset{(1)}{W_\beta} > \overset{(1)}{W_{-\beta}}$. It is obvious, that stationary point W_{st}, which corresponds to the crossing curve $Z = aHf(W)$ and the right line $Z = KW$, lies in the interval $\overset{(1)}{W}_{-\beta} < W_{st} = \overset{(1)}{W}_{u=0} < \overset{(1)}{W}_\beta$. For such disposition of the roots of the cubic equation at $u(t) = \pm\beta$ the qualitative picture of the behaviour solutions (4) is given *fig. 1.* Here a part of the trajectories is shown in negative domaines of the phasal coordinates. The points of the local minimum of curves at $u = \beta$ coincide with the points of the local maximum of curves at $u = -\beta$ $(W = -\varepsilon)$.

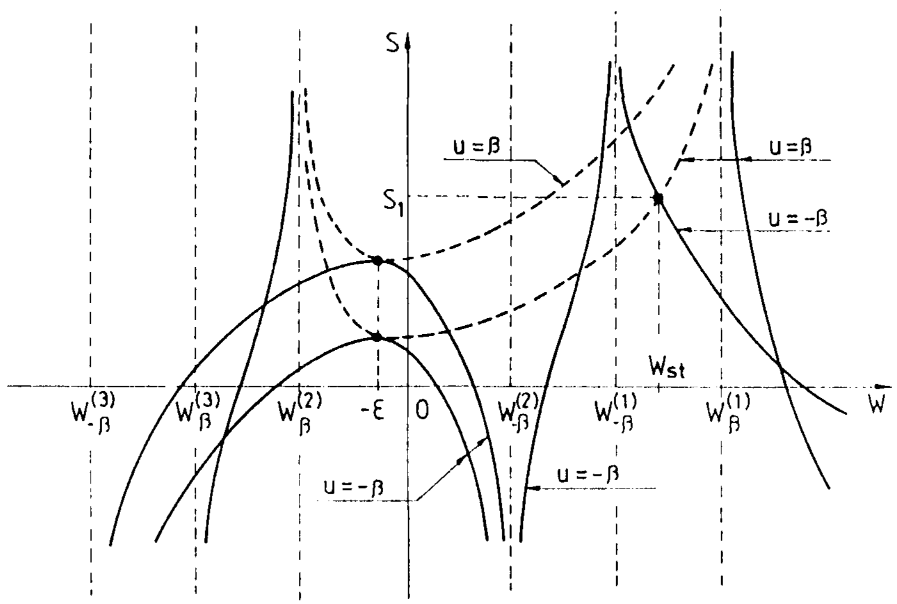

Figure 1.
The qualitative picture of the behavior solutions (4) in the phasal plane (W, S)

In the case $W^{(1)}_{-\beta} = W^{(2)}_{-\beta}$ picture of the character of the curves at $u = \beta$ does not change, two different branches of the curves at $u = -\beta$ in the interval $W^{(3)}_{-\beta} < W < W^{(1)}_{-\beta}$ transform into one branch in this interval with the vertical asymptotes $W = W^{(3)}_{-\beta}$ and $W = W^{(1)}_{-\beta}$, while this branch is monotone increasing function $S(W)$ in this interval with the point of inflection. If now $W^{(2)}_{-\beta} < \varepsilon$, $W^{(1)}_{\beta} > W^{(1)}_{-\beta} > 0$, so this branch transforms – over the interval $W^{(2)}_{-\beta} < W < W^{(1)}_{-\beta}$ – into a curve with the local minimum $W = -\varepsilon$ and with the vertical asymptotes, which are the borders of this interval.

Let us consider in detail the case corresponding fig. 1. and construct synthesis of the optimal controls in the positive domain $W, S \geq 0$. The curve at u = β is taken as the switching line (it is passing through a given final point (W_{st}, S_1)). The equation of this curve follows from the integral (4).

$$S = -\frac{B}{K}\left[A'ln\,|W - W^{(1)}_\beta| + B'ln\,|W - W^{(2)}_\beta| + C'ln\,|W - W^{(3)}_\beta|\right] + C_1$$

$$C_1 = S_1 + \frac{B}{K}\left[A'ln\,|W_{st} - W^{(1)}_\beta| + B'ln\,|W_{st} - W^{(2)}_\beta| + C'ln\,|W_{st} - W^{(3)}_\beta|\right], \tag{5}$$

where A', B' and C' are constants.

The coordinate of the crossing point of the switching line with the axis ordinates (at W = 0) has a form

$$S(0) = S_1 + \frac{B}{K}\left[A'ln\,\left|\frac{W_{st} - W^{(1)}_\beta}{W^{(1)}_\beta}\right| + B'ln\,\left|\frac{W_{st} - W^{(2)}_\beta}{W^{(2)}_\beta}\right| + C'ln\,\left|\frac{W_{st} - W^{(3)}_\beta}{W^{(3)}_\beta}\right|\right]. \tag{6}$$

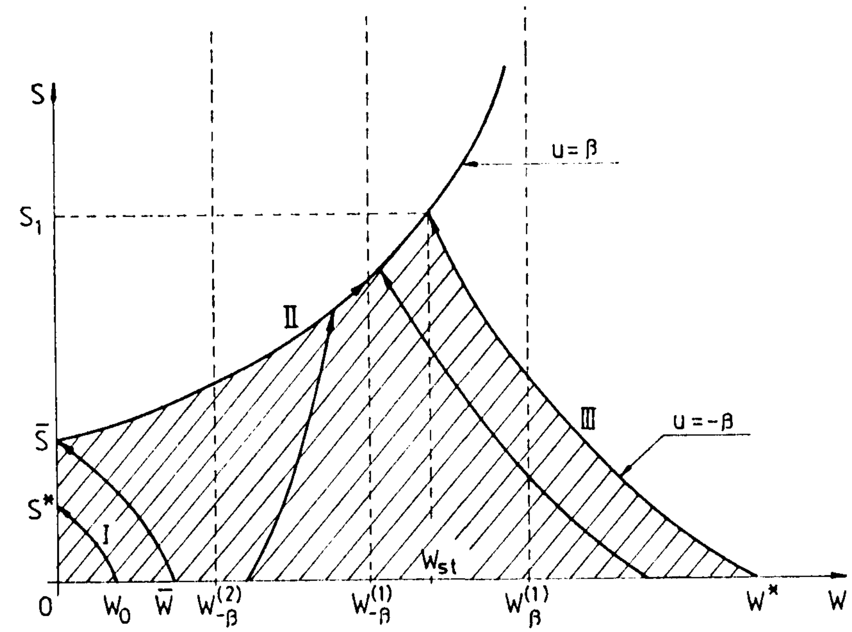

Figure 2.
The qualitative synthesis of the optimal controls for the dynamic system (2) for one of out of the cases

Here the most interesting case is $S > 0$ *fig. 2,* when it is possible a movement along the border of the right line at $W = 0$, $0 < S < \bar{S}$. It will take place at $W(0) = W_0 \leq \overline{W}$, where $\overline{W}$ corresponds to the phasal trajectory hitting in the point $(W = 0, S = \overline{S})$. The phasal trajectory issuing from the point $(W_0, 0)$ (cur. I) is described by equation

$$s(w) = -\frac{B}{K}\left[A_1 \ln|W - W^{(1)}_{-\beta}| + B_1 \ln|W - W^{(2)}_{-\beta}| + C_1 \ln|W - W^{(3)}_{-\beta}|\right] + C(W_0)\,, \quad (7)$$

where $C(W_0)$ is given by condition $S(W_0) = 0$.

Our searching region of the synthesis of the optimal controls is restricted by the following curves (fig. 2):

1. $0 \leq S \leq \overline{S}$, $W = 0$ is the right line of segment;
2. the curve II – the switching line (the form (5));
3. the curve III is described by equation (7), in which $C(W_0)$ is given by condition $S(W_{st}) = S_1$; the value W^* (fig. 2) is given by condition $S(W) = 0$;
4. $S = 0$, $0 \leq W \leq W^*$ is the right line of segment.

We can hit in a given point (W_{st}, S_1) from any point of the last segment. The curve I of crossing with the axis ordinates (S^*) can be determined from expression (7): $S(W = 0) = S^*$, where $C(W_0)$ is obtained from condition $S(W_0) = 0$. The phasal trajectories issuing from the points lied on the segment $S = 0$, $\overline{W} < W < W^*$ are obtained by analogy to the equation of curve I (expression (7)). The coordinates of the switching points can be determined with the help of solution transcendent equation, as the curve of crossing phasal with the switching line. We can write a general time solution to obtain the optimal times of the movement by considering curves:

$$C_2 + t = \int \frac{(W + r)^2 dW}{KW^3 + (2rK \pm\beta)W^2 + (Kr^2 - aHB \pm 2r\beta)W - aHB\varepsilon \pm\beta r^2}\,, \quad (8)$$

where sign plus, an before, corresponds to material removal $(u = -\beta)$, C_2 is a constant of integration.

The case $(W^{(1)}_{-\beta} > W^{(2)}_{-\beta} > 0,\ W^{(3)}_{-\beta} < 0)$ is obtained following time solution

$$t = -\frac{l}{K}\left[A^* \ln|W - W^{(1)}_{-\beta}| + B^* \ln|W - {}^{(2)}_{-\beta}| + C^* \ln|W - W^{(3)}_{-\beta}|\right] + C_2\,, \quad (9)$$

where A^*, B^* and C^* are constants.

The time of the transition from the initial point $(W_0, 0)$ to the switching point (W_{sw}, S_{sw}) can be determined from solution (8). Here C_2 is obtained from condition $t = 0$ at $W = W_0$, then searching time $t_{W_o, W_{sW}}$ is obtained from the form (9) when $W = W_{sw}$.

The time of the transition from the switching point to the final point is found by analogy (C_2 can be determined from condition $t = 0$ at $W = W_{sw}$, then $t_{W_{sW}, W_{st}}$ is obtained by substituting $W = W_{st}$ in the form (9)).

The general time of the transition from the point $(W_0, 0)$ has the form

$$T = t_{W_O, W_{W_s}} + t_{W_{sW}, W_{st}}\,.$$

Thus the value W_{sw} can be determined from the solution corresponding transcendent equation, as the phasal trajectory of the crossing at $u = -\beta$ with the switching line.

The solution (9) has the qualitative form shown at the *fig. 3*. We see that level $W^{(1)}_{-\beta}$ is stable, level $W^{(2)}_{-\beta}$ is unstable.

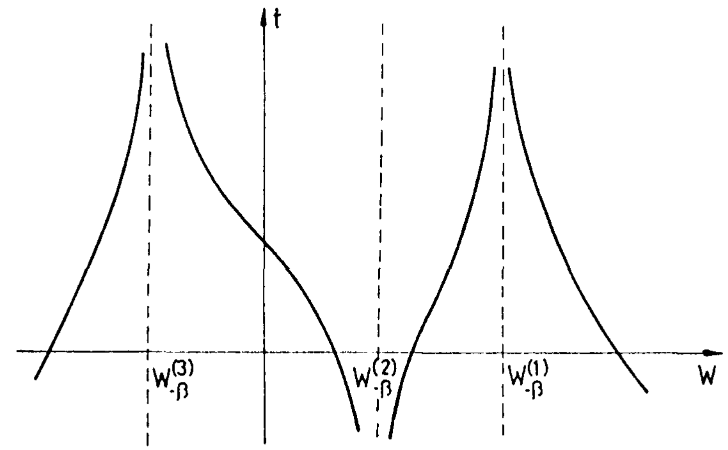

Figure 3.
The qualitative picture of the behavior times solutions (9)

The movement along the border $W = 0$, $0 < S < \overline{S}$ is described by equation $\frac{dS}{dt} = \frac{B\varepsilon}{r^2}$, it follows

$$S(t) = S^* + \frac{B\varepsilon}{r^2}t\,.$$

In this case we obtain a concrete value of the control factor from the balance equation for beach-forming material at $W = 0$.

$$\frac{dW}{dt} = \frac{aHB\,\varepsilon}{r^2} + u = 0 \ , \quad \text{its follows } u = -\frac{aHB\,\varepsilon}{r^2}$$

(removal of material), then the time of the movement along the border is equal to

$$t_{S^*,S} = \frac{S - S^*}{B\,\varepsilon}\, r^2$$

The second task of optimal control with the phasal restrictions occurs with linear function $f(W) = \gamma(W_m - W)$ and as the function $\varphi(W)$ a nonlinear law of attrition $\varphi(W) = \frac{CW}{\gamma_1 + W}$, where γ, W_m, C, γ_1 = const > 0, $0 \le W \le W_m$.

The initial system of equations has a form

$$\frac{dW}{dt} = aH\,\gamma\,(W_m - W) - \frac{CW}{\gamma_1 + W} + u(t) \qquad (10)$$

It is necessary to determine the transition of this system from the state $(W_0, 0)$ to the state (W_{st}, S_1) at minimal time.

At once the stationary volume of material is obtained from the first equation of the system at $u(t) = 0$, $\frac{dW}{dt} = 0$.

$$W_{st} = \frac{1}{2}\left(W_m - \gamma_1 - \frac{C}{aH\,\gamma}\right) + \frac{1}{2}\sqrt{\left(W_m - \gamma_1 - \frac{C}{aH\,\gamma}\right)^2 + 4\,W_m\,\gamma_1}\ , \qquad (11)$$

while the inequality $0 \le W_{st} \le W_m$ always holds.

In the paper it is shown, that the alternatival synthesis exists for both considered tasks, when the trajectory at $u = -\beta$ is taken as the switching line. The most optimal synthesis have been chosen taking into account the general time and the delivery and the removal of the volume of the material. For instance, this synthesis would less expedient for the second problem, since there would have been there much more of delivery of material volume.