Z. Geomorph. N. F.	Suppl. Bd. 25	106–109	Berlin · Stuttgart	September 1976

Some problems of the theory of moving water along a slope

by

A. M. TROFIMOV and V. M. MOSKOVKIN, Kazan

Zusammenfassung. In diesem Aufsatz wird die Definition eines stabilen Gleichgewichtsprofils unter dem Einfluß der Hangspülung untersucht sowie die Struktur eines mathematischen Modells, das die Rolle der den Hang bedeckenden Materialkorngrößen mit berücksichtigt.

Summary. This paper deals with the problem of a stable equilibrium profile that is being subjected to the action of sheet wash. In addition, it considers the structure of a mathematical model for debris on a slope.

Résumé. Cette étude traite le problème d'un profil en équilibre stable qui est soumis à l'action du ruissellement en nappe. En plus, il étudie la structure d'un modèle mathématique à propos du débris sur un versant.

First we examine the problem of defining the stable equilibrium profile of a slope subjected to sheet flood erosion.

The analogous problem for the equilibrium form of a sea shore was solved by S. S. GRIGORYAN (1965). We may consider a slope to be made of similar fragments in a composition and dimension of loose material.

The equation for thin film water movement in general coordinates is

(1) $$V \frac{dV}{dS} = g \sin \alpha - \frac{KV^2}{h},$$

where the coordinate S is parallel to the slope surface.

The velocity V is taken as the average over the flow depth (hydrological set of a problem). h is the average depth of water flow.

We take the coefficient for the force of flow friction as (GRIGORYAN 1965):

(2) $$K = \frac{AC_x}{2(Md_1)^2},$$

where A is the exposed area of the particle major axis Md_1 taking part in the formation of a flow friction force, and C_x is the drag coefficient.

Taking into account that the particle protrudes less than completely from among its neighbours, we write the expression for A

(3) $$A = \frac{1}{4} \pi \, (Md_1) \, (Md_2) \, \xi; \text{ for some } \xi < 1$$

where Md_2 is the median value for the small axis of a fragment. The equilibrium particle equation on a slope has the form

(4) $$CV^2 - mgf\cos \alpha + mg\sin \alpha = 0$$

where f is the tangent of the angle of friction.

Combined with equation (1) it defines the stable equilibrium profile of the slope. In equation (4)

(5) $$C = \frac{1}{2} \varrho A C_x$$

(6) $$m = \frac{1}{6} \pi \, (Md_1)^2 \, (Md_2) \, (\varrho_1 - \varrho),$$

and ϱ, ϱ_1 are respectively the density of loose material and water.

From (4) we obtain:

(7) $$V^2 = \tau g(f\cos \alpha - \sin \alpha), \text{ where } \quad \tau = \frac{m}{c}.$$

Writing equation (1) in the form:

(8) $$\frac{d(V^2)}{dS} = 2 \, g\sin \alpha - \frac{2K}{h} V^2;$$

and putting (7) into (8) we obtain equation (9) after integration:

(9) $$\int_{\alpha_0}^{\alpha} \frac{\cos \alpha + f\sin \alpha}{f\cos \alpha - (\frac{h}{K\tau} + 1) \sin \alpha} \, d\alpha = \frac{2K}{h} S \qquad \text{where } \tan \alpha_0 = f$$

Assuming that at

$$S \to \infty \quad \frac{dV}{dS} \to 0 \quad \begin{cases} \lim\limits_{S \to \infty} \sin \alpha = \sin \alpha \infty \\ \lim\limits_{S \to \infty} \cos \alpha = \cos \alpha \infty \end{cases}$$

From (7) and (8) we obtain:

(10) $$\tan \alpha \infty = \frac{f}{\frac{h}{\tau K} + 1}$$

Substituting (10), the integral (9) may be we written as:

(11) $$\frac{\sin \alpha \infty}{\sin \alpha_0} \int_{\alpha_0}^{\alpha} \frac{\cos \alpha_0 \cos \alpha + \sin \alpha_0 \sin \alpha}{\cos \alpha \sin \alpha_\infty - \cos \alpha_\infty \sin\alpha} \cdot d\alpha = \frac{2K}{h} \cdot S$$

Making some changes in the expression within the integral using known trigonometrical formulas, we write:

$$\cos (\alpha - \alpha_0) = \cos [(\alpha - \alpha_\infty) + (\alpha_\infty - \alpha_0)] = \cos (\alpha - \alpha_\infty) \cos (\alpha_\infty - \alpha_0) - \sin (\alpha - \alpha_\infty) \sin (\alpha_\infty - \alpha_0),$$

giving the final solution of the equation after integration:

(12) $$\frac{\sin\alpha_\infty}{\sin \alpha_0} \left\{ \cos (\alpha_\infty - \alpha_0) \ln \left[\frac{\sin (\alpha_0 - \alpha_\infty)}{\sin (\alpha - \alpha_\infty)} \right] - (\alpha - \alpha_0) \sin (\alpha_\infty - \alpha_0) \right\} = \frac{2K}{h} \cdot S$$

The expression (12) will define the stable equilibrium profile of slope subjected to the action of sheet flood erosion. It follows from (10) that $\tan \alpha_0 > \tan \alpha\infty$, and that this profile is concave.

For structuring more complex theories of water flow along the slope in equation (1), h may be considered as changing along the slope. Then it is necessary to use the storage equation.

Taking onto the account the differences of grain size along the slope we may conclude that $K = K(S)$. In the storage equation water infiltration may be taken into account which is related to the differences in debris grain size through the equation:

(13) $$\frac{d(Vh)}{dS} = J(S)$$

where $J(S)$ represents the rate of water infiltration.

Setting $\sin \alpha$ as a known function of S

(14) $$\sin \alpha = \varphi(S)$$

Then the equation system defining V and h will be the general nonstationary equations:

(15) $$\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial S} = \mathrm{gf}(S) - \frac{K(S) V^2}{h}$$

$$\frac{\partial h}{\partial t} + \frac{\partial (Vh)}{\partial S} = J(S)$$

Together with equation (1) for water film movement it is possible to find out the equation for fragments movement along the slope under the action of water, i.e. to structure the mathematical model of fragment drag along the slope. The equation of motion for one fragment we can write as (SCHEIDEGGER 1964):

(16) $$\frac{1}{6} \pi d^3 \delta \cdot V_s \frac{dV_s}{d_s} = C_a \frac{\pi}{8} d^2 \varrho (V - V_s)^2 + \frac{1}{6} \pi d^3 (\delta - \varrho) g \cdot \sin \alpha - \frac{1}{6} \pi d^3 (\delta - \varrho) g \cos \alpha \cdot f.$$

A. E. SCHEIDEGGER considered only the first part of this equation. Here δ and φ are respectively the fragment and water density; d is the diameter of fragment; C_a is the hydraulic drag coefficient. The equations (1) and (16) form the complete system for definition of V_s for the velocity of fragment and water (V) movement

(17) $$V \frac{dV}{dS} = g \sin \alpha - \frac{K}{h} V^2$$

$$V_s \frac{dV}{dS} = C_{a\varrho} \frac{3}{4} \frac{1}{d\delta} (V - V_s)^2 + (1 - \frac{\varrho}{\delta}) g \sin \alpha - g(1 - \frac{\varrho}{\delta}) f \cos \alpha.$$

Subtracting the second equation from the first one of (17) we obtain:

(18) $$\frac{1}{2} \frac{d(V - V_s)^2}{dS} = - \frac{K}{h} V^2 - C_{a\varrho} \frac{3}{4} \frac{1}{d\delta} (V - V_s)^2 + \frac{\varrho}{\delta} g \sin \alpha + g(1 - \frac{\varrho}{\delta}) f \cos \alpha.$$

Assuming the slope profile as $\sin \alpha = \varphi(S)$ and solving the first equation of the system (17) one gets V as the function S. Transforming the variable in (18) to

(19) $$(V - V_s)^2 = y,$$

we get a general differential equation for obtaining y and consequently V_s,

(20) $$\frac{1}{2} \cdot \frac{dS}{dy} = - C_{a\varrho} \frac{3}{4d\delta} y + M(S)$$

where $M(S)$ is known as function S.

Thus we have the solution of the problem of obtaining the debris velocity, in a surface water flow over the given slope.

Bibliography

GRIGORJAN C. C. (1965): Ob ustojtschivoj forme ravnovesija galetschnogo berega, podverzsiennogo dejstviju priboja. – Prikladnaja matematika i mechanika, **29,** 2.
SCHEIDEGGER A. E. (1964): Teoretitscheskaja geomorfologija. – Izd - Vo "Progress", 1964.