DERIVATION OF EQUATIONS OF SEISMIC AND ACOUSTIC WAVE PROPAGATION AND EQUATIONS OF FILTRATION VIA HOMOGENIZATION OF PERIODIC STRUCTURES

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Abstract. A linear system of differential equations describing a joint motion of elastic porous body and fluid occupying porous space is considered. Although the problem is linear, it is very hard to tackle due to the fact that its main differential equations involve nonsmooth oscillatory coefficients, both big and small, under the differentiation operators. The rigorous justification, under various conditions imposed on physical parameters, is fulfilled for homogenization procedures as the dimensionless size of the pores tends to zero, while the porous body is geometrically periodic. As the results for different ratios between physical parameters, we derive Biot's equations of poroelasticity, a system consisting of nonisotropic Lamé's equations for the solid component and acoustic equations for the liquid component, nonisotropic Lamé's equations or equations of viscoelasticity for one-velocity continuum, decoupled system consisting of Darcy's system of filtration or acoustic equations for the liquid component (first approximation) and nonisotropic Lamé's equations for the solid component (second approximation), a system consisting of nonisotropic Stokes equations for the liquid component and acoustic equations for the solid component, nonisotropic Stokes equations for one-velocity continuum, or, finally a different type of acoustic equations for one- or two-velocity continuum. The proofs are based on Nguetseng's two-scale convergence method of homogenization in periodic structures.

Introduction

In this article, a problem of modelling of small perturbations in elastic deformable medium, perforated by a system of channels (pores) filled with liquid or gas, is considered. Such media are called *elastic porous media* and they are a rather good approximation to real consolidated grounds. In present-day literature, the field of study in mechanics corresponding to these media is called *poromechanics* [1]. The solid component of such a medium has the name *skeleton*, and the domain that is filled with a fluid is named a *porous space*. The exact mathematical model of elastic porous medium consists of the classical equations of momentum and mass balance, which are stated in Euler variables, of the equations determining stress fields in both solid and liquid phases, and of an endowing relation determining the behavior of the interface between liquid and solid components. The latter relation expresses the fact that the interface is a material surface, which amounts to the condition that it consists of the same material particles all the time. Denoting by ρ the density of medium, by $\mathbf{v}$ the velocity, by $\mathbb{P}^f$ the stress tensor in the liquid component, by $\mathbb{P}^s$ the stress tensor in the rigid skeleton, and by $\tilde{\chi}$ the characteristic (indicator) function of porous space, we write the fundamental differential equations of the nonlinear model in the form

$$\rho \frac{d\boldsymbol{v}}{dt} = \operatorname{div}_x\{\tilde{\chi}\mathbf{P}^f + (1-\tilde{\chi})\mathbf{P}^s\} + \rho\mathbf{F}, \quad \frac{d\rho}{dt} + \rho \cdot \operatorname{div}_x \boldsymbol{v} = 0, \quad \frac{d\tilde{\chi}}{dt} = 0,$$

where d/dt stands for the material derivative with respect to the time variable.

Clearly the above stated original model is a model with an unknown (free) boundary. The more precise formulation of the nonlinear problem is not the focus of our present work. Instead, we aim to study the problem linearized at the rest state. In continuum mechanics the methods of linearization are developed rather deeply. The so-obtained linear model is a commonly accepted and basic one for description of filtration and seismic acoustics in elastic porous media (see, e.g., [2–4]). In this model, the characteristic function of the porous space $\tilde{\chi}$ is a known function for $t > 0$. It is assumed that this function coincides with the characteristic function of the porous space $\bar{\chi}$ given at the initial moment. In dimensionless variables (without primes)

$$\mathbf{x}' = L\boldsymbol{x}, \quad t' = \tau t, \quad \boldsymbol{w}' = L\boldsymbol{w}, \quad \rho_s' = \rho_0\rho_s, \quad \rho_f' = \rho_0\rho_f, \quad \mathbf{F}' = g\mathbf{F}$$

differential equations of the problem in a domain $\Omega \in \mathbb{R}^3$ for the dimensionless displacement vector $\mathbf{w}$ of the continuum medium have the form

$$\alpha_\tau \bar{\rho} \frac{\partial^2 \boldsymbol{w}}{\partial t^2} = \operatorname{div}_x \mathbf{P} + \bar{\rho}\boldsymbol{F}, \tag{0.1}$$

$$p = -\alpha_p \bar{\chi} \operatorname{div}_x \boldsymbol{w}. \tag{0.2}$$

Here the stress tensor of whole continuum

$$\mathbf{P} = \bar{\chi}\mathbf{P}^f + (1 - \bar{\chi})\mathbf{P}^s$$

coincides with an elastic stress tensor

$$\mathbf{P}^s = \alpha_\lambda D(x, \boldsymbol{w}) + \alpha_\lambda (\operatorname{div}_x \boldsymbol{w})\mathbb{I}$$

in the solid skeleton ($\mathbb{I}$ is a spherical tensor) and with a viscous tensor

$$\mathbf{P}^f = \alpha_\mu D\Big(x, \frac{\partial \boldsymbol{w}}{\partial t}\Big) - \Big(p - \alpha_\nu \operatorname{div}_x \frac{\partial \boldsymbol{w}}{\partial t}\Big)\mathbb{I}$$

in the porous space and

$$\bar{\rho} = \bar{\chi}\rho_f + (1 - \bar{\chi})\rho_s, \quad D(x, \boldsymbol{u}) = \frac{1}{2}\big(\nabla_x \boldsymbol{u} + (\nabla_x \boldsymbol{u})^{\mathrm{T}}\big).$$

The dimensionless constants α_i $(i = \tau, \nu, \ldots)$ are defined by the formulas

$$\alpha_\tau = \frac{L}{g\tau^2}, \quad \alpha_\nu = \frac{\nu}{\tau L g \rho_0}, \quad \alpha_\mu = \frac{2\mu}{\tau L g \rho_0}, \quad \alpha_p = \frac{c^2 \rho_f}{Lg}, \quad \alpha_\eta = \frac{\eta}{L g \rho_0}, \quad \alpha_\lambda = \frac{2\lambda}{L g \rho_0},$$

where μ is the viscosity of fluid (gas), ν is the bulk viscosity of fluid (gas), λ and η are elastic Lamé's constants, c is the speed of sound in fluid (gas), L is the characteristic size of the domain in consideration, τ is the characteristic time of the process, ρ_f and ρ_s are respectively mean dimensionless densities of liquid and rigid phases, correlated with mean density of water, and g is the value of acceleration of gravity.

The problem is endowed with homogeneous initial and boundary conditions

$$\boldsymbol{w}|_{t=0} = 0, \quad \frac{\partial \boldsymbol{w}}{\partial t}\Big|_{t=0} = 0, \quad \boldsymbol{x} \in \Omega, \tag{0.3}$$

$$\boldsymbol{w} = 0, \quad \boldsymbol{x} \in S = \partial\Omega, \quad t \geq 0. \tag{0.4}$$

The corresponding mathematical model, described by system (0.1), (0.2), contains a natural small parameter ε, which is the characteristic size of pores l divided by the characteristic size L of the entire porous body:

$$\varepsilon = \frac{l}{L}.$$

Our aim is to derive all possible limiting regimes (homogenized equations) as $\varepsilon \searrow 0$. Such an approximation significantly simplifies the original problem and at the same time preserves all of its main features. But even this approach is too hard to work out, and some additional simplifying assumptions are necessary. In terms of geometrical properties of the medium, the most appropriate is to simplify the problem postulating that the porous structure is periodic. We accept the following constraints.

Assumption 1. Domain $\Omega = (0,1)^3$ is a periodic repetition of an elementary cell $Y^\varepsilon = \varepsilon Y$, where $Y = (0,1)^3$ and quantity $1/\varepsilon$ is integer, so that Ω always contains an integer number of elementary cells Y_i^ε. Let Y_s be the "solid part" of Y, and the "liquid part" Y_f is its open complement. We denote $\gamma = \partial Y_f \cap \partial Y_s$, and γ is a piecewise C^1-surface. The porous space Ω_f^ε is the periodic repetition of the elementary cell εY_f, and the solid skeleton Ω_s^ε is the periodic repetition of the elementary cell εY_s. The boundary $\Gamma^\varepsilon = \partial\Omega_s^\varepsilon \cap \partial\Omega_f^\varepsilon$ is the periodic repetition in Ω of the boundary $\varepsilon\gamma$. *The "solid skeleton" Ω_s is a connected domain.*

In these assumptions

$$\bar{\chi}(\boldsymbol{x}) = \chi^\varepsilon(\boldsymbol{x}) = \chi\left(\frac{\boldsymbol{x}}{\varepsilon}\right), \quad \bar{\rho} = \rho^\varepsilon(\boldsymbol{x}) = \chi^\varepsilon(\boldsymbol{x})\rho_f + \big(1 - \chi^\varepsilon(\boldsymbol{x})\big)\rho_s,$$

where $\chi(\mathbf{y})$ is a characteristic function of Y_f in Y. In our model $\chi(\mathbf{y})$ is a known function.

We say that a *porous space is disconnected* (isolated pores) if $\gamma \cap \partial Y = \varnothing$.

Suppose that all dimensionless parameters depend on the small parameter ε of the model and there exist limits (finite or infinite)

$$\lim_{\varepsilon\searrow 0} \alpha_\mu(\varepsilon) = \mu_0, \quad \lim_{\varepsilon\searrow 0} \alpha_\lambda(\varepsilon) = \lambda_0, \quad \lim_{\varepsilon\searrow 0} \alpha_\tau(\varepsilon) = \tau_0, \quad \lim_{\varepsilon\searrow 0} \frac{\alpha_\mu}{\varepsilon^2} = \mu_1,$$
$$\lim_{\varepsilon\searrow 0} \frac{\alpha_\lambda}{\varepsilon^2} = \lambda_1, \quad \lim_{\varepsilon\searrow 0} \alpha_p(\varepsilon) = p_*, \quad \lim_{\varepsilon\searrow 0} \alpha_\eta(\varepsilon) = \eta_0.$$

The first research with the aim of finding limiting regimes in the case where the skeleton was assumed to be an absolutely rigid body was carried out by E. Sanchez-Palencia and L. Tartar. Sanchez-Palencia [3, Sec. 7.2] formally obtained Darcy's law of filtration using the method of two-scale asymptotic expansions, and L. Tartar [3, Appendix] mathematically rigorously justified the homogenization procedure. Using the same method of two-scale expansions J. Keller and R. Burridge [2] derived formally the system of Biot's equations [5] from the system (0.1)–(0.4) in the case

$$\mu_0 = 0, \quad 0 < \mu_1,\ \lambda_0,\ \tau_0 < \infty.$$

Under the same assumptions as in [2], a rigorous justification of Biot's model was given by G. Nguetseng [6] and later by Th. Clopeaut *et al* in [7]. Also R. Gilbert *et al* [4] derived a system of equations of viscoelasticity in which all the physical parameters were fixed independent of ε. The most complete results have been obtained by author [8,9] for the cases

$$\tau_0,\ \mu_0,\ \nu_0,\ p_*^{-1},\ \eta_0^{-1} < \infty.$$

In these works, Nguetseng's two-scale convergence method [10,11] was the main method of investigation and has been applied recently to a wide range of homogenization problems (see, e.g., [12–14]).

The present publication is a detailed summary of author's works [8, 9]. We consider all possible situations where

$$\tau_0,\ \mu_0 < \infty$$

and any of the following situations takes place:

$$\begin{aligned}
&\text{(I)} && \mu_0 = 0,\ 0 < \lambda_0 < \infty;\\
&\text{(II)} && 0 \le \mu_0,\ \lambda_0 = \infty;\\
&\text{(III)} && 0 < \mu_0,\ 0 < \lambda_0 < \infty;\\
&\text{(IV)} && 0 < \mu_0,\ \lambda_0 = 0;\\
&\text{(V)} && \mu_0 = 0,\ \lambda_0 = 0.
\end{aligned}$$

If $\tau_0 = \infty$ then, re-normalizing the displacement vector by setting $\mathbf{w} \to \alpha_\tau \mathbf{w}$, we reduce the problem to one of the cases (I)–(V). Note that this last case $\tau_0 = \infty$ appears if we model short-time processes like seismic or acoustic wave propagation or hydraulic fracturing, when the duration of the process does not exceed some seconds.

Let us briefly describe the content of the paper. Theorem 1 is devoted to the derivation of bounds of the solutions of the problem (0.1)–(0.4), uniform with respect to the small parameter ε. Here a very simple estimate of the solutions for the case $\tau_0 > 0$ and where all other criteria are bounded becomes nontrivial if $\tau_0 = 0$, or if $\lambda_0 = \infty$ (the skeleton is an absolutely rigid body), or if $p_* = \infty$ (incompressible liquid), or, finally, if $\eta_0 = \infty$ (the solid skeleton is an incompressible body). Further we show that for the case (I) (Theorem 2) the homogenized equations are Biot's system of poroelasticity for the two-velocity continuum ($0 < \mu_1 < \infty$, $\tau_0 = 0$), or the similar system consisting of nonisotropic Lamé's equations for the solid component coupled with acoustic equations for the liquid component ($\mu_1 = 0$), or nonisotropic Lamé's equations for the one-velocity continuum (the case of disconnected porous space, or the case where $\mu_1 = \infty$). For the case (IV) (Theorem 5) the homogenized equations are Stokes system of equations for the liquid component coupled with acoustic equations for the solid component ($\lambda_1 < \infty$), or the similar system consisting of nonisotropic Lamé's equations for the solid component and acoustic equations for the liquid component ($\mu_1 = 0$), or nonisotropic Stokes equations for the one-velocity continuum ($\lambda_1 = \infty$). For the case (V) (Theorem 6) the homogenized equations are different types of acoustic equations for one- or two-velocity continuum. This is a very interesting fact—that the initially one-velocity continuum becomes a two-velocity continuum after homogenization might be explained by the different smoothness of the solution in the solid and in the liquid components:

$$\int\limits_{\Omega} \alpha_\mu(\varepsilon)\chi^\varepsilon |\nabla \boldsymbol{w}^\varepsilon|^2 \, dx \le C_0, \quad \int\limits_{\Omega} \alpha_\lambda(\varepsilon)(1-\chi^\varepsilon)|\nabla \boldsymbol{w}^\varepsilon|^2) \, dx \le C_0,$$

where C_0 is a constant independent of the small parameter ε. To preserve the best properties of the solution we extend the solution from the chosen component onto the whole domain. At this stage, the criteria μ_1 (if $\mu_0 = 0$) and λ_1 (if $\lambda_0 = 0$) become crucial. Namely, if, for example,

$$\int\limits_{\Omega} \alpha_\mu(\varepsilon)|\nabla \boldsymbol{w}^\varepsilon|^2 \, dx < C_0, \quad \lim_{\varepsilon \searrow 0} \alpha_\mu(\varepsilon) = \mu_0 = 0, \quad \lim_{\varepsilon \searrow 0} \frac{\alpha_\mu}{\varepsilon^2} = \mu_1 = \infty$$

and $\lambda_0 > 0$, then the sequence $\{\boldsymbol{w}^\varepsilon\}$ converges strongly in $L^2\big(\Omega \times (0,T)\big)$ (more precisely, is strongly compact in $L^2\big(\Omega \times (0,T)\big)$) and the limiting system of differential equations describes a one-velocity continuum. The same conclusion is valid for the case

$$\int\limits_{\Omega} \alpha_\lambda(\varepsilon)|\nabla \boldsymbol{w}^\varepsilon|^2 \, dx < C_0, \quad \mu_0 > 0, \quad \lambda_0 = 0, \quad \lambda_1 = \infty.$$

But if, for example, for the first case $\lambda_0 = \infty$, then the sequence $\{\boldsymbol{w}^\varepsilon\}$ strongly converges in $L^2\big(\Omega \times (0,T)\big)$ to zero and we lose all information about solution. To extract this hidden information we have to re-normalize the solution. The exact form of the re-normalizing multiplier depends on different factors. For example, one of the possible situations might be the re-normalization

$$\boldsymbol{w}^\varepsilon \to \frac{\alpha_\mu}{\varepsilon^2}\boldsymbol{w}^\varepsilon.$$

All these situations are collected in the case (II) (Theorem 3). Here homogenized equations are Darcy's system of filtration ($0 < \mu_1 < \infty$, $\tau_0 = 0$), or acoustic system for the liquid component ($\mu_1 = 0$). For this first approximation the solid component behaves as an absolutely rigid body. More precise asymptotics (second approximation) show that the re-normalized displacements of the solid component is described by a nonisotropic Lamé's system of elasticity. If $\mu_1 = \infty$, then the limiting regimes are two-velocity media for re-normalized displacements, which are described in Theorem 2. Finally, for the case (III) the homogenized equations are nonlocal equations of viscoelastisity (connected porous space) or nonisotropic Lamé's system of elasticity (disconnected porous space) for one-velocity continuum (Theorem 4).

1. Formulation of the Main Results

As usual, Eq. (0.1) is understood in the sense of distributions. It involves the proper equations (0.1), (0.2) in the usual sense in the domains Ω_f^ε and Ω_s^ε and the boundary conditions

$$[\boldsymbol{w}] = 0, \quad [\boldsymbol{P} \cdot \mathbf{n}] = 0, \quad \boldsymbol{x}_0 \in \Gamma^\varepsilon, \quad t \geq 0, \tag{1.1}$$

on the boundary Γ^ε, where $\mathbf{n}$ is a unit normal to the boundary and

$$[\varphi](\boldsymbol{x}_0) = \varphi_{(s)}(\boldsymbol{x}_0) - \varphi_{(f)}(\boldsymbol{x}_0),$$
$$\varphi_{(s)}(\boldsymbol{x}_0) = \lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}_0\\ \boldsymbol{x}\in\Omega_s^\varepsilon}} \varphi(\boldsymbol{x}), \quad \varphi_{(f)}(\boldsymbol{x}_0) = \lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}_0\\ \boldsymbol{x}\in\Omega_f^\varepsilon}} \varphi(\boldsymbol{x}).$$

There are various equivalents in the sense of distribution forms of representation of Eq. (0.1) and boundary conditions (1.1). In what follows, it is convenient to write them in the form of the integral equalities.

Definition 1. We say that four functions $(\boldsymbol{w}^\varepsilon,\, p^\varepsilon,\, q^\varepsilon,\, \pi^\varepsilon)$ are called a generalized solution of the problem (0.1)–(0.4) if they satisfy the regularity conditions

$$\boldsymbol{w}^\varepsilon,\ D(x,\boldsymbol{w}^\varepsilon),\ \operatorname{div}_x \boldsymbol{w}^\varepsilon,\ q^\varepsilon,\ p^\varepsilon,\ \frac{\partial p^\varepsilon}{\partial t},\ \pi^\varepsilon \in L^2(\Omega_T)$$

in the domain $\Omega_T = \Omega \times (0,T)$, boundary conditions (0.4) in the trace sense, the equations

$$p^\varepsilon = -\alpha_p \chi^\varepsilon \operatorname{div}_x \boldsymbol{w}^\varepsilon, \quad \pi^\varepsilon = -\alpha_\eta (1-\chi^\varepsilon) \operatorname{div}_x \boldsymbol{w}^\varepsilon, \quad q^\varepsilon = p^\varepsilon + \frac{\alpha_\nu}{\alpha_p} \frac{\partial p^\varepsilon}{\partial t} \tag{1.2}$$

a.e. in Ω_T, and the integral identity

$$\int_{\Omega_T} \bigg(\alpha_\tau \rho^\varepsilon \boldsymbol{w}^\varepsilon \cdot \frac{\partial^2 \boldsymbol{\varphi}}{\partial t^2} - \chi^\varepsilon \alpha_\mu D(x,\boldsymbol{w}^\varepsilon) : D\Big(x, \frac{\partial \boldsymbol{\varphi}}{\partial t}\Big) - \rho^\varepsilon \boldsymbol{F} \cdot \boldsymbol{\varphi} + \{(1-\chi^\varepsilon)\alpha_\lambda D(x,\boldsymbol{w}^\varepsilon) - (q^\varepsilon + \pi^\varepsilon)\mathbb{I}\} : D(x,\boldsymbol{\varphi}) \bigg)\, dx\, dt = 0 \tag{1.3}$$

for all smooth vector-functions $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\boldsymbol{x},t)$ such that $\boldsymbol{\varphi}|_{\partial\Omega} = \boldsymbol{\varphi}|_{t=T} = \partial\boldsymbol{\varphi}/\partial t|_{t=T} = 0$.

In this definition, we have changed the form of representation of the stress tensor $\mathbf{P}$ in the integral identity (1.3) by introducing two new functions q^ε and π^ε. The main goal of such kind innovation is just to make easy the derivation and analysis of homogenized equations. Functions q^ε and π^ε behave as a pressure. Therefore, for brevity, we will call them pressure, the first two equations in (1.2) we will call continuity equations, and the last one in (1.2) we will call a state equation.

In (1.3) by $A : B$ we denote the convolution (or, equivalently, the inner tensor product) of two second-rank tensors along the both indexes, i.e.,

$$A : B = \operatorname{tr}(B^* \circ A) = \sum_{i,j=1}^{3} A_{ij} B_{ji}.$$

Suppose additionally that there exist limits (finite or infinite)

$$\lim_{\varepsilon\searrow 0} \alpha_\nu(\varepsilon) = \nu_0, \quad \lim_{\varepsilon\searrow 0} \frac{\varepsilon^2 \alpha_p}{\alpha_\mu} = p_1, \quad \lim_{\varepsilon\searrow 0} \frac{\alpha_\eta \varepsilon^2}{\alpha_\mu} = \eta_1,$$
$$\lim_{\varepsilon\searrow 0} \frac{\alpha_\lambda \varepsilon^2}{\alpha_\mu} = \lambda_2, \quad \lim_{\varepsilon\searrow 0} \frac{\alpha_\eta}{\alpha_\lambda} = \eta_2, \quad \lim_{\varepsilon\searrow 0} \frac{\alpha_p}{\alpha_\lambda} = p_2.$$

We also assume that

Assumption 2.

(1) $\mathbf{F}, \partial\mathbf{F}/\partial t \in L^2(\Omega_T)$;

(2) the dimensionless parameters satisfy the restrictions

$$p_*^{-1},\, \nu_0,\, \eta_0^{-1} < \infty; \quad 0 < \tau_0 + \mu_1.$$

In what follows, all parameters can take all permitted values. For example, if $\tau_0 = 0$, or $p_*^{-1} = 0$ (incompressible liquid), or $\eta_0^{-1} = 0$ (incompressible solid skeleton), then all terms in the final equations containing these parameters disappear.

Note that we do not consider the cases $p_* = 0$ and $\eta_0 = 0$ because they have no physical or mathematical importance.

The following Theorems 1–6 are the main results of this paper.

Theorem 1. *For all $\varepsilon > 0$ on the arbitrary time interval $[0,T]$ there exists a unique generalized solution of the problem* (0.1)–(0.4)*.*

(I) *If $\lambda_0 < \infty$, then*

$$\max_{0\le t\le T} \| \, |\boldsymbol{w}^\varepsilon(t)| + \sqrt{\alpha_\mu}\chi^\varepsilon|\nabla_x \boldsymbol{w}^\varepsilon(t)| + \sqrt{\alpha_\lambda}(1-\chi^\varepsilon)|\nabla_x \boldsymbol{w}^\varepsilon(t)| \, \|_{2,\Omega} \le I_F, \tag{1.4}$$

$$\|q^\varepsilon\|_{2,\Omega_T} + \|p^\varepsilon\|_{2,\Omega_T} + \|\pi^\varepsilon\|_{2,\Omega_T} \le I_F, \tag{1.5}$$

where $I_F = C\| \, |\mathbf{F}| + |\partial \mathbf{F}/\partial t| \, \|_{2,\Omega_T}$ and C is a constant independent of ε.

(II) *If $\lambda_0 = \infty$, then for displacements $\boldsymbol{w}^\varepsilon$ estimates* (1.4) *hold true, and under the condition*

$$p_* < \infty, \tag{1.6}$$

for pressures q^ε and p^ε in the liquid component estimates (1.5) *hold true.*

If instead of restriction (1.6) *the conditions*

$$p_2,\, p_2^{-1},\, \eta_2 < \infty; \quad \boldsymbol{F} = \nabla \Phi, \quad \frac{\partial \Phi}{\partial t}, \left|\frac{\partial \mathbf{F}}{\partial t}\right| \in L^2(\Omega_T) \tag{1.7}$$

hold true, then

$$\max_{0\le t\le T} \left(\left\|(1-\chi^\varepsilon)\nabla_x\big(\alpha_\lambda \boldsymbol{w}^\varepsilon(t)\big)\right\|_{2,\Omega} + \left\|\chi^\varepsilon \operatorname{div}_x\big(\alpha_\lambda \boldsymbol{w}^\varepsilon(t)\big)\right\|_{2,\Omega} \le I_F^{(1)}, \tag{1.8}$$

where $I_F^{(1)} = C\| \, |\mathbf{F}| + |\partial\Phi/\partial t| + |\partial \mathbf{F}/\partial t| \, \|_{2,\Omega_T}$ and C is a constant independent of ε. These last estimates imply (1.5)*.*

(III) *If $\lambda_0 = \infty$, $\mu_1 = \infty$, $0 < \lambda_2 < \infty$, then for re-normalized displacements*

$$w^\varepsilon \to \alpha_\lambda w^\varepsilon$$

with re-normalized parameters

$$\alpha_\mu \to \frac{\alpha_\mu}{\alpha_\lambda}, \quad \alpha_\lambda \to 1, \quad \alpha_\tau \to \frac{\alpha_\tau}{\alpha_\lambda}, \quad \alpha_\nu \to \frac{\alpha_\nu}{\alpha_\lambda}, \quad \alpha_p \to \frac{\alpha_p}{\alpha_\lambda}$$

situation (I) *of the present theorem holds true.*

(IV) *If $\lambda_0 = \infty$, $\mu_1 = \infty$, $\lambda_2 = \infty$, then for re-normalized displacements*

$$w^\varepsilon \to \varepsilon^{-2}\alpha_\mu w^\varepsilon$$

with re-normalized parameters

$$\alpha_\mu \to \varepsilon^2, \quad \alpha_\lambda \to \varepsilon^2\frac{\alpha_\lambda}{\alpha_\mu}, \quad \alpha_\tau \to \varepsilon^2\frac{\alpha_\tau}{\alpha_\mu}, \quad \alpha_\nu \to \varepsilon^2\frac{\alpha_\nu}{\alpha_\mu}, \quad \alpha_p \to \varepsilon^2\frac{\alpha_p}{\alpha_\mu}$$

situation (II) *of the present theorem holds true.*

Theorem 2. *Let $\lambda_0 < \infty$, $\mu_0 = 0$. Then functions $\boldsymbol{w}^\varepsilon$ admit an extension $\boldsymbol{u}^\varepsilon$ from $\Omega^\varepsilon_{s,T} = \Omega^\varepsilon_s \times (0,T)$ into Ω_T such that the sequence $\{\boldsymbol{u}^\varepsilon\}$ converges strongly in $L^2(\Omega_T)$ and weakly in $L^2\big((0,T); W^1_2(\Omega)\big)$ to the function $\boldsymbol{u}$. At the same time, sequences $\{\boldsymbol{w}^\varepsilon\}$, $\{p^\varepsilon\}$, $\{q^\varepsilon\}$, and $\{\pi^\varepsilon\}$ converge weakly in $L^2(\Omega_T)$ to $\boldsymbol{w}$, p, q, and π, respectively.*

The following assertions for these limiting functions hold true:

(I) *If* $\mu_1 = \infty$ *or the porous space is disconnected (the case of isolated pores), then* $\mathbf{w} = \mathbf{u}$ *and the functions* $\boldsymbol{u}$, p, q, *and* π *satisfy in the domain* Ω_T *the following initial-boundary-value problem:*

$$\tau_0 \hat{\rho} \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \operatorname{div}_x \{\lambda_0 \mathbb{A}_0^s : D(x, \boldsymbol{u}) + B_0^s \operatorname{div}_x \boldsymbol{u} + B_1^s q - (q + \pi) \cdot \mathbb{I}\} + \hat{\rho} \boldsymbol{F}, \tag{1.9}$$

$$\frac{1}{\eta_0} \pi + C_0^s : D(x, \boldsymbol{u}) + a_0^s \operatorname{div}_x \boldsymbol{u} + a_1^s q = 0, \tag{1.10}$$

$$\frac{1}{p_*} p + \frac{1}{\eta_0} \pi + \operatorname{div}_x \boldsymbol{u} = 0, \quad p + \nu_0 p_*^{-1} \frac{\partial p}{\partial t} = q, \tag{1.11}$$

where

$$\hat{\rho} = m \rho_f + (1 - m) \rho_s, \quad m = \int_Y \chi(\boldsymbol{y})\, dy.$$

The symmetric strictly positively definite constant fourth-rank tensor $\mathbb{A}_0^s$, *matrices* C_0^s, B_0^s, *and* B_1^s, *and constants* a_0^s *and* a_1^s *are defined below by formulas* (4.23), (4.25), (4.26).

Differential equations (1.9) *are endowed with the homogeneous initial and boundary conditions*

$$\tau_0 \boldsymbol{u}(\boldsymbol{x}, 0) = \tau_0 \frac{\partial \boldsymbol{u}}{\partial t}(\boldsymbol{x}, 0) = 0, \quad \boldsymbol{x} \in \Omega, \quad \boldsymbol{u}(\boldsymbol{x}, t) = 0, \quad \boldsymbol{x} \in S, \quad t > 0. \tag{1.12}$$

(II) *If* $\mu_1 < \infty$, *then the weak limits* $\boldsymbol{u}$, $\boldsymbol{w}^f$, p, q, *and* π *of sequences* $\{\boldsymbol{u}^\varepsilon\}$, $\{\chi^\varepsilon \boldsymbol{w}^\varepsilon\}$, $\{p^\varepsilon\}$, $\{q^\varepsilon\}$, *and* $\{\pi^\varepsilon\}$ *satisfy the initial-boundary-value problem consisting of the balance of momentum equation*

$$\tau_0 \rho_f \frac{\partial \boldsymbol{v}}{\partial t} + \tau_0 \rho_s (1 - m) \frac{\partial^2 \boldsymbol{u}}{\partial t^2} + \nabla(q + \pi) - \hat{\rho} \boldsymbol{F} = \operatorname{div}_x \{\lambda_0 \mathbb{A}_0^s : D(x, \boldsymbol{u}) + B_0^s \operatorname{div}_x \boldsymbol{u} + B_1^s q\} \tag{1.13}$$

and the continuity equations (1.10) *for the solid component, the continuity equation and the state equation*

$$\frac{1}{p_*} p + \frac{1}{\eta_0} \pi + \operatorname{div}_x \boldsymbol{w}^f = (m - 1) \operatorname{div}_x \boldsymbol{u}, \quad p + \nu_0 p_*^{-1} \frac{\partial p}{\partial t} = q \tag{1.14}$$

for the liquid component, and the relations

$$\boldsymbol{v} = m \frac{\partial \boldsymbol{u}}{\partial t} + \int_0^t B_1(\mu_1, t - \tau) \cdot \boldsymbol{z}(\boldsymbol{x}, \tau)\, d\tau, \tag{1.15}$$

$$\boldsymbol{z}(\boldsymbol{x}, t) = -\frac{1}{m} \nabla q(\boldsymbol{x}, t) + \rho_f \boldsymbol{F}(\boldsymbol{x}, t) - \tau_0 \rho_f \frac{\partial^2 \boldsymbol{u}}{\partial t^2}(\boldsymbol{x}, t)$$

in the case $\tau_0 > 0$ *and* $\mu_1 > 0$, *or Darcy's law in the form*

$$\boldsymbol{v} = m \frac{\partial \boldsymbol{u}}{\partial t} + B_2(\mu_1) \cdot \left(-\frac{1}{m} \nabla q + \rho_f \boldsymbol{F}\right) \tag{1.16}$$

in the case $\tau_0 = 0$ *and* $\mu_1 > 0$, *or, finally, the balance of momentum equation for the liquid component in the form*

$$\tau_0 \rho_f \frac{\partial \boldsymbol{v}}{\partial t} = \tau_0 \rho_f B_3 \cdot \frac{\partial^2 \boldsymbol{u}}{\partial t^2} + (m\mathbb{I} - B_3) \cdot \left(-\frac{1}{m} \nabla q + \rho_f \boldsymbol{F}\right) \tag{1.17}$$

in the case $\tau_0 > 0$ *and* $\mu_1 = 0$. *Here* $\boldsymbol{v} = \partial \boldsymbol{w}^f / \partial t$ *and* $\mathbb{A}_0^s$, B_0^s, *and* B_1^s *are the same as in* (1.9).

This problem is endowed with initial and boundary conditions (1.12) *for the displacements of the solid component and the homogeneous initial condition and boundary condition*

$$\boldsymbol{v}(\boldsymbol{x}, t) \cdot n(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in S, \quad t > 0, \tag{1.18}$$

for the velocity $\boldsymbol{v}$ *of the fluid component.*

In (1.15)–(1.18) $n(\boldsymbol{x})$ is the unit normal vector to S at a point $\boldsymbol{x}\in S$; matrix $B_1(\mu_1,t)$ and symmetric strictly positively defined matrices $B_2(\mu_1)$ and $(m\mathbb{I}-B_3)$ are defined below by formulas (4.34)–(4.38).

Theorem 3. *Let $\lambda_0=\infty$.*

(I) *If $\mu_1<\infty$ and one of conditions (1.6) or (1.7) holds true, then the sequences $\{\chi^\varepsilon \boldsymbol{w}^\varepsilon\}$, $\{p^\varepsilon\}$, and $\{q^\varepsilon\}$ converge weakly in $L^2(\Omega_T)$ to $\boldsymbol{w}^f$, p, and q, respectively. The functions $\boldsymbol{w}^\varepsilon$ admit an extension $\boldsymbol{u}^\varepsilon$ from $\Omega^\varepsilon_s\times(0,T)$ into Ω_T such that the sequence $\{\boldsymbol{u}^\varepsilon\}$ converges strongly in $L^2(\Omega_T)$ and weakly in $L^2\big((0,T);W^1_2(\Omega)\big)$ to zero and*

(1) *if $\tau_0>0$ and $\mu_1>0$, then functions $\boldsymbol{v}=\partial\boldsymbol{w}^f/\partial t$, p, and q solve in the domain Ω_T the problem (F_1), where*

$$\boldsymbol{v}=\int\limits_0^t B_1(\mu_1,t-\tau)\cdot\boldsymbol{z}_0(\boldsymbol{x},\tau)\,d\tau,\quad \boldsymbol{z}_0=-\frac{1}{m}\nabla q+\rho_f\boldsymbol{F},\tag{1.19}$$

$$p+\nu_0 p_*^{-1}\frac{\partial p}{\partial t}=q,\quad \frac{1}{p_*}\frac{\partial p}{\partial t}+\operatorname{div}_x\boldsymbol{v}=0;\tag{1.20}$$

(2) *if $\tau_0=0$ and $\mu_1>0$, then functions $\boldsymbol{v}$, p, and q solve in the domain Ω_T the problem (F_2), where $\boldsymbol{v}$ satisfies Darcy's law in the form*

$$\boldsymbol{v}=B_2(\mu_1)\cdot\Big(-\frac{1}{m}\nabla q+\rho_f\boldsymbol{F}\Big),\tag{1.21}$$

and pressures p and q satisfy Eqs. (1.20);

(3) *if $\tau_0>0$ and $\mu_1=0$, then functions $\boldsymbol{v}$, p, and q solve in the domain Ω_T the problem (F_3), where $\boldsymbol{v}$ satisfies the balance of momentum equation for the liquid component in the form*

$$\tau_0\rho_f\frac{\partial\boldsymbol{v}}{\partial t}=(m\mathbb{I}-B_3)\cdot\Big(-\frac{1}{m}\nabla q+\rho_f\boldsymbol{F}\Big),\tag{1.22}$$

and pressures p and q satisfy Eqs. (1.20).

Problems F_1–F_3 are endowed with homogeneous initial conditions and boundary condition (1.18). In (1.19), (1.21), and (1.22), matrices $B_1(\mu_1,t)$, $B_2(\mu_1)$, and B_3 are the same as in Theorem 2.

(II) *If $\mu_1<\infty$ and conditions (1.7) hold true, then the sequence $\{\alpha_\lambda\boldsymbol{u}^\varepsilon\}$ converges strongly in $L^2(\Omega_T)$ and weakly in $L^2\big((0,T);W^1_2(\Omega)\big)$ to function $\boldsymbol{u}$ and the sequence $\{\pi^\varepsilon\}$ converges weakly in $L^2(\Omega_T)$ to the function π. The limiting functions $\boldsymbol{u}$ and π satisfy the boundary value problem in the domain Ω*

$$0=\operatorname{div}_x\{\mathbb{A}^s_0:D(x,\boldsymbol{u})+B^s_0\operatorname{div}_x\boldsymbol{u}+B^s_1 q-(q+\pi)\cdot\mathbb{I}\}+\hat{\rho}\boldsymbol{F},\quad \boldsymbol{x}\in\Omega,\tag{1.23}$$

$$\frac{1}{\eta_2}\pi+C^s_0:D(x,\boldsymbol{u})+a^s_0\operatorname{div}_x\boldsymbol{u}+a^s_1 q=0,\quad \boldsymbol{x}\in\Omega,\tag{1.24}$$

where the function q is assumed given. It is defined from the corresponding Problems F_1–F_3 (the choice of the problem depends on τ_0 and μ_1). The symmetric strictly positively definite constant fourth-rank tensor $\mathbb{A}^s_0$, matrices C^s_0, B^s_0, and B^s_1, and constants a^s_0 and a^s_1 are defined below by formulas (4.23), (4.25), (4.26), in which we have $\eta_0=\eta_2$ and $\lambda_0=1$.

This problem is endowed with the homogeneous boundary conditions.

(III) *If $\mu_1=\infty$, $p_1^{-1},\eta_1^{-1}<\infty$, and $0<\lambda_2<\infty$, then there exist weak limits $\tilde{\boldsymbol{w}}$, p, and π of the sequences $\{\alpha_\mu\varepsilon^{-2}\chi^\varepsilon\boldsymbol{w}^\varepsilon\}$, $\{p^\varepsilon\}$, and $\{\pi^\varepsilon\}$ and a strong limit $\tilde{\boldsymbol{u}}$ of the sequence $\{\alpha_\mu\varepsilon^{-2}\boldsymbol{u}^\varepsilon\}$ in $L^2(\Omega_T)$, which satisfy in Ω_T the following initial-boundary-value problem:*

$$
\left.\begin{aligned}
\operatorname{div}_x\{\lambda_1\mathbb{A}^s_0 : D(x,\tilde{\boldsymbol{u}}) + B^s_0\operatorname{div}_x\tilde{\boldsymbol{u}} + B^s_1 p - (p+\pi)\cdot\mathbb{I}\} + \hat{\rho}\boldsymbol{F} = 0,\\
\frac{\partial\tilde{\boldsymbol{w}}}{\partial t} = \frac{\partial\tilde{\boldsymbol{u}}}{\partial t} + B_2(1)\cdot\left(-\frac{1}{m}\nabla p + \rho_f\boldsymbol{F}\right),\\
\frac{1}{p_1}p + \frac{1}{\eta_1}\pi + \operatorname{div}_x\tilde{\boldsymbol{w}} = (m-1)\operatorname{div}_x\tilde{\boldsymbol{u}},\\
\frac{1}{\eta_1}\pi + C^s_0 : D(x,\tilde{\boldsymbol{u}}) + a^s_0\operatorname{div}_x\tilde{\boldsymbol{u}} + a^s_1 p = 0.
\end{aligned}\right\}\tag{1.25}
$$

Here the symmetric strictly positively definite constant fourth-rank tensor $\mathbb{A}^s_0$, *matrices* C^s_0, B^s_0, *and* B^s_1, *and constants* a^s_0 *and* a^s_1 *are defined below by formulas* (4.23), (4.25), (4.26), *in which we have* $\eta_0 = \eta_1$ *and* $\lambda_0 = \lambda_2$.

This problem is endowed with the homogeneous initial and boundary conditions.

(IV) *If* $\mu_1 = \infty$ *and* $\lambda_1 = \infty$, *then the corresponding problem for displacements* $\{\alpha_\mu\varepsilon^{-2}\boldsymbol{w}^\varepsilon\}$ *is considered in parts* (I) *and* (II) *of the present theorem.*

Theorem 4. *Let* $0 < \mu_0,\ \lambda_0 < \infty$. *Then the weak and strong limits* $\boldsymbol{w}$, p, q, *and* π *of sequences* $\{\boldsymbol{w}^\varepsilon\}$, $\{p^\varepsilon\}$, $\{\pi^\varepsilon\}$, *and* $\{q^\varepsilon\}$ *satisfy in* Ω_T *the following initial-boundary-value problem:*

$$
\begin{aligned}
\tau_0\hat{\rho}\frac{\partial^2\boldsymbol{w}}{\partial t^2} + \nabla(q+\pi) - \hat{\rho}\boldsymbol{F} = \operatorname{div}_x\Bigg(\mathbb{A}_2 : D\left(x,\frac{\partial\boldsymbol{w}}{\partial t}\right) + \mathbb{A}_3 : D(x,\boldsymbol{w}) + B_4\operatorname{div}_x\boldsymbol{w}\\
+ \int_0^t\Big(\mathbb{A}_4(t-\tau) : D\big(x,\mathbf{w}(\boldsymbol{x},\tau)\big) + B_5(t-\tau)\operatorname{div}_x\boldsymbol{w}(\boldsymbol{x},\tau)\Big)\,d\tau\Bigg),
\end{aligned}\tag{1.26}
$$

$$
\frac{1}{p_*}p + m\operatorname{div}_x\boldsymbol{w} = -\int_0^t\Big(C_2(t-\tau) : D\big(x,\boldsymbol{w}(\boldsymbol{x},\tau)\big) + a_2(t-\tau)\operatorname{div}_x\boldsymbol{w}(\boldsymbol{x},\tau)\Big)\,d\tau,\tag{1.27}
$$

$$
\frac{1}{\eta_0}\pi + (1-m)\operatorname{div}_x\boldsymbol{w} = -\int_0^t\Big(C_3(t-\tau) : D\big(x,\boldsymbol{w}(\boldsymbol{x},\tau)\big) + a_3(t-\tau)\operatorname{div}_x\boldsymbol{w}(\boldsymbol{x},\tau)\Big)\,d\tau,\tag{1.28}
$$

$$
q = p + \frac{\nu_0}{p_*}\frac{\partial p}{\partial t}.\tag{1.29}
$$

Here $\mathbb{A}_2$, $\mathbb{A}_2$, *and* $\mathbb{A}_4$ *are fourth-rank tensors,* B_4, B_5, C_2, *and* C_3 *are matrices, and* a_2 *and* a_3 *are scalars. The exact expressions for these objects are given below by formulas* (6.15)–(6.20).

The problem is supplemented by the homogeneous initial and boundary conditions.

If the porous space is connected, then $\mathbb{A}_2$ *is a strictly positively definite symmetric tensor.*

If the porous space is disconnected, which is the case of isolated pores, then $\mathbb{A}_2 = 0$ *and system* (1.26) *degenerates into nonlocal anisotropic Lamé's system with a strictly positively definite and symmetric tensor* $\mathbb{A}_3$.

Theorem 5. *Let the porous space be a connected set and*

$$
\mu_0 > 0, \quad \lambda_0 = 0.
$$

Then functions $\partial\boldsymbol{w}^\varepsilon/\partial t$ *admit an extension* $\boldsymbol{v}^\varepsilon$ *from* $\Omega^\varepsilon_{f,T} = \Omega^\varepsilon_f \times (0,T)$ *into* Ω_T *such that the sequence* $\{\boldsymbol{v}^\varepsilon\}$ *converges strongly in* $L^2(\Omega_T)$ *and weakly in* $L^2\big((0,T);W^1_2(\Omega)\big)$ *to the function* $\boldsymbol{v}$. *At the same time, sequences* $\{\boldsymbol{w}^\varepsilon\}$, $\{(1-\chi^\varepsilon)\boldsymbol{w}^\varepsilon\}$, $\{p^\varepsilon\}$, $\{q^\varepsilon\}$, *and* $\{\pi^\varepsilon\}$ *converge weakly in* $L^2(\Omega_T)$ *to* $\boldsymbol{w}$, $\boldsymbol{w}^s$, p, q, *and* π, *respectively.*

(I) *If* $\lambda_1 = \infty$, *then* $\partial \boldsymbol{w}^s/\partial t = (1-m)\boldsymbol{v} = (1-m)\partial \boldsymbol{w}/\partial t$ *and the weak and strong limits* q, p, π, *and* $\boldsymbol{v}$ *satisfy in* Ω_T *the initial-boundary-value problem*

$$\hat{\rho}\frac{\partial \boldsymbol{v}}{\partial t} = \operatorname{div}_x\Big\{\mu_0 A^f : D(x,\boldsymbol{v}) + B_0^f \operatorname{div}_x \boldsymbol{v} + C_0^f \pi + \int_0^t \big(A_1^f(t-\tau) : D\big(x,\boldsymbol{v}(\boldsymbol{x},\tau)\big) + B_1^f(t-\tau)\operatorname{div}_x \boldsymbol{v}(\boldsymbol{x},\tau) + C_1^f(t-\tau)\pi(\boldsymbol{x},\tau)\big)\,d\tau\Big\} - \nabla(q+\pi) + \hat{\rho}\boldsymbol{F}, \tag{1.30}$$

$$\frac{1}{p_{0,f}}\frac{\partial p_f}{\partial t} + \mathbb{E}_0^f : \mathbb{D}(x,\boldsymbol{v}) + c_0^f p_s + (m + b_0^f)\operatorname{div}\boldsymbol{v} + \int_0^t \big(\mathbb{E}_1^f(t-\tau) : \mathbb{D}\big(x,\boldsymbol{v}(\boldsymbol{x},\tau)\big) + c_1^f(t-\tau)p_s(\boldsymbol{x},\tau) + b_1^f(t-\tau)\operatorname{div}\boldsymbol{v}(\boldsymbol{x},\tau)\big)\,d\tau = 0, \tag{1.31}$$

$$q = p + \frac{\nu_0}{p_*}\frac{\partial p}{\partial t}, \quad \frac{1}{p_*}\frac{\partial p}{\partial t} + \frac{1}{\eta_0}\frac{\partial \pi}{\partial t} + \operatorname{div}_x \boldsymbol{v} = 0, \tag{1.32}$$

where the mean density $\hat{\rho}$ *and the porosity* m *have been defined above in Theorem 2, and the symmetric strictly positively definite constant fourth-rank tensor* A^f, *fourth-rank tensor* $A_1^f(t)$, *constant matrices* C_0^f, B_0^f, *and* E_0^f, *matrices* $C_1^f(t)$, $B_1^f(t)$, *and* $E_1^f(t)$, *scalars* b_0^f *and* c_0^f, *and functions* $b_1^f(t)$ *and* $c_1^f(t)$ *are defined below by formulas (7.27)–(7.29) and (7.31).*

Differential equations (1.30) are endowed with homogeneous initial and boundary conditions

$$\boldsymbol{v}(\boldsymbol{x},0) = 0, \quad \boldsymbol{x} \in \Omega, \quad \boldsymbol{v}(\boldsymbol{x},t) = 0, \quad \boldsymbol{x} \in S, \quad t > 0. \tag{1.33}$$

(II) *If* $\lambda_1 < \infty$, *then the weak and strong limits* $\boldsymbol{w}^s$, q, p, π, *and* $\boldsymbol{v}$ *satisfy in* Ω_T *the initial-boundary-value problem, which consists of the nonisotropic Stokes system*

$$\rho_f m\frac{\partial \boldsymbol{v}}{\partial t} + \rho_s\frac{\partial^2 \boldsymbol{w}^s}{\partial t^2} + \nabla(q+\pi) - \hat{\rho}\boldsymbol{F} = \operatorname{div}_x\Big\{\mu_0 A^f : D(x,\boldsymbol{v}) + B_0^f \operatorname{div}_x \boldsymbol{v} + C_0^f \pi + \int_0^t \big(A_1^f(t-\tau) : D\big(x,\boldsymbol{v}(\boldsymbol{x},\tau)\big) + B_1^f(t-\tau)\operatorname{div}_x \boldsymbol{v}(\boldsymbol{x},\tau) + C_1^f(t-\tau)\pi(\boldsymbol{x},\tau)\big)\,d\tau\Big\}, \tag{1.34}$$

$$\frac{1}{p_{0,f}}\frac{\partial p_f}{\partial t} + \mathbb{E}_0^f : \mathbb{D}(x,\boldsymbol{v}) + c_0^f p_s + (m + b_0^f)\operatorname{div}\boldsymbol{v} + \int_0^t \big(\mathbb{E}_1^f(t-\tau) : \mathbb{D}\big(x,\boldsymbol{v}(\boldsymbol{x},\tau)\big) + c_1^f(t-\tau)p_s(\boldsymbol{x},\tau) + b_1^f(t-\tau)\operatorname{div}\boldsymbol{v}(\boldsymbol{x},\tau)\big)\,d\tau = 0, \tag{1.35}$$

$$q = p + \frac{\nu_0}{p_*}\frac{\partial p}{\partial t} \tag{1.36}$$

for the liquid component coupled with the continuity equation

$$\frac{1}{p_*}\frac{\partial p}{\partial t} + \frac{1}{\eta_0}\frac{\partial \pi}{\partial t} + \operatorname{div}_x \frac{\partial \boldsymbol{w}^s}{\partial t} + m\operatorname{div}_x \boldsymbol{v} = 0 \tag{1.37}$$

and the relation

$$\frac{\partial \boldsymbol{w}^s}{\partial t} = (1-m)\boldsymbol{v}(\boldsymbol{x},t) + \int_0^t B_1^s(t-\tau)\cdot\tilde{\boldsymbol{z}}(\boldsymbol{x},\tau)\,d\tau, \tag{1.38}$$

$$\tilde{\boldsymbol{z}}(\boldsymbol{x},t) = -\frac{1}{1-m}\nabla_x \pi(\boldsymbol{x},t) + \rho_s \boldsymbol{F}(\boldsymbol{x},t) - \rho_s\frac{\partial \boldsymbol{v}}{\partial t}(\boldsymbol{x},t)$$

in the case $\lambda_1 > 0$, *or the balance of momentum equation in the form*

$$\rho_s \frac{\partial^2 \boldsymbol{w}^s}{\partial t^2} = \rho_s B_2^s \cdot \frac{\partial \boldsymbol{v}}{\partial t} + \big((1-m)I - B_2^s\big) \cdot \Big(-\frac{1}{1-m}\nabla_x \pi + \rho_s \boldsymbol{F}\Big) \tag{1.39}$$

in the case $\lambda_1 = 0$ *for the solid component. The problem is supplemented by boundary and initial conditions* (1.33) *for the velocity* $\boldsymbol{v}$ *of the liquid component and by the homogeneous initial conditions and the boundary condition*

$$\boldsymbol{w}^s(\boldsymbol{x},t)\cdot \boldsymbol{n}(\boldsymbol{x}) = 0, \quad (\boldsymbol{x},t) \in S, \quad t > 0, \tag{1.40}$$

for the displacements $\boldsymbol{w}^s$ *of the solid component. In Eqs.* (1.38)–(1.40) $\boldsymbol{n}(\boldsymbol{x})$ *is the unit normal vector to* S *at a point* $\boldsymbol{x} \in S$, *and matrices* $B_1^s(t)$ *and* B_2^s *are given below by Eqs.* (7.38) *and* (7.40), *where the matrix* $\big((1-m)I - B_2^s\big)$ *is symmetric and strictly positively definite.*

Theorem 6. *Let the porous space be a connected set and*

$$\mu_0 = \lambda_0 = 0; \quad p_*,\, \eta_0 < \infty.$$

Then there exist functions $\boldsymbol{w}_f^\varepsilon$, $\boldsymbol{w}_s^\varepsilon \in L^\infty\big(0,T; W_2^1(\Omega)\big)$ *such that*

$$\boldsymbol{w}_f^\varepsilon = \boldsymbol{w}^\varepsilon \quad in \quad \Omega_f^\varepsilon \times (0,T), \qquad \boldsymbol{w}_s^\varepsilon = \boldsymbol{w}^\varepsilon \quad in \quad \Omega_s^\varepsilon \times (0,T)$$

and sequences $\{p^\varepsilon\}$, $\{q^\varepsilon\}$, $\{\pi^\varepsilon\}$, $\{\boldsymbol{w}^\varepsilon\}$, $\{\chi^\varepsilon \boldsymbol{w}^\varepsilon\}$, $\{(1-\chi^\varepsilon)\boldsymbol{w}^\varepsilon\}$, $\{\boldsymbol{w}_f^\varepsilon\}$, *and* $\{\boldsymbol{w}_s^\varepsilon\}$ *converge as* $\varepsilon \searrow 0$ *weakly in* $L^2(\Omega_T)$ *to functions* p, q, π, $\boldsymbol{w}$, $\boldsymbol{w}^f$, $\boldsymbol{w}^s$, $\boldsymbol{w}_f$, *and* $\boldsymbol{w}_s$, *respectively.*

(I) *If* $\mu_1 = \lambda_1 = \infty$, *then* $\boldsymbol{w}_f = \boldsymbol{w}_s = \boldsymbol{w}$ *and the functions* $\boldsymbol{w}$, p, q, *and* π *satisfy in* Ω_T *the system of acoustic equations*

$$\hat{\rho}\frac{\partial^2 \boldsymbol{w}}{\partial t^2} = -\frac{1}{(1-m)}\nabla_x \pi + \hat{\rho}\boldsymbol{F}, \tag{1.41}$$

$$\frac{1}{p_*}p + \frac{1}{\eta_0}\pi + \operatorname{div}_x \boldsymbol{w} = 0, \tag{1.42}$$

$$q = p + \frac{\nu_0}{p_*}\frac{\partial p}{\partial t}, \quad \frac{1}{m}q = \frac{1}{1-m}\pi, \tag{1.43}$$

homogeneous initial conditions

$$\boldsymbol{w}(\boldsymbol{x},0) = \frac{\partial \boldsymbol{w}}{\partial t}(\boldsymbol{x},0) = 0, \quad \boldsymbol{x} \in \Omega, \tag{1.44}$$

and homogeneous boundary condition

$$\boldsymbol{w}(\boldsymbol{x},t)\cdot \boldsymbol{n}(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in S, \quad t > 0. \tag{1.45}$$

(II) *If* $\mu_1 = \infty$ *and* $\lambda_1 < \infty$, *then the functions* $\boldsymbol{w}_f = \boldsymbol{w}$, $\boldsymbol{w}^s$, p, q, *and* π *satisfy in* Ω_T *the system of acoustic equations, which consists of the state equations* (1.43) *and the balance of momentum equation*

$$\rho_f m \frac{\partial^2 \boldsymbol{w}_f}{\partial t^2} + \rho_s \frac{\partial^2 \boldsymbol{w}^s}{\partial t^2} = -\frac{1}{m}\nabla_x q + \hat{\rho}\boldsymbol{F} \tag{1.46}$$

for the liquid component, the continuity equation

$$\frac{1}{p_*}p + \frac{1}{\eta_0}\pi + m \operatorname{div}_x \boldsymbol{w}_f + \operatorname{div}_x \boldsymbol{w}^s = 0, \tag{1.47}$$

and the relation

$$\frac{\partial \boldsymbol{w}^s}{\partial t} = (1-m)\frac{\partial \boldsymbol{w}_f}{\partial t} + \int_0^t B_1^s(t-\tau)\cdot \boldsymbol{z}^s(\boldsymbol{x},\tau)\, d\tau, \tag{1.48}$$

$$\boldsymbol{z}^s(\boldsymbol{x},t) = -\frac{1}{1-m}\nabla_x \pi(\boldsymbol{x},t) + \rho_s \boldsymbol{F}(\boldsymbol{x},t) - \rho_s \frac{\partial^2 \boldsymbol{w}_f}{\partial t^2}(\boldsymbol{x},t)$$

in the case $\lambda_1 > 0$, *or the balance of momentum equation in the form*

$$\rho_s \frac{\partial^2 \boldsymbol{w}^s}{\partial t^2} = \rho_s B_2^s \cdot \frac{\partial^2 \boldsymbol{w}_f}{\partial t^2} + \big((1-m)I - B_2^s\big) \cdot \Big(-\frac{1}{1-m}\nabla_x \pi + \rho_s \boldsymbol{F}\Big) \tag{1.49}$$

in the case $\lambda_1 = 0$ *for the solid component. The problem* (1.43), (1.46)–(1.49) *is supplemented by homogeneous initial conditions* (1.44) *for the displacements in the liquid and solid components and homogeneous boundary condition* (1.45) *for the displacements* $\boldsymbol{w} = m\boldsymbol{w}_f + \boldsymbol{w}^s$.

In Eqs. (1.48), (1.49) *matrices* $B_1^s(t)$ *and* B_2^s *are the same as in Theorem* 5.

(III) *If* $\mu_1 < \infty$ *and* $\lambda_1 = \infty$, *then the functions* $\boldsymbol{w}^f$, $\boldsymbol{w}_s = \boldsymbol{w}$, p, q, *and* π *satisfy in* Ω_T *the system of acoustic equations, which consists of the state equations* (1.43) *and the balance of momentum equation*

$$\rho_f \frac{\partial^2 \boldsymbol{w}^f}{\partial t^2} + \rho_s(1-m)\frac{\partial^2 \boldsymbol{w}_s}{\partial t^2} = -\frac{1}{(1-m)}\nabla_x \pi + \hat{\rho}\boldsymbol{F} \tag{1.50}$$

for the solid component, the continuity equation

$$\frac{1}{p_*}p + \frac{1}{\eta_0}\pi + \operatorname{div}_x \boldsymbol{w}^f + (1-m)\operatorname{div}_x \boldsymbol{w}_s = 0, \tag{1.51}$$

and the relation

$$\frac{\partial \boldsymbol{w}^f}{\partial t} = m\frac{\partial \mathbf{w}_s}{\partial t} + \int_0^t B_1^f(t-\tau)\cdot \boldsymbol{z}^f(\boldsymbol{x},\tau)\,d\tau, \tag{1.52}$$

$$\boldsymbol{z}^f(\boldsymbol{x},t) = -\frac{1}{m}\nabla_x q(\boldsymbol{x},t) + \rho_f \boldsymbol{F}(\boldsymbol{x},t) - \rho_f\frac{\partial^2 \boldsymbol{w}_s}{\partial t^2}(\boldsymbol{x},t)$$

in the case $\mu_1 > 0$, *or the balance of momentum equation in the form*

$$\rho_f \frac{\partial^2 \boldsymbol{w}^f}{\partial t^2} = \rho_f B_2^f \cdot \frac{\partial^2 \boldsymbol{w}_s}{\partial t^2} + (mI - B_2^f)\cdot\Big(-\frac{1}{m}\nabla_x q + \rho_f \boldsymbol{F}\Big) \tag{1.53}$$

in the case $\mu_1 = 0$ *for the displacements in the liquid component. The problem* (1.43), (1.50)–(1.53) *is supplemented by homogeneous initial conditions* (1.44) *for the displacements in the liquid and solid components and homogeneous boundary condition* (1.45) *for the displacements* $\boldsymbol{w} = \boldsymbol{w}^f + (1-m)\boldsymbol{w}_s$.

In Eqs. (1.52), (1.53) *matrices* $B_1^f(t)$ *and* B_2^f *are given below by formulas* (8.32), (8.33), *where the matrix* $(mI - B_2^f)$ *is symmetric and strictly positively definite.*

(IV) *If* $\mu_1 < \infty$ *and* $\lambda_1 < \infty$, *then the functions* $\boldsymbol{w}$, p, q, *and* π *satisfy in* Ω_T *the system of acoustic equations, which consists of the continuity and the state equations* (1.42) *and* (1.43) *and the relation*

$$\frac{\partial \boldsymbol{w}}{\partial t} = \int_0^t B^\pi(t-\tau)\cdot\nabla\pi(\boldsymbol{x},\tau)\,d\tau + \boldsymbol{f}(\boldsymbol{x},t), \tag{1.54}$$

where $B^\pi(t)$ *and* $\boldsymbol{f}(\boldsymbol{x},t)$ *are given below by* (8.40) *and* (8.41).

The problem (1.42), (1.43), (1.54) *is supplemented by homogeneous initial and boundary conditions* (1.44) *and* (1.45).

2. Preliminaries

2.1. Two-Scale Convergence. Justification of Theorems 2–6 relies on systematic use of the method of two-scale convergence proposed by Nguetseng [10].

Definition 2. A sequence $\{w^\varepsilon\} \subset L^2(\Omega_T)$ is said to be *two-scale convergent* to a limit $W \in L^2(\Omega_T \times Y)$ if and only if for any 1-periodic in $\boldsymbol{y}$ function $\sigma = \sigma(\boldsymbol{x}, t, \boldsymbol{y})$ the limiting relation

$$\lim_{\varepsilon \searrow 0} \int\limits_{\Omega_T} w^\varepsilon(\boldsymbol{x}, t)\varphi\left(\boldsymbol{x}, t, \frac{\boldsymbol{x}}{\varepsilon}\right) dx\, dt = \int\limits_{\Omega_T}\int\limits_{Y} W(\boldsymbol{x}, t, \boldsymbol{y})\varphi(\boldsymbol{x}, t, \boldsymbol{y})\, dy\, dx\, dt \tag{2.1}$$

holds.

Existence and main properties of weakly convergent sequences are established by the following fundamental theorem [10, 11].

Theorem 7 (Nguetseng's theorem).

(1) *Any bounded in $L^2(Q)$ sequence contains a subsequence, two-scale convergent to some limit $W \in L^2(\Omega_T \times Y)$.*
(2) *Let sequences $\{w^\varepsilon\}$ and $\{\varepsilon\nabla_x w^\varepsilon\}$ be uniformly bounded in $L^2(\Omega_T)$. Then there exist a 1-periodic in $\boldsymbol{y}$ function $W = W(\boldsymbol{x}, t, \boldsymbol{y})$ and a subsequence $\{w^\varepsilon\}$ such that $W, \nabla_y W \in L^2(\Omega_T \times Y)$, and the subsequences $\{w^\varepsilon\}$ and $\{\varepsilon\nabla_x w^\varepsilon\}$ two-scale converge to W and $\nabla_y W$, respectively.*
(3) *Let sequences $\{w^\varepsilon\}$ and $\{\nabla_x w^\varepsilon\}$ be bounded in $L^2(Q)$. Then there exist functions $w \in L^2(\Omega_T)$ and $W \in L^2(\Omega_T \times Y)$ and a subsequence from $\{\nabla_x w^\varepsilon\}$ such that the function W is 1-periodic in $\boldsymbol{y}$, $\nabla_x w \in L^2(\Omega_T)$, $\nabla_y W \in L^2(\Omega_T \times Y)$, and the subsequence $\{\nabla_x w^\varepsilon\}$ two-scale converges to the function $\nabla_x w(\boldsymbol{x}, t) + \nabla_y W(\boldsymbol{x}, t, \boldsymbol{y})$.*

Corollary 2.1. *Let $\sigma \in L^2(Y)$ and $\sigma^\varepsilon(\boldsymbol{x}) = \sigma(\boldsymbol{x}/\varepsilon)$. Assume that a sequence $\{w^\varepsilon\} \subset L^2(\Omega_T)$ two-scale converges to $W \in L^2(\Omega_T \times Y)$. Then the sequence $\{\sigma^\varepsilon w^\varepsilon\}$ two-scale converges to the function σW.*

2.2. An Extension Lemma. The typical difficulty in homogenization problems, like problem (0.1)–(0.4), while passing to a limit as $\varepsilon \searrow 0$ arises because of the fact that the bounds on the gradient of displacement $\nabla_x \boldsymbol{w}^\varepsilon$ may be distinct in liquid and rigid components. The classical approach in overcoming this difficulty consists of constructing extension to the whole Ω of the displacement field defined merely on Ω_s or Ω_f. The following lemma is valid due to the well-known results from [15, 16]. We formulate it in appropriate form for us.

Lemma 2.1. *Suppose that Assumption 1 on the geometry of periodic structure holds, $w^\varepsilon \in W_2^1(\Omega_s^\varepsilon)$ and $w^\varepsilon = 0$ on $S_s^\varepsilon = \partial\Omega_s^\varepsilon \cap \partial\Omega$ in the trace sense. Then there exists a function $u^\varepsilon \in W_2^1(\Omega)$ such that its restriction on the sub-domain Ω_s^ε coincides with w^ε, i.e.,*

$$\big(1 - \chi^\varepsilon(\boldsymbol{x})\big)\big(u^\varepsilon(\boldsymbol{x}) - w^\varepsilon(\boldsymbol{x})\big) = 0, \quad \boldsymbol{x} \in \Omega, \tag{2.2}$$

and, moreover, the estimate

$$\|u^\varepsilon\|_{2,\Omega} \le C\|w^\varepsilon\|_{2,\Omega_s^\varepsilon}, \quad \|D(x, u^\varepsilon)\|_{2,\Omega} \le C\|D(x, w^\varepsilon)\|_{2,\Omega_s^\varepsilon} \tag{2.3}$$

holds true, where the constant C depends only on geometry Y and does not depend on ε.

2.3. Friedrichs–Poincaré's Inequality in Periodic Structure. The following lemma was proved by L. Tartar in [3, Appendix]. It specifies Friedrichs–Poincaré's inequality for ε-periodic structure.

Lemma 2.2. *Suppose that assumptions on the geometry of Ω_f^ε hold true. Then for any function $w \in \overset{\circ}{W}{}_2^1(\Omega_f^\varepsilon)$ the inequality*

$$\int\limits_{\Omega_f^\varepsilon} |w|^2\, dx \le C\varepsilon^2 \int\limits_{\Omega_f^\varepsilon} |\nabla_x w|^2\, dx \tag{2.4}$$

holds true with some constant C independent of ε.

Further we denote

(1)
$$\langle\Phi\rangle_Y=\int\limits_Y \Phi\,dy,\quad \langle\Phi\rangle_{Y_f}=\int\limits_Y \chi\Phi\,dy,\quad \langle\Phi\rangle_{Y_s}=\int\limits_Y (1-\chi)\Phi\,dy,$$
$$\langle\varphi\rangle_\Omega=\int\limits_\Omega \varphi\,dx,\quad \langle\varphi\rangle_{\Omega_T}=\int\limits_{\Omega_T}\varphi\,dx\,dt;$$

(2) if $\mathbf{a}$ and $\mathbf{b}$ are two vectors, then the matrix $\mathbf{a}\otimes\mathbf{b}$ is defined by the formula
$$(\mathbf{a}\otimes\mathbf{b})\cdot\mathbf{c}=\mathbf{a}(\mathbf{b}\cdot\mathbf{c})$$
for any vector $\mathbf{c}$;

(3) if B and C are two matrices, then $B\otimes C$ is a fourth-rank tensor such that its convolution with any matrix A is defined by the formula
$$(B\otimes C):A=B(C:A);$$

(4) by $\mathbb{I}^{ij}$ we denote the (3×3)-matrix with just one nonvanishing entry, which is equal to 1 and stands in the ith row and the jth column;

(5) we also introduce
$$J^{ij}=\frac{1}{2}(\mathbb{I}^{ij}+\mathbb{I}^{ji})=\frac{1}{2}(\boldsymbol{e}_i\otimes\boldsymbol{e}_j+\boldsymbol{e}_j\otimes\boldsymbol{e}_i),\quad \mathbb{J}=\sum_{i,j=1}^{3}J^{ij}\otimes J^{ij},$$
where $(\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3)$ are the standard Cartesian basis vectors.

3. Proof of Theorem 1

3.1. Let $\lambda_0<\infty$. If restriction $\tau_0>0$ holds, then (1.4) follows from the inequality
$$\max_{0<t<T}\Big(\sqrt{\alpha_\eta}\Big\|\operatorname{div}_x\frac{\partial\boldsymbol{w}^\varepsilon}{\partial t}(t)\Big\|_{2,\Omega_s^\varepsilon}+\sqrt{\alpha_\lambda}\Big\|\nabla_x\frac{\partial\boldsymbol{w}^\varepsilon}{\partial t}(t)\Big\|_{2,\Omega_s^\varepsilon}+\sqrt{\alpha_\tau}\Big\|\frac{\partial^2 w^\varepsilon}{\partial t^2}(t)\Big\|_{2,\Omega}+\sqrt{\alpha_p}\Big\|\operatorname{div}_x\frac{\partial\boldsymbol{w}^\varepsilon}{\partial t}(t)\Big\|_{2,\Omega_f^\varepsilon}\Big)$$
$$+\sqrt{\alpha_\mu}\Big\|\chi^\varepsilon\nabla_x\frac{\partial^2\boldsymbol{w}^\varepsilon}{\partial t^2}\Big\|_{2,\Omega_T}+\sqrt{\alpha_\nu}\Big\|\chi^\varepsilon\operatorname{div}_x\frac{\partial^2\boldsymbol{w}^\varepsilon}{\partial t^2}\Big\|_{2,\Omega_T}\leq\frac{C_0}{\sqrt{\alpha_\tau}},\quad (3.1)$$
where C_0 is independent of ε. The last estimates are obtained if we differentiate equation for $\boldsymbol{w}^\varepsilon$ with respect to time, multiply by $\partial^2\boldsymbol{w}^\varepsilon/\partial t^2$, and integrate by parts. The same estimates guarantee the existence and uniqueness of the generalized solution for the problem (0.1)–(0.4). To do that we consider Galerkin's method, taking as a basic space the Sobolev's space $\overset{\circ}{W}{}^1_2(\Omega)$, and as a basis any basis that is orthonormal in the space $L^2(\Omega)$.

Let $p_*<\infty$ and $\eta_0<\infty$. Then pressures p^ε and π^ε are bounded from continuity equations (1.2) with the help of estimates (3.1). The pressure q^ε is bounded from the state equation (1.2) rewritten in the form
$$q^\varepsilon=-\alpha_p\chi^\varepsilon\operatorname{div}_x w^\varepsilon-\alpha_\nu\chi^\varepsilon\operatorname{div}_x\frac{\partial\boldsymbol{w}^\varepsilon}{\partial t}\quad (3.2)$$
with the help of the same estimates (3.1).

If $p_*=\infty$, then estimate (1.5) for the sum of pressures $q^\varepsilon+\pi^\varepsilon$ follows from the basic integral identity (1.3) and estimates (3.1) as an estimate for the corresponding functional, if we re-normalize the pressures $q^\varepsilon+\pi^\varepsilon$ such that
$$\int\limits_\Omega\big(q^\varepsilon(\boldsymbol{x},t)+\pi^\varepsilon(\boldsymbol{x},t)\big)\,dx=0.$$

Indeed, the basic integral identity (1.3) and estimates (3.1) imply

$$\left| \int_{\Omega} (q^{\varepsilon} + \pi^{\varepsilon}) \operatorname{div}_x \boldsymbol{\psi} \, dx \right| \leq C \|\nabla \boldsymbol{\psi}\|_{2,\Omega}.$$

Choosing now ψ such that $q^{\varepsilon} + \pi^{\varepsilon} = \operatorname{div}_x \boldsymbol{\psi}$, we get the desired estimate for the sum of pressures $q^{\varepsilon} + \pi^{\varepsilon}$. Such a choice is always possible (see [17]) if we put

$$\boldsymbol{\psi} = \nabla \varphi + \boldsymbol{\psi}_0, \quad \operatorname{div}_x \boldsymbol{\psi}_0 = 0,$$
$$\Delta \varphi = q^{\varepsilon} + \pi^{\varepsilon}, \quad \varphi|_{\partial\Omega} = 0, \quad (\nabla \varphi + \boldsymbol{\psi}_0)|_{\partial\Omega} = 0.$$

Note that the re-normalization of the pressures $(q^{\varepsilon} + \pi^{\varepsilon})$ transforms continuity equations (1.2) for pressures into

$$\frac{1}{\alpha_p} p^{\varepsilon} + \chi^{\varepsilon} \operatorname{div}_x \boldsymbol{w}^{\varepsilon} = -\frac{1}{m} \beta^{\varepsilon} \chi^{\varepsilon}, \tag{3.3}$$

$$\frac{1}{\alpha_\eta} \pi^{\varepsilon} + (1 - \chi^{\varepsilon}) \operatorname{div}_x \boldsymbol{w}^{\varepsilon} = \frac{1}{1-m} (1 - \chi^{\varepsilon}) \, \beta^{\varepsilon}, \tag{3.4}$$

where

$$\beta^{\varepsilon} = \langle (1 - \chi^{\varepsilon}) \operatorname{div}_x \boldsymbol{w}^{\varepsilon} \rangle_{\Omega}.$$

But for all that, estimates (3.1) have also been changed (all terms disappear with multipliers α_ν, α_p, and α_η), and to get these estimates we must to take into account the inequalities

$$\left| \frac{\beta^{\varepsilon}}{\sqrt{m}} \right| \leq \| \operatorname{div}_x \boldsymbol{w}^{\varepsilon} \|_{2,\Omega_f^{\varepsilon}}, \qquad \left| \frac{\beta^{\varepsilon}}{\sqrt{1-m}} \right| \leq \| \operatorname{div}_x \boldsymbol{w}^{\varepsilon} \|_{2,\Omega_s^{\varepsilon}}.$$

The case $\eta_0 = \infty$ is considered in the same way. Note that for all situations the basic integral identity (1.3) permits one to bound only the sum $q^{\varepsilon} + \pi^{\varepsilon}$. But thanks to the property that the product of these two functions is equal to zero, it is enough to get bounds for each of these functions. The pressure p^{ε} is bounded from the state equation (1.2).

Estimating $\boldsymbol{w}^{\varepsilon}$ in the case $\tau_0 = 0$ is not simple, and we outline it in greater detail.

Let $\mu_1 > 0$ and $\tau_0 = 0$. As usual, we obtain the basic estimates if we multiply the equations for $\boldsymbol{w}^{\varepsilon}$ by $\partial \boldsymbol{w}^{\varepsilon}/\partial t$ and then integrate by parts all obtained terms. Only one term,

$$\rho^{\varepsilon} \boldsymbol{F} \cdot \frac{\partial \boldsymbol{w}^{\varepsilon}}{\partial t},$$

needs additional consideration here. First of all, on the strength of Lemma 2.1, we construct an extension $\boldsymbol{u}^{\varepsilon}$ of the function $\boldsymbol{w}^{\varepsilon}$ from Ω_s^{ε} into Ω_f^{ε} such that $\boldsymbol{u}^{\varepsilon} = \boldsymbol{w}^{\varepsilon}$ in Ω_s^{ε}, $\boldsymbol{u}^{\varepsilon} \in W_2^1(\Omega)$ and

$$\|\boldsymbol{u}^{\varepsilon}\|_{2,\Omega} \leq C \|\nabla_x \boldsymbol{u}^{\varepsilon}\|_{2,\Omega} \leq \frac{C}{\sqrt{\alpha_\lambda}} \|(1 - \chi^{\varepsilon}) \sqrt{\alpha_\lambda} \, \nabla_x \boldsymbol{w}^{\varepsilon}\|_{2,\Omega}.$$

In this inequality the first estimate is a some version of Friedrichs–Poincaré's inequality. The constant here does not depend on the small parameter ε due to the boundary condition for the function u^{ε}. This function vanishes on some ε-periodic set on the boundary with a positive measure.

After that we estimate $\|\boldsymbol{w}^{\varepsilon}\|_{2,\Omega}$ with the help of Friedrichs–Poincaré's inequality in periodic structure (Lemma 2.2) for the difference $(\boldsymbol{u}^{\varepsilon} - \boldsymbol{w}^{\varepsilon})$:

$$\begin{aligned}
\|\boldsymbol{w}^{\varepsilon}\|_{2,\Omega} &\leq \|\boldsymbol{u}^{\varepsilon}\|_{2,\Omega} + \|\boldsymbol{u}^{\varepsilon} - \boldsymbol{w}^{\varepsilon}\|_{2,\Omega} \leq \|\boldsymbol{u}^{\varepsilon}\|_{2,\Omega} + C\varepsilon \|\chi^{\varepsilon} \nabla_x (\boldsymbol{u}^{\varepsilon} - \boldsymbol{w}^{\varepsilon})\|_{2,\Omega} \\
&\leq \|\boldsymbol{u}^{\varepsilon}\|_{2,\Omega} + C\varepsilon \|\nabla_x \boldsymbol{u}^{\varepsilon}\|_{2,\Omega} + C\big(\varepsilon \alpha_\mu^{-\frac{1}{2}}\big) \|\chi^{\varepsilon} \sqrt{\alpha_\mu} \, \nabla_x \boldsymbol{w}^{\varepsilon}\|_{2,\Omega} \\
&\leq \frac{C}{\sqrt{\alpha_\lambda}} \|(1 - \chi^{\varepsilon}) \sqrt{\alpha_\lambda} \, \nabla_x \boldsymbol{w}^{\varepsilon}\|_{2,\Omega} + C\big(\varepsilon \alpha_\mu^{-\frac{1}{2}}\big) \|\chi^{\varepsilon} \sqrt{\alpha_\mu} \, \nabla_x \boldsymbol{w}^{\varepsilon}\|_{2,\Omega}.
\end{aligned}$$

Next we pass to the derivative with respect to time from $\partial \boldsymbol{w}^{\varepsilon}/\partial t$ to $\rho^{\varepsilon} \boldsymbol{F}$ and bound all positive terms (including the term $\alpha_\nu \chi^{\varepsilon} \operatorname{div}_x \partial \boldsymbol{w}^{\varepsilon}/\partial t$) in the usual way with the help of Hölder and Grownwall's inequalities.

The rest of the proof is the same as for the case $\tau_0 > 0$ if we use a consequence of (3.1):

$$\max_{0<t<T} \alpha_\tau \left\| \frac{\partial^2 \boldsymbol{w}^\varepsilon}{\partial t^2}(t) \right\|_{2,\Omega} \leq C.$$

3.2. Let $\lambda_0 = \infty$ and conditions (1.7) hold true. It is obvious that estimates (1.4) are still valid.

The desired estimates (1.8) follow from the basic integral identity (1.3) for $\alpha_\lambda \boldsymbol{w}^\varepsilon$ in the same way as in the case of estimates (1.4). The main difference here is in the term

$$\rho^\varepsilon \boldsymbol{F} \cdot \alpha_\lambda \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t},$$

which now transforms to

$$\Upsilon \equiv \rho_f \boldsymbol{F} \cdot \alpha_\lambda \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t} + (\rho_f - \rho_f)(1-\chi^\varepsilon) \boldsymbol{F} \cdot \alpha_\lambda \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}.$$

The integral of first term in Υ transforms as

$$\rho_f \int_0^t \int_\Omega \nabla \Phi \cdot \alpha_\lambda \frac{\partial \boldsymbol{w}^\varepsilon}{\partial \tau}\, dx\, d\tau = -\rho_f \int_0^t \int_\Omega \Phi \alpha_\lambda \operatorname{div}_x \frac{\partial \boldsymbol{w}^\varepsilon}{\partial \tau}\, dx\, d\tau$$

$$= -\rho_f \int_0^t \int_\Omega \left(\chi^\varepsilon \cdot \Phi \alpha_\lambda \operatorname{div}_x \frac{\partial \boldsymbol{w}^\varepsilon}{\partial \tau} + (1-\chi^\varepsilon) \cdot \Phi \alpha_\lambda \operatorname{div}_x \frac{\partial \boldsymbol{w}^\varepsilon}{\partial \tau} \right) dx\, d\tau$$

$$= -\rho_f \int_\Omega (\chi^\varepsilon \cdot \Phi \alpha_\lambda \operatorname{div}_x \boldsymbol{w}^\varepsilon + (1-\chi^\varepsilon) \cdot \Phi \alpha_\lambda \operatorname{div}_x \boldsymbol{u}^\varepsilon)\, dx$$

$$+ \rho_f \int_0^t \int_\Omega (\chi^\varepsilon \cdot \Phi_\tau \alpha_\lambda \operatorname{div}_x \boldsymbol{w}^\varepsilon + (1-\chi^\varepsilon) \cdot \Phi_\tau \alpha_\lambda \operatorname{div}_x \boldsymbol{u}^\varepsilon)\, dx\, d\tau$$

and is bounded with the help of positive terms

$$\int_\Omega (\chi^\varepsilon (\alpha_p \alpha_\lambda^{-1})(\alpha_\lambda \operatorname{div}_x \boldsymbol{w}^\varepsilon)^2 + (1-\chi^\varepsilon)|\alpha_\lambda \nabla_x \boldsymbol{u}^\varepsilon|^2)\, dx.$$

(The first term appears in the basic identity after using the continuity equation (1.2) in the liquid component.)

The integral of the second term in Υ is bounded with the help of the term

$$\int_\Omega (1-\chi^\varepsilon)|\alpha_\lambda \nabla_x \boldsymbol{u}^\varepsilon|^2\, dx$$

in the same way as before.

Estimates (1.5) follow now from (1.8). The pressures p^ε and π^ε are bounded from the continuity equations (1.2), and the pressure q^ε is bounded from the state equation (1.2) if we use the continuity equation for the liquid component.

If instead of conditions (1.7) one has condition (1.6), then bounds (1.5) for pressures follow from Eqs. (1.2) and bounds (3.1). Note that for all these cases $\beta^\varepsilon = 0$. □

4. Proof of Theorem 2

4.1. Weak and Two-Scale Limits of Sequences of Displacement and Pressures. On the strength of Theorem 1, the sequences $\{p^\varepsilon\}$, $\{q^\varepsilon\}$, $\{\pi^\varepsilon\}$, and $\{\boldsymbol{w}^\varepsilon\}$ are uniformly in ε bounded in $L^2(\Omega_T)$. Hence there exist a subsequence of small parameters $\{\varepsilon > 0\}$ and functions p, q, π, and $\boldsymbol{w}$ such that

$$p^\varepsilon \to p, \quad \pi^\varepsilon \to \pi, \quad q^\varepsilon \to q, \quad \boldsymbol{w}^\varepsilon \to \boldsymbol{w} \ \text{ weakly in } \ L^2(\Omega_T).$$

as $\varepsilon \searrow 0$. Moreover, the sequence $\{(1-\chi^\varepsilon)\nabla \boldsymbol{w}^\varepsilon\}$ is uniformly in ε bounded in $L^2(\Omega_T)$. Due to Lemma 2.1 there is a function $\boldsymbol{u}^\varepsilon \in L^\infty\big(0,T;W_2^1(\Omega)\big)$ such that $\boldsymbol{u}^\varepsilon = \boldsymbol{w}^\varepsilon$ in $\Omega_s \times (0,T)$, and the family $\{\boldsymbol{u}^\varepsilon\}$ is uniformly in ε bounded in $L^\infty\big(0,T;W_2^1(\Omega)\big)$. Therefore, it is possible to extract a subsequence of $\{\varepsilon > 0\}$ such that

$$\boldsymbol{u}^\varepsilon \to \boldsymbol{u} \quad \text{weakly in} \quad L^2\big(0,T;W_2^1(\Omega)\big)$$

as $\varepsilon \searrow 0$. Moreover,

$$\chi^\varepsilon \alpha_\mu D(x,\boldsymbol{w}^\varepsilon) \to 0 \tag{4.1}$$

strongly in $L^2(\Omega_T)$ as $\varepsilon \searrow 0$.

Relabeling if necessary, we assume that the sequences themselves converge.

On the strength of Nguetseng's theorem, there exist 1-periodic in $\boldsymbol{y}$ functions $P(\boldsymbol{x},t,\boldsymbol{y})$, $\Pi(\boldsymbol{x},t,\boldsymbol{y})$, $Q(\boldsymbol{x},t,\boldsymbol{y})$, $\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$, and $\mathbf{U}(\boldsymbol{x},t,\boldsymbol{y})$ such that the sequences $\{p^\varepsilon\}$, $\{\pi^\varepsilon\}$, $\{q^\varepsilon\}$, $\{\boldsymbol{w}^\varepsilon\}$, and $\{\nabla_x \boldsymbol{u}^\varepsilon\}$ two-scale converge to $P(\boldsymbol{x},t,\boldsymbol{y})$, $\Pi(\boldsymbol{x},t,\boldsymbol{y})$, $Q(\boldsymbol{x},t,\boldsymbol{y})$, $\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$ and $\nabla_x \boldsymbol{u} + \nabla_y \boldsymbol{U}(\boldsymbol{x},t,\boldsymbol{y})$, respectively.

Note that the sequence $\{\operatorname{div}_x \boldsymbol{w}^\varepsilon\}$ weakly converges to $\operatorname{div}_x \boldsymbol{w}$ and $\boldsymbol{u} \in L^2\big(0,T;\mathring{W}_2^1(\Omega)\big)$. The last assertion for a disconnected porous space follows from inclusion $\boldsymbol{u}^\varepsilon \in L^2\big(0,T;\mathring{W}_2^1(\Omega)\big)$ and for the connected porous space it follows from the Friedrichs–Poincaré's inequality for $\boldsymbol{u}^\varepsilon$ in the ε-layer of the boundary S and from convergence of sequence $\{\boldsymbol{u}^\varepsilon\}$ to $\boldsymbol{u}$ strongly in $L^2(\Omega_T)$ and weakly in $L^2\big((0,T);W_2^1(\Omega)\big)$.

4.2. Micro- and Macroscopic Equations I.

Lemma 4.1. *For all $\boldsymbol{x} \in \Omega$ and $\boldsymbol{y} \in Y$, weak and two-scale limits of the sequences $\{p^\varepsilon\}$, $\{\pi^\varepsilon\}$, $\{q^\varepsilon\}$, $\{\boldsymbol{w}^\varepsilon\}$, and $\{\boldsymbol{u}^\varepsilon\}$ satisfy the relations*

$$P = \frac{p\chi}{m}, \quad Q = \frac{q\chi}{m}; \tag{4.2}$$

$$\frac{\Pi}{\eta_0} + (1-\chi)(\operatorname{div}_x \boldsymbol{u} + \operatorname{div}_y \mathbf{U}) = \frac{\beta(1-\chi)}{1-m}; \tag{4.3}$$

$$\operatorname{div}_y \boldsymbol{W} = 0; \tag{4.4}$$

$$\boldsymbol{W} = \chi(\boldsymbol{y})\boldsymbol{W} + (1-\chi)\boldsymbol{u}; \tag{4.5}$$

$$q = p + \nu_0 p_*^{-1}\frac{\partial p}{\partial t}; \tag{4.6}$$

$$\frac{p}{p_*} + \operatorname{div}_x \boldsymbol{w} = (1-m)\operatorname{div}_x \boldsymbol{u} + \langle \operatorname{div}_y \mathbf{U}\rangle_{Y_s} - \beta; \tag{4.7}$$

$$\frac{\pi}{\eta_0} + (1-m)\operatorname{div}_x \boldsymbol{u} + \langle \operatorname{div}_y \mathbf{U}\rangle_{Y_s} = \beta, \tag{4.8}$$

where $\beta = \int_\Omega \langle \operatorname{div}_y \boldsymbol{U}\rangle_{Y_s}\,dx$, if $p_* + \eta_0 = \infty$ and $\beta = 0$ if $p_* + \eta_0 < \infty$.

Proof. In order to prove Eq. (4.2), into Eq. (1.3) insert a test function $\boldsymbol{\psi}^\varepsilon = \varepsilon \boldsymbol{\psi}(\boldsymbol{x},t,\boldsymbol{x}/\varepsilon)$, where $\boldsymbol{\psi}(\boldsymbol{x},t,\boldsymbol{y})$ is an arbitrary 1-periodic and finite on Y_f function in $\boldsymbol{y}$. Passing to the limit as $\varepsilon \searrow 0$, we get

$$\nabla_y Q(\boldsymbol{x},t,\boldsymbol{y}) = 0, \quad \boldsymbol{y} \in Y_f. \tag{4.9}$$

The weak and two-scale limiting passage in the state equation (1.2) yield that Eq. (4.6) and the equation

$$Q = P + \frac{\nu_0}{p_*}\frac{\partial P}{\partial t} \tag{4.10}$$

hold. Taking into account Eqs. (4.9) and (4.10), we get $\nabla_y P(\boldsymbol{x},t,\boldsymbol{y}) = 0$, $\boldsymbol{y} \in Y_f$. Next, fulfilling the two-scale limiting passage in equalities $(1-\chi^\varepsilon)p^\varepsilon = 0$ and $(1-\chi^\varepsilon)q^\varepsilon = 0$, we arrive at $(1-\chi)P = 0$ and $(1-\chi)Q = 0$, which along with Eqs. (4.9) and (4.10) justifies (4.2).

Equations (4.3), (4.4), (4.7), and (4.8) appear as the results of two-scale limiting passages in Eqs. (3.3), (3.4) (where $\beta^\varepsilon = \langle (1-\chi^\varepsilon)\operatorname{div}_x \boldsymbol{w}^\varepsilon\rangle_\Omega$ if $p_* + \eta_0 = \infty$ and $\beta^\varepsilon = 0$ if $p_* + \eta_0 < \infty$) with the proper test functions being involved. Thus, for example, Eq. (4.7) arises if we represent Eq. (3.3) in the form

$$\frac{1}{\alpha_p}p^\varepsilon + \operatorname{div}_x \boldsymbol{w}^\varepsilon = (1-\chi^\varepsilon)\operatorname{div}_x \boldsymbol{u}^\varepsilon - \frac{1}{m}\beta^\varepsilon\chi^\varepsilon, \tag{4.11}$$

multiply it by an arbitrary function independent of the "fast" variable $\boldsymbol{x}/\varepsilon$, and then pass to the limit as $\varepsilon \searrow 0$. Equation (4.8) is derived quite similarly. In order to prove Eq. (4.4), it is sufficient to consider the two-scale limiting relations in Eq. (4.11) as $\varepsilon \searrow 0$ with the test functions $\varepsilon\psi(\boldsymbol{x}/\varepsilon)h(\boldsymbol{x},t)$, where ψ and h are arbitrary smooth test functions.

In order to prove Eq. (4.5) it is sufficient to consider the two-scale limiting relations in $(1-\chi^\varepsilon)\times(\boldsymbol{w}^\varepsilon - \boldsymbol{u}^\varepsilon) = 0$. □

Corollary 4.1. *If $p_* + \eta_0 = \infty$, then the weak limits p, π, and q satisfy the relations*

$$\langle p\rangle_\Omega = \langle \pi\rangle_\Omega = \langle q\rangle_\Omega = 0. \tag{4.12}$$

Lemma 4.2. *For all $(\boldsymbol{x},t)\in\Omega_T$ the relation*

$$\operatorname{div}_y\Big\{\lambda_0(1-\chi)\big(D(y,\mathbf{U}) + D(x,\boldsymbol{u})\big) - \Big(\Pi + \frac{1}{m}q\chi\Big)\cdot\mathbb{I}\Big\} = 0 \tag{4.13}$$

holds true.

Proof. Substituting a test function of the form $\boldsymbol{\psi}^\varepsilon = \varepsilon\boldsymbol{\psi}(\boldsymbol{x},t,\boldsymbol{x}/\varepsilon)$, where $\boldsymbol{\psi}(\boldsymbol{x},t,\boldsymbol{y})$ is an arbitrary 1-periodic in $\boldsymbol{y}$ function vanishing on the boundary S, into Eq. (1.3) and passing to the limit as $\varepsilon\searrow 0$, we arrive at the desired microscopic relation on the cell Y. □

Lemma 4.3. *Let $\hat\rho = m\rho_f + (1-m)\rho_s$, $\boldsymbol{V} = \chi\partial\boldsymbol{w}/\partial t$, and $\boldsymbol{v} = \langle\boldsymbol{V}\rangle_Y$. Then for all $0\le\tau_0<\infty$ the quadruple of functions*

$$\tau_0\rho_f\frac{\partial\boldsymbol{v}}{\partial t} + \tau_0\rho_s(1-m)\frac{\partial^2\boldsymbol{u}}{\partial t^2} = \operatorname{div}_x\big\{\lambda_0\big((1-m)D(x,\boldsymbol{u}) + \langle D(y,\boldsymbol{U})\rangle_{Y_s}\big) - (q+\pi)\cdot\mathbb{I}\big\} + \hat\rho\boldsymbol{F}. \tag{4.14}$$

Proof. Equations (4.14) arise as the limit of Eqs. (1.3) with the test functions being finite in Ω_T and independent of ε. □

4.3. Micro- and Macroscopic Equations II.

Lemma 4.4. *If $\mu_1=\infty$, then the weak and two-scale limits of $\{\boldsymbol{u}^\varepsilon\}$ and $\{\boldsymbol{w}^\varepsilon\}$ coincide.*

Proof. In order to verify this, it suffices to consider the difference $\boldsymbol{u}^\varepsilon - \boldsymbol{w}^\varepsilon$ and apply Friedrichs–Poincaré's inequality, just like in the proof of Theorem 1. □

Lemma 4.5. *Let $\mu_1<\infty$. Then the weak and two-scale limits of $\{q^\varepsilon\}$ and $\{\chi^\varepsilon\boldsymbol{w}^\varepsilon\}$ satisfy the microscopic relations*

$$\tau_0\rho_f\frac{\partial\boldsymbol{V}}{\partial t} = \mu_1\Delta_y\boldsymbol{V} - \nabla_y R - \frac{1}{m}\nabla_x q + \rho_f\boldsymbol{F}, \quad \boldsymbol{y}\in Y_f, \tag{4.15}$$

$$\boldsymbol{V} = \frac{\partial\boldsymbol{u}}{\partial t}, \quad \boldsymbol{y}\in\gamma, \tag{4.16}$$

in the case $\mu_1>0$, and relations

$$\tau_0\rho_f\frac{\partial\boldsymbol{V}}{\partial t} = -\nabla_y R - \frac{1}{m}\nabla_x q + \rho_f\boldsymbol{F}, \quad \boldsymbol{y}\in Y_f, \tag{4.17}$$

$$\Big(\boldsymbol{V} - \frac{\partial\boldsymbol{u}}{\partial t}\Big)\cdot\mathbf{n} = 0, \quad \boldsymbol{y}\in\gamma, \tag{4.18}$$

in the case $\mu_1=0$. In Eq. (4.18) $\boldsymbol{n}$ is the unit normal to γ.

Proof. Differential equations (4.15) and (4.17) follow as $\varepsilon \searrow 0$ from integral equality (1.3) with the test function $\boldsymbol{\psi} = \boldsymbol{\varphi}(x\varepsilon^{-1}) \cdot h(\boldsymbol{x}, t)$, where $\boldsymbol{\varphi}$ is solenoidal and finite in Y_f.

Boundary conditions (4.16) are the consequences of the two-scale convergence of $\{\alpha_\mu^{\frac{1}{2}} \nabla_x \boldsymbol{w}^\varepsilon\}$ to the function $\mu_1^{\frac{1}{2}} \nabla_y \mathbf{W}(\boldsymbol{x}, t, \boldsymbol{y})$. On the strength of this convergence, the function $\nabla_y \boldsymbol{W}(\boldsymbol{x}, t, \boldsymbol{y})$ is L^2-integrable in Y. The boundary conditions (4.18) follow from Eq. (4.4). □

Lemma 4.6. *If the porous space is disconnected, which is the case of isolated pores, then the weak and two-scale limits of sequences* $\{\boldsymbol{u}^\varepsilon\}$ *and* $\{\boldsymbol{w}^\varepsilon\}$ *coincide.*

Proof. Indeed, in the case $0 \leq \mu_1 < \infty$ the systems of equations (4.4), (4.15), (4.16) or (4.4), (4.17), (4.18) have the unique solution $\boldsymbol{V} = \partial \boldsymbol{u}/\partial t$. □

4.4. Homogenized Equations I.

Lemma 4.7. *If* $\mu_1 = \infty$ *or the porous space is disconnected, then* $\boldsymbol{w} = \boldsymbol{u}$ *and the weak limits* $\boldsymbol{u}$, p, q, *and* π *satisfy in* Ω_T *the initial-boundary-value problem*

$$\tau_0 \hat{\rho} \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \operatorname{div}_x \{\lambda_0 \mathbb{A}_0^s : D(x, \boldsymbol{u}) + B_0^s \operatorname{div}_x \boldsymbol{u} + B_1^s q - (q + \pi) \cdot \mathbb{I}\} + \hat{\rho} \boldsymbol{F}, \tag{4.19}$$

$$\frac{1}{\eta_0} \pi + C_0^s : D(x, \boldsymbol{u}) + a_0^s \operatorname{div}_x \boldsymbol{u} + a_1^s q = 0, \tag{4.20}$$

$$q = p + \nu_0 p_*^{-1} \frac{\partial p}{\partial t}, \quad \frac{1}{p_*} p + \frac{1}{\eta_0} \pi + \operatorname{div}_x \boldsymbol{u} = 0, \tag{4.21}$$

where the symmetric strictly positively definite constant fourth-rank tensor $\mathbb{A}_0^s$, *matrices* C_0^s, B_0^s, *and* B_1^s, *and constants* a_0^s *and* a_1^s *are defined below by formulas* (4.23), (4.25), (4.26).

Differential equations (4.19) *are endowed with homogeneous initial and boundary conditions*

$$\tau_0 \boldsymbol{u}(\boldsymbol{x}, 0) = \tau_0 \frac{\partial \boldsymbol{u}}{\partial t}(\boldsymbol{x}, 0) = 0, \quad \boldsymbol{x} \in \Omega, \qquad \boldsymbol{u}(\boldsymbol{x}, t) = 0, \quad \boldsymbol{x} \in S, \quad t > 0. \tag{4.22}$$

Proof. In the first place, let us note that $\boldsymbol{u} = \boldsymbol{w}$ due to Lemmas 4.4 and 4.6.

The homogenized equations (4.19) follow from the macroscopic equations (4.14), after we insert in them the expression

$$\lambda_0 \langle D(y, \mathbf{U}) \rangle_{Y_s} = \lambda_0 \mathbb{A}_1^s : D(x, \boldsymbol{u}) + B_0^s \operatorname{div}_x \boldsymbol{u} + B_1^s q.$$

In turn, this expression follows by virtue of solutions of Eqs. (4.3) and (4.13) on the pattern cell Y_s. Indeed, setting

$$\boldsymbol{U} = \sum_{i,j=1}^{3} \boldsymbol{U}^{ij}(\boldsymbol{y}) D_{ij} + \boldsymbol{U}_0(\boldsymbol{y}) \operatorname{div}_x \boldsymbol{u} + \frac{1}{m} \boldsymbol{U}_1(\boldsymbol{y}) q,$$

$$\Pi = \lambda_0 \sum_{i,j=1}^{3} \Pi^{ij}(\boldsymbol{y}) D_{ij} + \Pi_0(\boldsymbol{y}) \operatorname{div}_x \boldsymbol{u} + \frac{1}{m} \Pi_1(\boldsymbol{y}) q,$$

where

$$D_{ij}(\boldsymbol{x}, t) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j}(\boldsymbol{x}, t) + \frac{\partial u_j}{\partial x_i}(\boldsymbol{x}, t) \right),$$

we arrive at the following periodic-boundary value problems in Y_s:

$$\left.\begin{aligned} \operatorname{div}_y \{(1 - \chi)(D(y, \boldsymbol{U}^{ij}) + J^{ij}) - \Pi^{ij} \cdot \mathbb{I}\} = 0, \\ \frac{\lambda_0}{\eta_0} \Pi^{ij} + (1 - \chi) \operatorname{div}_y \boldsymbol{U}^{ij} = 0; \end{aligned}\right\}$$

$$\left.\begin{aligned} \operatorname{div}_y\{\lambda_0(1-\chi)D(y,\boldsymbol{U}_0)-\Pi_0\cdot\mathbb{I}\}=0,\\ \frac{1}{\eta_0}\Pi_0+(1-\chi)(\operatorname{div}_y\boldsymbol{U}_0+1)=0; \end{aligned}\right\}$$

$$\left.\begin{aligned} \operatorname{div}_y\{\lambda_0(1-\chi)D(y,\boldsymbol{U}_1)-(\Pi_1+\chi)\cdot\mathbb{I}\}=0,\\ \frac{1}{\eta_0}\Pi_1+(1-\chi)\operatorname{div}_y\boldsymbol{U}_1=0. \end{aligned}\right\}$$

Note that $\beta=0$ even if $p_*+\eta_0=\infty$ due to homogeneous boundary condition for $\boldsymbol{u}(\boldsymbol{x},t)$ and relations (4.12).

On the strength of the assumptions on the geometry of the pattern "solid" cell Y_s, the above-mentioned problems have unique solution up to an arbitrary constant vector. In order to discard the arbitrary constant vectors we demand

$$\langle\boldsymbol{U}^{ij}\rangle_{Y_s}=\langle\boldsymbol{U}_0\rangle_{Y_s}=\langle\boldsymbol{U}_1\rangle_{Y_s}=0.$$

Thus,

$$\mathbb{A}_0^s=(1-m)\sum_{i,j=1}^{3}J^{ij}\otimes J^{ij}+\mathbb{A}_1^s,\quad \mathbb{A}_1^s=\sum_{i,j=1}^{3}\langle D(y,\boldsymbol{U}^{ij})\rangle_{Y_s}\otimes J^{ij}. \tag{4.23}$$

Symmetry of the tensor $\mathbb{A}_0^s$ follows from symmetry of the tensor $\mathbb{A}_1^s$, and symmetry of the latter follows from the equality

$$\langle D(y,\boldsymbol{U}^{ij})\rangle_{Y_s}:J^{kl}=-\langle D(y,\boldsymbol{U}^{ij}):D(y,\boldsymbol{U}^{kl})\rangle_{Y_s}-\frac{\lambda_0}{\eta_0}\Pi^{ij}\Pi^{kl}, \tag{4.24}$$

which appears by means of multiplication of the equation for $\boldsymbol{U}^{ij}$ by $\boldsymbol{U}^{kl}$ and by integration by parts.

This equality also implies positive definiteness of the tensor $\mathbb{A}_0^s$. Indeed, let ζ be an arbitrary symmetric constant matrix. Setting

$$\mathbb{Z}=\sum_{i,j=1}^{3}\boldsymbol{U}^{ij}\zeta_{ij},\quad \tilde{\Pi}=\sum_{i,j=1}^{3}\Pi^{ij}\zeta_{ij}$$

and taking into account Eq. (4.24), we have

$$\langle D(y,\mathbb{Z})\rangle_{Y_s}:\zeta=-\langle D(y,\mathbb{Z}):D(y,\mathbb{Z})\rangle_{Y_s}-\frac{\lambda_0}{\eta_0}\tilde{\Pi}^2.$$

This equality and the definition of the tensor A_0^s give

$$(\mathbb{A}_0^s:\zeta):\zeta=\langle(D(y,\mathbb{Z})+\zeta):(D(y,\mathbb{Z})+\zeta)\rangle_{Y_s}+\frac{\lambda_0}{\eta_0}\tilde{\Pi}^2.$$

Now the strict positive definiteness of the tensor $\mathbb{A}_0^s$ follows immediately from the equality above and from the geometry of the elementary cell Y_s. Namely, suppose that $(\mathbb{A}_0^s:\zeta):\zeta=0$ for some constant matrix ζ such that $\zeta:\zeta=1$. Then $\mathbb{D}(y,\mathbb{Z})+\zeta=0$, which is possible if and only if $\mathbb{Z}$ is a linear function in $\boldsymbol{y}$. On the other hand, all linear periodic functions on Y_s are constant. Finally, the normalization condition $\langle\boldsymbol{U}^{ij}\rangle_{Y_s}=0$ yields that $\mathbb{Z}=0$. However, this is impossible, because $\zeta:\zeta=1$.

Finally, Eqs. (4.20) and (4.21) for the pressures follow from Eqs. (4.6)–(4.8) and

$$B_0^s=\lambda_0\langle D(y,\boldsymbol{U}_0)\rangle_{Y_s},\quad B_1^s=\frac{\lambda_0}{m}\langle D(y,\boldsymbol{U}_1)\rangle_{Y_s},\quad a_1^s=\frac{1}{m}\langle\operatorname{div}_y\boldsymbol{U}_1\rangle_{Y_s}, \tag{4.25}$$

$$C_0^s=\sum_{i,j=1}^{3}\langle\operatorname{div}_y\boldsymbol{U}^{ij}\rangle_{Y_s}J^{ij},\quad a_0^s=1-m+\langle\operatorname{div}_y\boldsymbol{U}_0\rangle_{Y_s}. \tag{4.26}$$

□

4.5. Homogenized Equations II. Let $\mu_1 < \infty$. In the same manner as above, we verify that the limit $\boldsymbol{u}$ of the sequence $\{\boldsymbol{u}^\varepsilon\}$ satisfies the initial-boundary-value problem like (4.19)–(4.22). The main difference here is that, in general, the weak limit $\boldsymbol{w}$ of the sequence $\{\boldsymbol{w}^\varepsilon\}$ differs from $\boldsymbol{u}$. More precisely, the following statement is true.

Lemma 4.8. *If $\mu_1 < \infty$, then the weak limits $\boldsymbol{u}$, $\boldsymbol{w}^f$, p, q, and π of the sequences $\{\boldsymbol{u}^\varepsilon\}$, $\{\chi^\varepsilon \boldsymbol{w}^\varepsilon\}$, $\{p^\varepsilon\}$, $\{q^\varepsilon\}$, and $\{\pi^\varepsilon\}$ satisfy the initial-boundary-value problem in Ω_T, consisting of the balance of momentum equation*

$$\tau_0\Big(\rho_f\frac{\partial \boldsymbol{v}}{\partial t}+\rho_s(1-m)\frac{\partial^2\boldsymbol{u}}{\partial t^2}\Big)+\nabla(q+\pi)-\hat{\rho}\boldsymbol{F}=\operatorname{div}_x\{\lambda_0 A_0^s : D(x,\boldsymbol{u})+B_0^s\operatorname{div}_x\boldsymbol{u}+B_1^s q\} \tag{4.27}$$

and the continuity equation (4.20) for the solid component, where $\boldsymbol{v}=\partial\boldsymbol{w}^f/\partial t$ and $\mathbb{A}_0^s$, B_0^s, and B_1^s are the same as in (4.19), the state and continuity equations

$$p+\nu_0 p_*^{-1}\frac{\partial p}{\partial t}=q,\qquad \frac{1}{p_*}p+\frac{1}{\eta_0}\pi+\operatorname{div}_x\boldsymbol{w}^f=(m-1)\operatorname{div}_x\boldsymbol{u} \tag{4.28}$$

for the liquid component, and the relation

$$\boldsymbol{v}=m\frac{\partial\boldsymbol{u}}{\partial t}+\int_0^t B_1(\mu_1,t-\tau)\cdot\boldsymbol{z}(\boldsymbol{x},\tau)\,d\tau, \tag{4.29}$$

$$\boldsymbol{z}(\boldsymbol{x},t)=-\frac{1}{m}\nabla q(\boldsymbol{x},t)+\rho_f\boldsymbol{F}(\boldsymbol{x},t)-\tau_0\rho_f\frac{\partial^2\boldsymbol{u}}{\partial t^2}(\boldsymbol{x},t)$$

in the case $\tau_0>0$ and $\mu_1>0$, or Darcy's law in the form

$$\boldsymbol{v}=m\frac{\partial\boldsymbol{u}}{\partial t}+B_2(\mu_1)\cdot\Big(-\frac{1}{m}\nabla q+\rho_f\boldsymbol{F}\Big) \tag{4.30}$$

in the case $\tau_0=0$ and $\mu_1>0$, or, finally, the balance of momentum equation for the liquid component in the form

$$\tau_0\rho_f\frac{\partial\boldsymbol{v}}{\partial t}=\tau_0\rho_f B_3\cdot\frac{\partial^2\boldsymbol{u}}{\partial t^2}+(m\mathbb{I}-B_3)\cdot\Big(-\frac{1}{m}\nabla q+\rho_f\boldsymbol{F}\Big) \tag{4.31}$$

in the case $\tau_0>0$ and $\mu_1=0$. The problem is supplemented by boundary and initial conditions (4.22) for the displacement $\boldsymbol{u}$ of the rigid component and by the boundary condition

$$\boldsymbol{v}(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x})=0,\quad \boldsymbol{x}\in S,\quad t>0, \tag{4.32}$$

for the velocity $\boldsymbol{v}$ of the liquid component. In Eqs. (4.29)–(4.32) $\boldsymbol{n}(\boldsymbol{x})$ is the unit normal vector to S at a point $\boldsymbol{x}\in S$, and matrices $B_1(\mu_1,t)$, $B_2(\mu_1)$, and B_3 are given below by Eqs. (4.34)–(4.38).

Proof. The homogenized equations of balance of momentum and balance of mass derive exactly as (4.19), (4.20). For example, to get Eq. (4.28) we just express $\operatorname{div}_x\boldsymbol{w}$ as a sum of Eqs. (4.7) and (4.8) using Eq. (4.5) after homogenization: $\boldsymbol{w}=\boldsymbol{w}^f+(1-m)\boldsymbol{u}$. Therefore, we omit the relevant proofs now and focus only on derivation of homogenized equations for the velocity $\boldsymbol{v}$. The derivation of boundary condition (4.32) is standard [3].

(a) If $\mu_1>0$ and $\tau_0>0$, then the solution of the system of microscopic equations (4.4), (4.15), and (4.16), provided with the homogeneous initial data, is given by the formula

$$\boldsymbol{V}=\frac{\partial\boldsymbol{u}}{\partial t}+\int_0^t \mathbf{B}_1^f(\boldsymbol{y},t-\tau)\cdot\boldsymbol{z}(\boldsymbol{x},\tau)\,d\tau,\quad R=\int_0^t R_f(\boldsymbol{y},t-\tau)\cdot\boldsymbol{z}(\boldsymbol{x},\tau)\,d\tau,$$

in which

$$\mathbf{B}_1^f(\boldsymbol{y},t)=\sum_{i=1}^3\boldsymbol{V}^i(\boldsymbol{y},t)\otimes\boldsymbol{e}_i,\quad R_f(\boldsymbol{y},t)=\sum_{i=1}^3 R^i(\boldsymbol{y},t)\boldsymbol{e}_i,$$

and the functions $\mathbf{V}^i(\boldsymbol{y},t)$ and $R^i(\boldsymbol{y},t)$ are defined by virtue of the periodic initial-boundary-value problem

$$\left.\begin{aligned} &\tau_0\rho_f\frac{\partial \boldsymbol{V}^i}{\partial t}-\mu_1\Delta \boldsymbol{V}^i+\nabla R^i=0, \quad \operatorname{div}_y \boldsymbol{V}^i=0, \quad \boldsymbol{y}\in Y_f, \quad t>0,\\ &\boldsymbol{V}^i=0, \quad \boldsymbol{y}\in\gamma, \quad t>0; \quad \tau_0\rho_f\boldsymbol{V}^i(y,0)=\boldsymbol{e}_i, \quad \boldsymbol{y}\in Y_f. \end{aligned}\right\} \tag{4.33}$$

In Eq. (4.33) $\boldsymbol{e}_i$ is the standard Cartesian basis vector.

Therefore,

$$B_1(\mu_1,t)=\langle \mathbf{B}_1^f\rangle_{Y_f}(t). \tag{4.34}$$

(b) If $\tau_0=0$ and $\mu_1>0$, then the solution of the stationary microscopic equations (4.4), (4.15), and (4.16) is given by the formula

$$\boldsymbol{V}=\frac{\partial \boldsymbol{u}}{\partial t}+\mathbf{B}_2^f(\boldsymbol{y})\cdot(-\nabla q+\rho_f\boldsymbol{F}),$$

in which

$$\mathbf{B}_2^f(\boldsymbol{y})=\sum_{i=1}^{3}\boldsymbol{U}^i(\boldsymbol{y})\otimes\boldsymbol{e}_i,$$

and the functions $\mathbf{U}^i(\boldsymbol{y})$ are defined from the periodic boundary-value problem

$$\left.\begin{aligned} &-\mu_1\Delta\boldsymbol{U}^i+\nabla R^i=\boldsymbol{e}_i, \quad \operatorname{div}_y\boldsymbol{U}^i=0, \quad \boldsymbol{y}\in Y_f,\\ &\boldsymbol{U}^i=0, \quad \boldsymbol{y}\in\gamma. \end{aligned}\right\} \tag{4.35}$$

Thus,

$$B_2(\mu_1)=\langle \mathbf{B}_2^f((\boldsymbol{y}))\rangle_{Y_f}. \tag{4.36}$$

The matrix $B_2(\mu_1)$ is symmetric and strictly positively definite [3, Chap. 8].

(c) If $\tau_0>0$ and $\mu_1=0$, then in the process of solving the system (4.4), (4.17), and (4.18) we firstly find the pressure $R(\boldsymbol{x},t,\boldsymbol{y})$ by solving the Neumann problem for Laplace's equation in Y_f. If

$$R(\boldsymbol{x},t,\boldsymbol{y})=\sum_{i=1}^{3}R_i(\boldsymbol{y})\boldsymbol{e}_i\cdot\boldsymbol{z}(\boldsymbol{x},t),$$

where $R^i(\boldsymbol{y})$ is the solution of the problem

$$\Delta R_i=0, \quad \boldsymbol{y}\in Y_f; \quad \nabla R_i\cdot\boldsymbol{n}=\boldsymbol{n}\cdot\boldsymbol{e}_i, \quad \boldsymbol{y}\in\gamma, \tag{4.37}$$

then formula (4.31) appears as the result of homogenization of (4.17) and

$$B_3=\sum_{i=1}^{3}\langle\nabla R_i(\boldsymbol{y})\rangle_{Y_f}\otimes\boldsymbol{e}_i, \tag{4.38}$$

where the matrix $B=(m\mathbb{I}-B_3)$ is symmetric and positively definite. In fact, let

$$\tilde{R}=\sum_{i=1}^{3}R_i\xi_i$$

for any unit vector ξ. Then $(B\cdot\xi)\cdot\xi=\langle(\xi-\nabla\tilde{R})^2\rangle_{Y_f}>0$ due to the same reasons as in Lemma 4.7. □

5. Proof of Theorem 3

5.1. Weak and Two-Scale Limits of Sequences of Displacement and Pressures.

(I) Let one of the conditions (1.6) or (1.7) hold true. Then on the strength of Theorems 1 and 7 we conclude that sequences $\{\chi^\varepsilon \boldsymbol{w}^\varepsilon\}$, $\{p^\varepsilon\}$, and $\{q^\varepsilon\}$ two-scale converge to $\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$, $P(\boldsymbol{x},t,\boldsymbol{y})$, and $Q(\boldsymbol{x},t,\boldsymbol{y})$ and weakly converge in $L^2(\Omega_T)$ to $\boldsymbol{w}^f$, p, and q, respectively, and a sequence $\{\boldsymbol{u}^\varepsilon(\boldsymbol{x},t)\}$, where $\boldsymbol{u}^\varepsilon(\boldsymbol{x},t)$ is an extension of $\boldsymbol{w}^\varepsilon(\boldsymbol{x},t)$ from the domain Ω^ε_s into domain Ω, strongly converges in $L^2(\Omega_T)$ and weakly in $L^2\big((0,T);W^1_2(\Omega)\big)$ to zero.

(II) If $\mu_1<\infty$ and conditions (1.7) hold true, then due to estimates (1.5) and (1.8) the sequence $\{\alpha_\lambda \boldsymbol{u}^\varepsilon\}$ converges strongly in $L^2(\Omega_T)$ and weakly in $L^2\big((0,T);W^1_2(\Omega)\big)$ to a function $\boldsymbol{u}$, and the sequence $\{\pi^\varepsilon\}$ converges weakly in $L^2(\Omega_T)$ to a function π.

(III) If $\mu_1=\infty$, $p_1^{-1},\eta_1^{-1}<\infty$, and $0<\lambda_1<\infty$, then on the strength of part (III) of Theorem 1 the sequences $\{\alpha_\mu\varepsilon^{-2}\chi^\varepsilon \boldsymbol{w}^\varepsilon\}$, $\{p^\varepsilon\}$, $\{\pi^\varepsilon\}$, and $\{q^\varepsilon\}$ two-scale converge to functions $\chi(\boldsymbol{y})\tilde{\boldsymbol{W}}(\boldsymbol{x},t,\boldsymbol{y})$, $P(\boldsymbol{x},t,\boldsymbol{y})$, $\Pi(\boldsymbol{x},t,\boldsymbol{y})$, and $Q(\boldsymbol{x},t,\boldsymbol{y})$ and weakly in $L^2(\Omega_T)$ to functions $\tilde{\boldsymbol{w}}$, p, π, and q, respectively, and the sequence $\{\alpha_\mu\varepsilon^{-2}\boldsymbol{u}^\varepsilon\}$ strongly converges in $L^2(\Omega_T)$ and weakly in $L^2\big((0,T);W^1_2(\Omega)\big)$ to the function $\tilde{\boldsymbol{u}}$.

As before in Sec. 4, we conclude that $\tilde{\boldsymbol{u}}\in L^2\big(0,T;\mathring{W}^1_2(\Omega)\big)$.

5.2. Homogenized Equations.

(I) If $\mu_1<\infty$ and one of the conditions (1.6) or (1.7) holds true, then, as in the proof of Theorem 2, we construct a closed system of equations for the velocity $\boldsymbol{v}=\partial\boldsymbol{w}^f/\partial t$ in the liquid component and for the pressures p and q, consisting of the modifications of the momentum conservation law (4.29)–(4.31) and boundary condition (4.32), in which we have $\boldsymbol{u}(\boldsymbol{x},t)=0$, and of the equations

$$p+\nu_0 p_*^{-1}\frac{\partial p}{\partial t}=q,\quad \frac{1}{p_*}\frac{\partial p}{\partial t}+\operatorname{div}_x\boldsymbol{v}=0,\quad \boldsymbol{x}\in\Omega,\quad t>0. \tag{5.1}$$

We name the above-described systems Problem F_1, F_2, or F_3 depending on the form of the matrices B_1, B_2, or B_3, that occur in Eqs. (4.29)–(4.31).

(II) Let $\mu_1<\infty$ and condition (1.7) hold true. We observe that the limiting displacements in the rigid skeleton are equal to zero. In order to find a more accurate asymptotic of the solution of the original model, we use again re-normalization. Namely, let $\boldsymbol{w}^\varepsilon\to\alpha_\lambda\boldsymbol{w}^\varepsilon$. Then the new displacements satisfy the same problem as displacements before re-normalization, but with the new parameters

$$\alpha_\eta\to\alpha_\eta\alpha_\lambda^{-1},\quad \alpha_\lambda\to 1,\quad \alpha_\tau\to\alpha_\tau\alpha_\lambda^{-1}.$$

Thus, we arrive at the assumptions of Theorem 2. Namely, the limiting functions $\boldsymbol{u}(\boldsymbol{x},t)$, $\pi(\boldsymbol{x},t)$, $\Pi(\boldsymbol{x},t,\boldsymbol{y})$, and $\boldsymbol{U}(\boldsymbol{x},t,\boldsymbol{y})$ satisfy the same system of micro- and macroscopic equations defining the behavior of the solid component, in which the pressure q is given by virtue of one of Problems F_1–F_3. The only difference from the already considered case is in the micro- and macroscopic continuity equations, because this equation depends on the value η_2. These micro- and macroscopic continuity equations coincide with (4.3) and (4.8) if we put there $\eta_0=\eta_2$.

Hence for $\boldsymbol{u}(\boldsymbol{x},t)$ and $\pi(\boldsymbol{x},t)$ there hold true the homogenized momentum equation in the form

$$0=\operatorname{div}_x\{\mathbb{A}^s_0:D(x,\boldsymbol{u})+B^s_0\operatorname{div}_x\boldsymbol{u}+B^s_1 q-(q+\pi)\cdot\mathbb{I}\}+\hat{\rho}\boldsymbol{F},\quad \boldsymbol{x}\in\Omega, \tag{5.2}$$

the macroscopic continuity equation (4.20), in which we have $\eta_0=\eta_2$, and the boundary condition (1.12).

The tensor $\mathbb{A}^s_0$, the matrices C^s_0, B^s_0, and B^s_1, and the constants a^s_0 and a^s_1 are defined from Eqs. (4.23), (4.25), (4.26), in which we have $\eta_0=\eta_2$ and $\lambda_0=1$.

(III) If $\mu_1=\infty$, $p_1^{-1},\eta_1^{-1}<\infty$, and $0<\lambda_1<\infty$ then re-normalizing by $\boldsymbol{w}^\varepsilon\to\alpha_\mu\varepsilon^{-2}\boldsymbol{w}^\varepsilon$ we arrive at the assumptions of Theorem 2, when $\mu_1=1$, $\tau_0=0$, and $\lambda_0=\lambda_1$. Namely, functions $\tilde{\boldsymbol{w}}$, p, π, and $\tilde{\boldsymbol{u}}$ satisfy the following initial-boundary-value problem in Ω_T:

$$\left.\begin{aligned}
&\operatorname{div}_x\{\lambda_1\mathbb{A}^s_0 : D(x,\tilde{\boldsymbol{u}}) + B^s_0 \operatorname{div}_x \tilde{\boldsymbol{u}} + B^s_1 p - (p+\pi)\cdot\mathbb{I}\} + \hat{\rho}\boldsymbol{F} = 0,\\
&\frac{\partial \tilde{\boldsymbol{w}}}{\partial t} = \frac{\partial \tilde{\boldsymbol{u}}}{\partial t} + B_2(1)\cdot\left(-\frac{1}{m}\nabla p + \rho_f \boldsymbol{F}\right),\\
&\frac{1}{p_1}p + \frac{1}{\eta_1}\pi + \operatorname{div}_x \tilde{\boldsymbol{w}} = (m-1)\operatorname{div}_x \tilde{\boldsymbol{u}},\\
&\frac{1}{\eta_1}\pi + C^s_0 : D(x,\tilde{\boldsymbol{u}}) + a^s_0 \operatorname{div}_x \tilde{\boldsymbol{u}} + a^s_1 p = 0.
\end{aligned}\right\} \tag{5.3}$$

As before, the tensor $\mathbb{A}^s_0$, the matrices C^s_0, B^s_0, and B^s_1, and the constants a^s_0 and a^s_1 are defined by formulas (4.23), (4.25), (4.26), in which we have $\eta_0 = \eta_1$ and $\lambda_0 = \lambda_2$.

Note that here $\nu_0 = 0$. Therefore, the state equation $p + \nu_0 p_*^{-1}\partial p/\partial t = q$ becomes $p = q$.

The problem is endowed by the corresponding homogeneous initial and boundary conditions. $\square$

6. Proof of Theorem 4

6.1. Weak and Two-Scale Limits of Sequences of Displacement and Pressures. On the strength of Theorem 1, the sequences $\{p^\varepsilon\}$, $\{q^\varepsilon\}$, $\{\pi^\varepsilon\}$, and $\{\boldsymbol{w}^\varepsilon\}$ are uniformly in ε bounded in $L^2(\Omega_T)$. Then there exist a subsequence from $\{\varepsilon > 0\}$ and functions p, π, q, and $\boldsymbol{w}$ such that as $\varepsilon \searrow 0$

$$p^\varepsilon \to p, \quad q^\varepsilon \to q, \quad \pi^\varepsilon \to \pi, \quad \boldsymbol{w}^\varepsilon \to \boldsymbol{w} \text{ weakly in } L^2(\Omega_T).$$

Moreover, since $\lambda_0, \mu_0 > 0$, the bound (1.4) implies

$$\nabla_x \boldsymbol{w}^\varepsilon \underset{\varepsilon\searrow 0}{\longrightarrow} \nabla_x \boldsymbol{w} \text{ weakly in } L^2(\Omega_T).$$

Due to Ngutseng's theorem, there exist one more subsequence from $\{\varepsilon > 0\}$ and 1-periodic in $\boldsymbol{y}$ functions $P(\boldsymbol{x},t,\boldsymbol{y})$, $\Pi(\boldsymbol{x},t,\boldsymbol{y})$, $Q(\boldsymbol{x},t,\boldsymbol{y})$, and $\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$ such that the sequences $\{p^\varepsilon\}$, $\{\pi^\varepsilon\}$, $\{q^\varepsilon\}$, and $\{\nabla \boldsymbol{w}^\varepsilon\}$ two-scale converge as $\varepsilon \searrow 0$ respectively to P, Π, Q, and $\nabla_x \boldsymbol{w} + \nabla_y \mathbf{W}$.

6.2. Micro- and Macroscopic Equations. In the present section we do not consider functions of time t, which re-normalize pressures. As we have shown before, finally all these functions are equal to zero.

Lemma 6.1. *The two-scale limits of the sequences $\{p^\varepsilon\}$, $\{\pi^\varepsilon\}$, $\{q^\varepsilon\}$, and $\{\nabla \boldsymbol{w}^\varepsilon\}$ satisfy in $Y_T = Y \times (0,T)$ the following relations*:

$$\frac{1}{\eta_0}\Pi + (1-\chi)(\operatorname{div}_x \boldsymbol{w} + \operatorname{div}_y \boldsymbol{W}) = 0; \tag{6.1}$$

$$\frac{1}{p_*}P + \chi(\operatorname{div}_x \boldsymbol{w} + \operatorname{div}_y \boldsymbol{W}) = 0, \quad Q = P + \frac{\nu_0}{p_*}\frac{\partial P}{\partial t}; \tag{6.2}$$

$$\operatorname{div}_y\left(\chi\mu_0\left(D\left(x, \frac{\partial \boldsymbol{w}}{\partial t}\right) + D\left(y, \frac{\partial \boldsymbol{W}}{\partial t}\right)\right) + (1-\chi)\lambda_0\big(D(x,\boldsymbol{w}) + D(y,\boldsymbol{W})\big)\right) - \nabla_y(Q+\Pi) = 0. \tag{6.3}$$

Lemma 6.2. *The weak limits p, π, q, and $\boldsymbol{w}$ satisfy in Ω_T the following system of macroscopic equations*:

$$\frac{1}{\eta_0}\pi + (1-m)\operatorname{div}_x \boldsymbol{w} + \langle \operatorname{div}_y \boldsymbol{W}\rangle_{Y_s} = 0; \tag{6.4}$$

$$\frac{1}{p_*}p + m \operatorname{div}_x \boldsymbol{w} + \langle \operatorname{div}_y \boldsymbol{W}\rangle_{Y_f} = 0, \quad q = p + \frac{\nu_0}{p_*}\frac{\partial p}{\partial t}; \tag{6.5}$$

$$\begin{aligned}
\tau_0\hat{\rho}\frac{\partial^2 \boldsymbol{w}}{\partial t^2} = \operatorname{div}_x\Bigg(&\mu_0\left(mD\left(x, \frac{\partial \boldsymbol{w}}{\partial t}\right) + \left\langle D\left(y, \frac{\partial \boldsymbol{W}}{\partial t}\right)\right\rangle_{Y_f}\right)\\
&+ \lambda_0\big((1-m)D(x,\boldsymbol{w}) + \langle D(y,\boldsymbol{W})\rangle_{Y_s} - (q+\pi)\mathbb{I}\big) + \hat{\rho}\boldsymbol{F}.
\end{aligned} \tag{6.6}$$

Proofs of these statements are the same as in Lemmas 4.1–4.3.

6.3. Homogenized Equations.

Lemma 6.3. *The weak limits p, π, q, and $\boldsymbol{w}$ satisfy in Ω_T the following system of homogenized equations*:

$$\tau_0\hat{\rho}\frac{\partial^2\boldsymbol{w}}{\partial t^2}+\nabla(q+\pi)-\hat{\rho}\boldsymbol{F}=\operatorname{div}_x\bigg(\mathbb{A}_2:D\Big(x,\frac{\partial\boldsymbol{w}}{\partial t}\Big)+\mathbb{A}_3:D(x,\boldsymbol{w})+B_4\operatorname{div}_x\boldsymbol{w}$$
$$+\int_0^t\big(\mathbb{A}_4(t-\tau):D\big(x,\boldsymbol{w}(\boldsymbol{x},\tau)\big)+B_5(t-\tau)\operatorname{div}_x\boldsymbol{w}(\boldsymbol{x},\tau)\big)\,d\tau\bigg),\tag{6.7}$$

$$\frac{1}{p_*}p+m\operatorname{div}_x\boldsymbol{w}=-\int_0^t\big(C_2(t-\tau):D\big(x,\boldsymbol{w}(\boldsymbol{x},\tau)\big)+a_2(t-\tau)\operatorname{div}_x\boldsymbol{w}(\boldsymbol{x},\tau)\big)\,d\tau,\tag{6.8}$$

$$\frac{1}{\eta_0}\pi+(1-m)\operatorname{div}_x\boldsymbol{w}=-\int_0^t\big(C_3(t-\tau):D\big(x,\boldsymbol{w}(\boldsymbol{x},\tau)\big)+a_3(t-\tau)\operatorname{div}_x\boldsymbol{w}(\boldsymbol{x},\tau)\big)\,d\tau,\tag{6.9}$$

$$q=p+\frac{\nu_0}{p_*}\frac{\partial p}{\partial t}.\tag{6.10}$$

Here $\mathbb{A}_2$, $\mathbb{A}_3$, and $\mathbb{A}_4$ are fourth-rank tensors, B_4, B_5, C_2, and C_3 are matrices, and a_2 and a_3 are scalars. The exact expressions for these objects are given below by formulas (6.15)–(6.20).

Proof. Let

$$Z(\boldsymbol{x},t)=\mu_0D\Big(x,\frac{\partial\boldsymbol{w}}{\partial t}\Big)-\lambda_0D(x,\boldsymbol{w}),\quad Z_{ij}=\mathbf{e}_i\cdot(Z\cdot\mathbf{e}_j),\quad z(\boldsymbol{x},t)=\operatorname{div}_x\boldsymbol{w}.$$

As usual we look for the solution of the system of microscopic equations (6.1)–(6.3) in the form

$$\boldsymbol{W}=\int_0^t\Big[\boldsymbol{W}^0(y,t-\tau)z(\boldsymbol{x},\tau)+\sum_{i,j=1}^3\boldsymbol{W}^{ij}(\boldsymbol{y},t-\tau)Z_{ij}(\boldsymbol{x},\tau)\Big]\,d\tau,$$

$$P=\chi\int_0^t\Big[P^0(\boldsymbol{y},t-\tau)z(\boldsymbol{x},\tau)+\sum_{i,j=1}^3P^{ij}(\boldsymbol{y},t-\tau)Z_{ij}(\boldsymbol{x},\tau)\Big]\,d\tau,$$

$$Q=\chi\bigg(Q_0(y)\cdot z(\boldsymbol{x},t)+\sum_{i,j=1}^3Q_0^{ij}(\boldsymbol{y})\cdot Z_{ij}(\boldsymbol{x},t)+\int_0^t\Big[Q^0(\boldsymbol{y},t-\tau)z(\boldsymbol{x},\tau)+\sum_{i,j=1}^3Q^{ij}(\boldsymbol{y},t-\tau)Z_{ij}(\boldsymbol{x},\tau)\Big]\,d\tau\bigg),$$

$$\Pi=(1-\chi)\bigg(\int_0^t\Big[\Pi^0(\boldsymbol{y},t-\tau)z(\boldsymbol{x},\tau)+\sum_{i,j=1}^3\Pi^{ij}(\boldsymbol{y},t-\tau)Z_{ij}(\boldsymbol{x},\tau)\Big]\,d\tau\bigg),$$

where the 1-periodic in $\boldsymbol{y}$ functions $\boldsymbol{W}^0$, $\boldsymbol{W}^{ij}$, P^0, P^{ij}, Q_0, Q^0, Q^{ij}, Q_0^{ij}, Π^0, and Π^{ij} satisfy the following periodic initial-boundary-value problems in the elementary cell Y:

Problem (I)

$$\operatorname{div}_y\bigg(\chi\bigg(\mu_0D\Big(y,\frac{\partial\boldsymbol{W}^{ij}}{\partial t}\Big)+(1-\chi)(\lambda_0D(y,\boldsymbol{W}^{ij})-(\Pi^{ij}+Q^{ij})\mathbb{I}\bigg)\bigg)=0;\tag{6.11}$$

$$\frac{1}{p_*}P^{ij}+\chi\operatorname{div}_y\boldsymbol{W}^{ij}=0,\quad Q^{ij}=P^{ij}+\frac{\nu_0}{p_*}\frac{\partial P^{ij}}{\partial t};\tag{6.12}$$

$$\frac{1}{\eta_0}\Pi^{ij} + (1-\chi)\operatorname{div}_y \boldsymbol{W}^{ij} = 0, \quad \boldsymbol{W}^{ij}(\boldsymbol{y},0) = \boldsymbol{W}^{ij}_0(\boldsymbol{y}); \tag{6.13}$$

$$\operatorname{div}_y\big(\chi(\mu_0 D(y,\boldsymbol{W}^{ij}_0) + J^{ij} - Q^{ij}_0\mathbb{I})\big) = 0, \quad \chi(Q^{ij}_0 + \nu_0 \operatorname{div}_y \boldsymbol{W}^{ij}_0) = 0. \tag{6.14}$$

Problem (II)

$$\operatorname{div}_y\bigg(\chi\bigg(\mu_0 D\Big(y,\frac{\partial \boldsymbol{W}^0}{\partial t}\Big) + (1-\chi)(\lambda_0 D(y,\mathbf{W}^0) - (\Pi^0 + Q^0)\mathbb{I}\bigg)\bigg) = 0;$$

$$\frac{1}{p_*}P^0 + \chi(\operatorname{div}_y \mathbf{W}^0 + 1) = 0, \quad Q^0 = P^0 + \frac{\nu_0}{p_*}\frac{\partial P^0}{\partial t};$$

$$\frac{1}{\eta_0}\Pi^0 + (1-\chi)(\operatorname{div}_y \mathbf{W}^0 + 1) = 0, \quad \mathbf{W}^0(\boldsymbol{y},0) = \mathbf{W}^0_0(\boldsymbol{y});$$

$$\operatorname{div}_y\big(\chi(\mu_0 D(y,\boldsymbol{W}^0_0) - Q_0\mathbb{I})\big) = 0, \quad \chi\big(Q_0 + \nu_0(\operatorname{div}_y\ \mathbf{W}^0_0 + 1)\big) = 0.$$

Then

$$\mathbb{A}_2 = \mu_0 m\mathbb{J} + \mu_0\mathbb{A}^f_0, \quad \mathbb{A}^f_0 = \sum_{i,j=1}^{3}\langle\mu_0 D(y,\mathbf{W}^{ij}_0)\rangle_{Y_f}\otimes J^{ij}; \tag{6.15}$$

$$\mathbb{A}_3 = \lambda_0\big((1-m)\mathbb{J} - \mathbb{A}^f_0\big) + \mu_0\mathbb{A}^f_1(0), \quad \mathbb{A}_4(t) = \mu_0\left(\frac{d}{dt} - \lambda_0\right)\mathbb{A}^f_1(t); \tag{6.16}$$

$$\mathbb{A}^f_1(t) = \sum_{i,j=1}^{3}\bigg(\bigg\langle\mu_0 D\Big(y,\frac{\partial W^{ij}}{\partial t}(\boldsymbol{y},t)\Big)\bigg\rangle_{Y_f} + \big\langle\lambda_0 D\big(y,\boldsymbol{W}^{ij}(\boldsymbol{y},t)\big)\big\rangle_{Y_s}\bigg)\otimes J^{ij}; \tag{6.17}$$

$$B_5(t) = \bigg\langle\chi\mu_0 D\Big(y,\frac{\partial W^0}{\partial t}(y,t)\Big) + (1-\chi)\lambda_0 D\big(y,W^0(y,t)\big)\bigg\rangle_Y; \tag{6.18}$$

$$C_2(t) = -C_3(t) = \sum_{i,j=1}^{3}\langle\chi \operatorname{div}_y W^{ij}(y,t)\rangle_Y J^{ij}; \tag{6.19}$$

$$a_2(t) = -a_3(t) = \langle\chi\operatorname{div}_y W^0(y,t)\rangle_Y, \quad B_4 = \big\langle\chi\mu_0 D\big(y,W^0_0(y)\big)\big\rangle_Y. \tag{6.20}$$

The lemma is proved. □

Lemma 6.4. *The tensors $\mathbb{A}_2$–$\mathbb{A}_4$, the matrices B_4, B_5, C_2, and C_3, and the scalars a_2 and a_3 are well defined and infinitely smooth in time.*

If a porous space is connected, then the symmetric tensor $\mathbb{A}_2$ is strictly positively definite. For the case of disconnected porous space (isolated pores) $\mathbb{A}_2 = 0$ and the tensor $\mathbb{A}_2$ becomes strictly positively definite.

All these objects are well defined if Problem (I) and Problem (II) are well-posed. The solvability of the above-mentioned problems and smoothness with respect to time follow, due to linearity, from the standard a priori estimates (multiplication of the equation for the solution by the proper solution and integration by parts). Note that all these problems have a unique solution up to an arbitrary constant vector. In order to discard the arbitrary constant vectors, we demand that the average value of the solution over the domain Y be equal to zero. The smoothness with respect to time follows from the estimates of the solution at the initial time moment. Thus, for example, in the Problem (I) first of all we estimate χW^{ij}_0 as a solution to the problem (6.14). Further solving (6.11) together with continuity equation (6.13) at $t=0$ and using the continuity of the displacements on the boundary γ, we define and estimate $(1-\chi)W^{ij}_0$. After that, from (6.11) at $t=0$ we define and estimate $\chi(\partial W^{ij}/\partial t)(y,0)$. In the same way we estimate the second derivatives with respect to time after differentiation of all equations with respect to time.

The symmetry of $\mathbb{A}_2$ is proved in the same way as the symmetry of $\mathbb{A}_0$. If the porous space is disconnected, then the problem (6.14) has a unique solution linear in $\boldsymbol{y}$, such that

$$\chi(D(y,W^{ij}_0) + J^{ij}) = 0. \tag{6.21}$$

The last equality implies $\mathbb{A}_2 = 0$. In this case the tensor $\mathbb{A}_3$ becomes strictly positively definite. Indeed

$$\mathbb{A}_3 = \lambda_0 \sum_{i,j=1}^{3} J^{ij} \otimes J^{ij} + \mu_0 \mathbb{A}_1^f(0) = \lambda_0 \sum_{i,j=1}^{3} J^{ij} \otimes J^{ij} + \sum_{i,j=1}^{3} \Big\langle \chi\mu_0 D\Big(y, \frac{\partial W^{ij}}{\partial t}(y,0)\Big) + \frac{\lambda_0}{\mu_0} J^{ij} \Big\rangle_Y \otimes J^{ij}.$$

On the other hand, coming back to (6.11) at the initial time moment, we see that

$$\Big\langle \chi\mu_0 D\Big(y, \frac{\partial W^{ij}}{\partial t}(y,0)\Big) : D(y, W_0^{kl}) \Big\rangle_Y = -\lambda_0 \langle \chi D(y, W_0^{ij}) : D(y, W_0^{kl}) \rangle - \Big\langle \frac{1}{\eta_0} \Pi^{ij} \cdot \Pi^{kl} \Big\rangle_Y \Big|_{t=0}.$$

Moreover, due to (6.21)

$$\Big\langle \chi\mu_0 D\Big(y, \frac{\partial W^{ij}}{\partial t}(y,0)\Big) : D(y, W_0^{kl}) \Big\rangle_Y = -\Big\langle \chi D\Big(y, \frac{\partial W^{ij}}{\partial t}(y,0)\Big) : J^{kl} \Big\rangle_Y,$$

which proves our statement.

7. Proof of Theorem 5

7.1. Weak and Two-Scale Limits of Sequences of Displacement and Pressures. On the strength of Theorem 1, the sequences $\{p^\varepsilon\}$, $\{q^\varepsilon\}$, $\{\pi^\varepsilon\}$, and $\{\boldsymbol{w}^\varepsilon\}$ are uniformly in ε bounded in $L^2(\Omega_T)$. Hence there exist a subsequence of small parameters $\{\varepsilon > 0\}$ and functions p, q, π, and $\boldsymbol{w}$ such that

$$p^\varepsilon \to p, \quad q^\varepsilon \to q, \quad \pi^\varepsilon \to \pi, \quad \boldsymbol{w}^\varepsilon \to \boldsymbol{w} \tag{7.1}$$

weakly in $L^2(\Omega_T)$ as $\varepsilon \searrow 0$.

Moreover, due to Lemma 2.1 (an extension lemma) there is a function $\boldsymbol{v}^\varepsilon \in L^\infty\big(0,T; W_2^1(\Omega)\big)$, such that $\boldsymbol{v}^\varepsilon = \partial \boldsymbol{w}^\varepsilon/\partial t$ in $\Omega_f \times (0,T)$, and the family $\{\boldsymbol{v}^\varepsilon\}$ is uniformly in ε bounded in $L^2\big(0,T; W_2^1(\Omega)\big)$. Therefore, there exist a subsequence of $\{\varepsilon > 0\}$ and a function $\boldsymbol{v} \in L^2\big(0,T; W_2^1(\Omega)\big)$ such that

$$\boldsymbol{v}^\varepsilon \to \boldsymbol{v} \ \text{ weakly in } \ L^2\big(0,T; W_2^1(\Omega)\big) \tag{7.2}$$

as $\varepsilon \searrow 0$.

Note also that

$$(1-\chi^\varepsilon)\alpha_\lambda D(x, \boldsymbol{w}^\varepsilon) \to 0 \tag{7.3}$$

strongly in $L^2(\Omega_T)$ as $\varepsilon \searrow 0$.

Relabeling if necessary, we assume that the sequences themselves converge.

On the strength of Nguetseng's theorem, there exist 1-periodic in $\boldsymbol{y}$ functions $P(\boldsymbol{x},t,\boldsymbol{y})$, $\Pi(\boldsymbol{x},t,\boldsymbol{y})$, $Q(\boldsymbol{x},t,\boldsymbol{y})$, $\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$, and $\boldsymbol{V}(\boldsymbol{x},t,\boldsymbol{y})$ such that the sequences $\{p^\varepsilon\}$, $\{\pi^\varepsilon\}$, $\{q^\varepsilon\}$, $\{\boldsymbol{w}^\varepsilon\}$, and $\{\nabla_x \boldsymbol{v}^\varepsilon\}$ two-scale converge to $P(\boldsymbol{x},t,\boldsymbol{y})$, $\Pi(\boldsymbol{x},t,\boldsymbol{y})$, $Q(\boldsymbol{x},t,\boldsymbol{y})$, $\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$, and $\nabla_x \boldsymbol{v} + \nabla_y \boldsymbol{V}(\boldsymbol{x},t,\boldsymbol{y})$, respectively.

Note that the sequence $\{\operatorname{div}_x \boldsymbol{w}^\varepsilon\}$ weakly converges to $\operatorname{div}_x \boldsymbol{w}$ and $\boldsymbol{v} \in L^2\big(0,T; \mathring{W}_2^1(\Omega)\big)$. The last assertion follows from the Friedrichs–Poincaré's inequality for $\boldsymbol{v}^\varepsilon$ in the ε-layer of the boundary S and from convergence of sequence $\{\boldsymbol{v}^\varepsilon\}$ to $\boldsymbol{v}$ strongly in $L^2(\Omega_T)$ and weakly in $L^2\big((0,T); W_2^1(\Omega)\big)$.

7.2. Micro- and Macroscopic Equations I. We start this section with the macro- and microscopic equations connected with the continuity equations.

Lemma 7.1. *For all $\boldsymbol{x} \in \Omega$ and $\boldsymbol{y} \in Y$, the weak and two-scale limits of the sequences $\{p^\varepsilon\}$, $\{\pi^\varepsilon\}$, $\{q^\varepsilon\}$, $\{\boldsymbol{w}^\varepsilon\}$, and $\{\boldsymbol{v}^\varepsilon\}$ satisfy the relations*

$$\Pi = \frac{\pi(1-\chi)}{1-m}; \tag{7.4}$$

$$q = p + \nu_0 p_*^{-1} \frac{\partial p}{\partial t}, \quad Q = P + \nu_0 p_*^{-1} \frac{\partial P}{\partial t}; \tag{7.5}$$

$$p_*^{-1} \frac{\partial p}{\partial t} + m \operatorname{div}_x \boldsymbol{v} + \langle \operatorname{div}_y \boldsymbol{V} \rangle_{Y_f} = -\frac{\partial \beta}{\partial t}; \tag{7.6}$$

$$p_*^{-1}\frac{\partial P}{\partial t} + \chi(\operatorname{div}_x \boldsymbol{v} + \operatorname{div}_y \boldsymbol{V}) = -\frac{\chi}{m}\frac{\partial \beta}{\partial t}; \tag{7.7}$$
$$\frac{p}{p_*} + \frac{\pi}{\eta_0} + \operatorname{div}_x \boldsymbol{w} = 0; \tag{7.8}$$
$$\boldsymbol{w}(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x}) = 0, \quad \boldsymbol{x}\in S; \tag{7.9}$$
$$\operatorname{div}_y \boldsymbol{W} = 0; \tag{7.10}$$
$$\frac{\partial \boldsymbol{W}}{\partial t} = \chi\boldsymbol{v} + (1-\chi)\frac{\partial \boldsymbol{W}}{\partial t}, \tag{7.11}$$

where $\partial\beta/\partial t = -\langle\langle \operatorname{div}_y \boldsymbol{V}\rangle_{Y_f}\rangle_\Omega$ *if* $p_* + \eta_0 = \infty$ *and* $\beta = 0$ *if* $p_* + \eta_0 < \infty$ *and* $\boldsymbol{n}(\boldsymbol{x})$ *is the unit normal vector to* S *at a point* $\boldsymbol{x}\in S$.

Proof. In order to prove (7.4), into Eq. (1.3) insert a test function $\boldsymbol{\psi}^\varepsilon = \varepsilon\boldsymbol{\psi}(\boldsymbol{x},t,\boldsymbol{x}/\varepsilon)$, where $\boldsymbol{\psi}(\boldsymbol{x},t,\boldsymbol{y})$ is an arbitrary 1-periodic and finite on Y_s function in $\boldsymbol{y}$. Passing to the limit as $\varepsilon \searrow 0$, we get

$$\nabla_y \Pi(\boldsymbol{x},t,\boldsymbol{y}) = 0, \quad \boldsymbol{y}\in Y_s. \tag{7.12}$$

Next, fulfilling the two-scale limiting passage in the equality

$$\chi^\varepsilon \pi^\varepsilon = 0$$

we arrive at

$$\chi\Pi = 0,$$

which along with Eqs. (7.12) justifies Eq. (7.4).

Equations (7.5)–(7.9) appear as the result of two-scale limiting passages in Eqs. (1.2), (3.3), (3.4) with the proper test functions being involved. Thus, for example, Eqs. (7.8) and (7.9) arise if we consider the sum of Eqs. (3.3) and (3.4)

$$\frac{1}{\alpha_p}p^\varepsilon + \frac{1}{\alpha_\eta}\pi^\varepsilon + \operatorname{div}_x \boldsymbol{w}^\varepsilon = \frac{1}{m(1-m)}\beta^\varepsilon(\chi^\varepsilon - m), \tag{7.13}$$

multiply it by an arbitrary function independent of the “fast” variable $\boldsymbol{x}/\varepsilon$, and then pass to the limit as $\varepsilon \searrow 0$. In order to prove Eq. (7.10), it is sufficient to consider the two-scale limiting relations in Eq. (7.13) as $\varepsilon \searrow 0$ with the test functions $\varepsilon\psi(\boldsymbol{x}/\varepsilon)h(\boldsymbol{x},t)$, where ψ and h are arbitrary smooth functions. In order to prove Eq. (7.11) it is sufficient to consider the two-scale limiting relations in

$$\chi^\varepsilon\left(\frac{\partial \boldsymbol{w}^\varepsilon}{\partial t} - \boldsymbol{v}^\varepsilon\right) = 0.$$

□

Corollary 7.1. *If* $p_* + \eta_0 = \infty$, *then the weak limits* p, π, *and* q *satisfy relations* (4.12).

Lemma 7.2. *For all* $(\boldsymbol{x},t)\in\Omega_T$ *the relation*

$$\operatorname{div}_y\left\{\mu_0\chi\big(D(y,\boldsymbol{V}) + D(x,\boldsymbol{v})\big) - \left(Q + \frac{(1-\chi)}{1-m}\pi\right)\cdot I\right\} = 0 \tag{7.14}$$

holds true.

Proof. Substituting a test function of the form $\boldsymbol{\psi}^\varepsilon = \varepsilon\boldsymbol{\psi}(\boldsymbol{x},t,\boldsymbol{x}/\varepsilon)$, where $\boldsymbol{\psi}(\boldsymbol{x},t,\boldsymbol{y})$ is an arbitrary 1-periodic in $\boldsymbol{y}$ function vanishing on the boundary S, into integral identity (1.3) and, passing to the limit as $\varepsilon \searrow 0$, we arrive at Eq. (7.14) on the elementary cell Y. □

Lemma 7.3. *Let* $\hat\rho = m\rho_f + (1-m)\rho_s$. *Then the functions* $\boldsymbol{w}^s = \langle\boldsymbol{W}\rangle_{Y_s}$, $\boldsymbol{v}$, q, *and* π *satisfy in* Ω_T *the system of macroscopic equations*

$$\rho_f m\frac{\partial \boldsymbol{v}}{\partial t} + \rho_s\frac{\partial^2 \boldsymbol{w}^s}{\partial t^2} - \hat\rho\boldsymbol{F} = \operatorname{div}_x\big\{\mu_0\big(mD(x,\boldsymbol{v}) + \langle D(y,\boldsymbol{V})\rangle_{Y_f}\big) - (q+\pi)\cdot I\big\}. \tag{7.15}$$

Proof. Equations (7.15) arise as the limit of Eqs. (1.3) with the test functions being finite in Ω_T and independent of ε. □

7.3. Micro- and Macroscopic Equations II.

Lemma 7.4. *If $\lambda_1 = \infty$, then the weak limits of $\{\boldsymbol{v}^\varepsilon\}$ and $\{\partial \boldsymbol{w}^\varepsilon/\partial t\}$ coincide.*

Proof. Let $\boldsymbol{\Psi}(\boldsymbol{x},t,\boldsymbol{y})$ be an arbitrary function periodic in $\boldsymbol{y}$. The sequence $\{\sigma^\varepsilon\}$, where

$$\sigma^\varepsilon = \int\limits_\Omega \sqrt{\alpha_\lambda}\nabla \boldsymbol{w}^\varepsilon(\boldsymbol{x},t)\boldsymbol{\Psi}\left(\boldsymbol{x},t,\frac{\boldsymbol{x}}{\varepsilon}\right)dx,$$

is uniformly bounded in ε. Therefore,

$$\int\limits_\Omega \varepsilon\nabla\boldsymbol{w}^\varepsilon\boldsymbol{\Psi}\left(\boldsymbol{x},t,\frac{\boldsymbol{x}}{\varepsilon}\right)dx = \frac{\varepsilon}{\sqrt{\alpha_\lambda}}\sigma^\varepsilon \to 0$$

as $\varepsilon \searrow 0$, which is equivalent to

$$\int\limits_\Omega\int\limits_Y \boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})\nabla_y\boldsymbol{\Psi}(\boldsymbol{x},t,\boldsymbol{y})\,dx\,dy = 0,$$

or $\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y}) = \boldsymbol{w}(\boldsymbol{x},t)$. Thus, the sequence $\{\partial\boldsymbol{w}^\varepsilon/\partial t\}$ converges strongly in $L^2(\Omega_T)$ and, due to the equality

$$\chi^\varepsilon\left(\boldsymbol{v}^\varepsilon - \frac{\partial\boldsymbol{w}^\varepsilon}{\partial t}\right) = 0,$$

its limit $\partial\boldsymbol{w}/\partial t$ coincides with the limit $\boldsymbol{v}$ of the sequence $\{\boldsymbol{v}^\varepsilon\}$. □

Lemma 7.5. *Let $\lambda_1 < \infty$. Then the weak and two-scale limits π and $\boldsymbol{W}$ satisfy the microscopic relations*

$$\rho_s\frac{\partial^2\boldsymbol{W}}{\partial t^2} = \lambda_1\Delta_y\boldsymbol{W} - \nabla_y R - \frac{1}{1-m}\nabla_x\pi + \rho_s\boldsymbol{F}, \quad \boldsymbol{y}\in Y_s, \tag{7.16}$$

$$\frac{\partial\boldsymbol{W}}{\partial t} = \boldsymbol{v}, \quad \boldsymbol{y}\in\gamma, \tag{7.17}$$

in the case $\lambda_1 > 0$, and relations

$$\rho_s\frac{\partial^2\boldsymbol{W}}{\partial t^2} = -\nabla_y R - \frac{1}{1-m}\nabla_x\pi + \rho_s\boldsymbol{F}, \quad \boldsymbol{y}\in Y_s, \tag{7.18}$$

$$\left(\frac{\partial\boldsymbol{W}}{\partial t} - \boldsymbol{v}\right)\cdot\boldsymbol{n} = 0, \quad \boldsymbol{y}\in\gamma, \tag{7.19}$$

in the case $\lambda_1 = 0$.

In Eq. (7.19) *$\boldsymbol{n}$ is the unit normal to γ.*

Proof. Differential equations (7.16) and (7.18) follow as $\varepsilon\searrow 0$ from integral equality (1.3) with the test function $\boldsymbol{\psi} = \boldsymbol{\varphi}(x\varepsilon^{-1})\cdot h(\boldsymbol{x},t)$, where $\boldsymbol{\varphi}$ is solenoidal and finite in Y_s.

Boundary condition (7.17) is a consequence of the two-scale convergence of $\{\sqrt{\alpha_\lambda}\nabla_x\boldsymbol{w}^\varepsilon\}$ to the function $\sqrt{\lambda_1}\nabla_y\mathbf{W}(\boldsymbol{x},t,\boldsymbol{y})$. On the strength of this convergence, the function $\nabla_y\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$ is L^2-integrable in Y. The boundary condition (7.19) follows from Eqs. (7.10), (7.11). □

7.4. Homogenized Equations I. Here we derive homogenized equations for the liquid component.

Lemma 7.6. *If $\lambda_1 = \infty$, then $\partial\boldsymbol{w}/\partial t = \boldsymbol{v}$ and the weak limits $\boldsymbol{v}$, p, q, and π satisfy in Ω_T the initial-boundary-value problem*

$$\hat{\rho}\frac{\partial\boldsymbol{v}}{\partial t} = \operatorname{div}_x\Big\{\mu_0 A^f : D(x,\boldsymbol{v}) + B_0^f\operatorname{div}_x\boldsymbol{v} + C_0^f\pi$$
$$+\int\limits_0^t \big(A_1^f(t-\tau) : D\big(x,\boldsymbol{v}(\boldsymbol{x},\tau)\big) + B_1^f(t-\tau)\operatorname{div}_x\boldsymbol{v}(\boldsymbol{x},\tau) + C_1^f(t-\tau)\pi(\boldsymbol{x},\tau)\big)\,d\tau\Big\} - \nabla(q+\pi) + \hat{\rho}\boldsymbol{F}, \tag{7.20}$$

$$\frac{1}{p_{0,f}}\frac{\partial p_f}{\partial t} + \mathbb{E}^f_0 : \mathbb{D}(x, \boldsymbol{v}) + c^f_0 p_s + (m + b^f_0)\operatorname{div}\boldsymbol{v}$$

$$+ \int_0^t \big(\mathbb{E}^f_1(t-\tau) : \mathbb{D}\big(x, \boldsymbol{v}(\boldsymbol{x},\tau)\big) + c^f_1(t-\tau)p_s(\boldsymbol{x},\tau) + b^f_1(t-\tau)\operatorname{div}\boldsymbol{v}(\boldsymbol{x},\tau)\big)\,d\tau = 0, \tag{7.21}$$

$$q = p + \frac{\nu_0}{p_*}\frac{\partial p}{\partial t}, \quad \frac{1}{p_*}\frac{\partial p}{\partial t} + \frac{1}{\eta_0}\frac{\partial \pi}{\partial t} + \operatorname{div}_x \boldsymbol{v} = 0, \tag{7.22}$$

where the symmetric strictly positively definite constant fourth-rank tensor A^f, fourth-rank tensor $A^f_1(t)$, constant matrices C^f_0, B^f_0, and E^f_0, matrices $C^f_1(t)$, $B^f_1(t)$, and $E^f_1(t)$, scalars b^f_0 and c^f_0, and functions $b^f_1(t)$ and $c^f_1(t)$ are defined below by formulas (7.27)–(7.29) *and* (7.31)*.*

Differential equations (7.20) *are endowed with homogeneous initial and boundary conditions*

$$\boldsymbol{v}(\boldsymbol{x},0) = 0, \quad \boldsymbol{x} \in \Omega, \quad \boldsymbol{v}(\boldsymbol{x},t) = 0, \quad \boldsymbol{x} \in S, \quad t > 0. \tag{7.23}$$

Proof. In the first place let us note that $\boldsymbol{v} = \partial\boldsymbol{w}/\partial t$ due to Lemma 7.4. Let us consider for simplicity the case

$$p_* + \eta_0 < \infty.$$

The homogenized equations (7.20) follow from the macroscopic equations (7.15) after we insert in them the expression

$$\mu_0\langle D(y,\boldsymbol{V})\rangle_{Y_f} = \mu_0 A^f_0 : D(x,\boldsymbol{v}) + B^f_0 \operatorname{div}_x \boldsymbol{v} + C^f_0 \pi$$

$$+ \int_0^t \big(A^f_1(t-\tau) : D\big(x, \boldsymbol{v}(\boldsymbol{x},\tau)\big) + B^f_1(t-\tau)\operatorname{div}_x \boldsymbol{v}(\boldsymbol{x},\tau) + C^f_1(t-\tau)\pi(\boldsymbol{x},\tau)\big)\,d\tau.$$

In turn, this expression follows by virtue of solutions of Eq. (7.5) in the form

$$Q = P - \nu_0\chi(\operatorname{div}_x \boldsymbol{v} + \operatorname{div}_y \boldsymbol{V}) + \nu_0\left(\frac{\chi}{m}\right)\frac{\partial\beta}{\partial t}$$

and Eqs. (7.7) and (7.14) on the pattern cell Y_f. Indeed, setting

$$\boldsymbol{V} = \sum_{i,j=1}^3 \boldsymbol{V}^{(ij)}_0(\boldsymbol{y})D_{ij}(\boldsymbol{x},t) + \boldsymbol{V}^{(0)}_0(\boldsymbol{y})p_s(\boldsymbol{x},t) + \boldsymbol{V}^{(1)}_0(\boldsymbol{y})\operatorname{div}\boldsymbol{v}(\boldsymbol{x},t)$$

$$+ \int_0^t \Bigg(\sum_{i,j=1}^3 \boldsymbol{V}^{(ij)}(\boldsymbol{y},t-\tau)D_{ij}(\boldsymbol{x},\tau) + \boldsymbol{V}^{(0)}(\boldsymbol{y},t-\tau)p_s(\boldsymbol{x},\tau) + \boldsymbol{V}^{(1)}(\boldsymbol{y},t-\tau)\operatorname{div}\boldsymbol{v}(\boldsymbol{x},\tau)\,d\tau\Bigg),$$

$$Q = \sum_{i,j=1}^3 Q^{(ij)}_0(\boldsymbol{y})D_{ij}(\boldsymbol{x},t) + Q^{(0)}_0(\boldsymbol{y})p_s(\boldsymbol{x},t) + Q^{(1)}_0(\boldsymbol{y})\operatorname{div}\boldsymbol{v}(\boldsymbol{x},t)$$

$$+ \int_0^t \Bigg(\sum_{i,j=1}^3 Q^{(ij)}(\boldsymbol{y},t-\tau)D_{ij}(\boldsymbol{x},\tau) + Q^{(0)}(\boldsymbol{y},t-\tau)p_s(\boldsymbol{x},\tau) + Q^{(1)}(\boldsymbol{y},t-\tau)\operatorname{div}\boldsymbol{v}(\boldsymbol{x},\tau)\,d\tau\Bigg),$$

$$P_f = \int_0^t \Bigg(\sum_{i,j=1}^3 P^{(ij)}(\boldsymbol{y},t-\tau)D_{ij}(\boldsymbol{x},\tau) + P^{(0)}(\boldsymbol{y},t-\tau)p_s(\boldsymbol{x},\tau) + P^{(1)}(\boldsymbol{y},t-\tau)\operatorname{div}\boldsymbol{v}(\boldsymbol{x},\tau)\,d\tau\Bigg),$$

where

$$D_{ij}(\boldsymbol{x},t) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j}(\boldsymbol{x},t) + \frac{\partial v_j}{\partial x_i}(\boldsymbol{x},t)\right),$$

we arrive at the following periodic boundary-value problems in Y:

$$\left.\begin{array}{c}\operatorname{div}_y\Big(\chi\big(\mu_0\mathbb{D}(y,\boldsymbol{V}^{(ij)})-Q^{(ij)}\mathbb{I}\big)\Big)=0,\\ \dfrac{1}{p_{0,f}}\dfrac{\partial P^{(ij)}}{\partial t}+\operatorname{div}_y\boldsymbol{V}^{(ij)}=0,\\ Q^{(ij)}=P^{(ij)}+\dfrac{\nu_0}{p_{0,f}}\dfrac{\partial P^{(ij)}}{\partial t},\\ \operatorname{div}_y\big(\chi(\mu_0\mathbb{D}(y,\boldsymbol{V}_0^{(ij)})+\mathbb{J}^{ij}-Q_0^{(ij)}\mathbb{I})\big)=0,\\ \dfrac{1}{p_{0,f}}P|_{t=0}^{(ij)}+\operatorname{div}_y\boldsymbol{V}_0^{(ij)}=0,\quad Q_0^{(ij)}=\dfrac{\nu_0}{p_{0,f}}P|_{t=0}^{(ij)};\end{array}\right\}\tag{7.24}$$

$$\left.\begin{array}{c}\operatorname{div}_y\big(\chi(\mu_0\mathbb{D}(y,\boldsymbol{V}^{(0)})-Q^{(0)}\mathbb{I})\big)=0,\\ \dfrac{1}{p_{0,f}}\dfrac{\partial P^{(0)}}{\partial t}+\operatorname{div}_y\boldsymbol{V}^{(0)}=0,\\ Q^{(0)}=P^{(0)}+\dfrac{\nu_0}{p_{0,f}}\dfrac{\partial P^{(0)}}{\partial t},\\ \operatorname{div}_y\left(\chi(\mu_0\mathbb{D}(y,\boldsymbol{V}_0^{(0)})-Q_0^{(0)}\mathbb{I})-\dfrac{(1-\chi)}{(1-m)}\mathbb{I}\right)=0,\\ \dfrac{1}{p_{0,f}}P|_{t=0}^{(0)}+\operatorname{div}_y\boldsymbol{V}_0^{(0)}=0,\quad Q_0^{(0)}=\dfrac{\nu_0}{p_{0,f}}P|_{t=0}^{(0)};\end{array}\right\}\tag{7.25}$$

$$\left.\begin{array}{c}\operatorname{div}_y\big(\chi(\mu_0\mathbb{D}(y,\boldsymbol{V}^{(1)})-Q^{(1)}\mathbb{I})\big)=0,\\ \dfrac{1}{p_{0,f}}\dfrac{\partial P^{(1)}}{\partial t}+\operatorname{div}_y\boldsymbol{V}^{(1)}=0,\\ Q^{(1)}=P^{(1)}+\dfrac{\nu_0}{p_{0,f}}\dfrac{\partial P^{(1)}}{\partial t},\\ \operatorname{div}_y\big(\chi(\mu_0\mathbb{D}(y,\boldsymbol{V}_0^{(1)})-Q_0^{(1)}\mathbb{I})\big)=0,\\ \dfrac{1}{p_{0,f}}P|_{t=0}^{(1)}+\operatorname{div}_y\boldsymbol{V}_0^{(1)}+1=0,\quad Q_0^{(1)}=\dfrac{\nu_0}{p_{0,f}}P|_{t=0}^{(1)}.\end{array}\right\}\tag{7.26}$$

On the strength of the assumptions on the geometry of the pattern "liquid" cell Y_f, problems (7.24)–(7.26) have unique solution up to an arbitrary constant vector. In order to discard the arbitrary constant vectors we demand

$$\langle\boldsymbol{V}^{(ij)}\rangle_{Y_f}=\langle\boldsymbol{V}^{(0)}\rangle_{Y_f}=\langle\boldsymbol{V}^{(1)}\rangle_{Y_f}=\langle\boldsymbol{V}^{(2)}\rangle_{Y_f}=0.$$

Thus,

$$A^f=m\sum_{i,j=1}^{3}J^{ij}\otimes J^{ij}+A_0^f,\quad A_0^f=\sum_{i,j=1}^{3}\langle D(y,\boldsymbol{V}^{(ij)})\rangle_{Y_f}\otimes J^{ij},\tag{7.27}$$

$$\mathbb{A}_1^f(t)=\mu_0\sum_{i,j=1}^{3}\langle\mathbb{D}(y,\boldsymbol{V}^{(ij)})\rangle_{Y_f}\otimes\mathbb{J}^{ij},\tag{7.28}$$

$$\left.\begin{array}{ll}\mathbb{C}_0^f=\mu_0\langle\mathbb{D}(y,\boldsymbol{V}_0^{(0)})\rangle_{Y_f}, & \mathbb{C}_1^f(t)=\mu_0\langle\mathbb{D}(y,\boldsymbol{V}^{(0)})\rangle_{Y_f},\\ \mathbb{B}_0^f=\mu_0\langle\mathbb{D}(y,\boldsymbol{V}_0^{(1)})\rangle_{Y_f}, & \mathbb{B}_1^f(t)=\mu_0\langle\mathbb{D}(y,\boldsymbol{V}^{(1)})\rangle_{Y_f}.\end{array}\right\}\tag{7.29}$$

Symmetry of the tensor A^f follows from symmetry of the tensor A^f_0, and symmetry of the latter follows from the equality

$$\mu_0\langle\mathbb{D}(y,\boldsymbol{V}_0^{(ij)})\rangle_{Y_f} : J^{kl} = -\mu_0\langle\mathbb{D}(y,\boldsymbol{V}_0^{(ij)}) : \mathbb{D}(y,\boldsymbol{V}_0^{(kl)})\rangle_{Y_f} - \frac{\nu_0}{p^2_{0,f}}\langle P|^{(ij)}_{t=0}\cdot P|^{(kl)}_{t=0}\rangle_{Y_f}, \tag{7.30}$$

which appears by multiplication of Eq. (7.24) for $\boldsymbol{V}^{(ij)}$ by $\boldsymbol{V}^{(kl)}$ and by integration by parts using the corresponding continuity equation.

This equality also implies positive definiteness of the tensor A^f. Indeed, let $\mathbb{Z}=(Z_{ij})$ be an arbitrary symmetric constant matrix. Setting

$$\boldsymbol{Z}=\sum_{i,j=1}^{3}\boldsymbol{V}_0^{(ij)}Z_{ij},\quad \tilde{P}=\sum_{i,j=1}^{3}P|^{(ij)}_{t=0}Z_{ij}$$

and taking into account Eq. (7.30), we get

$$\mu_0\langle\mathbb{D}(y,\boldsymbol{Z})\rangle_{Y_f} : \mathbb{Z} = -\mu_0\langle\mathbb{D}(y,\boldsymbol{Z}) : \mathbb{D}(y,\boldsymbol{Z})\rangle_{Y_f} - \frac{\nu_0}{p^2_{0,f}}\langle\tilde{P}^2\rangle_{Y_f}.$$

This equality and the definition of the tensor A^f give

$$(\mathbb{A}^f : \mathbb{Z}) : \mathbb{Z} = \langle(\mathbb{D}(y,\boldsymbol{Z})+\mathbb{Z}) : (\mathbb{D}(y,\boldsymbol{Z})+\mathbb{Z})\rangle_{Y_f} + \frac{\nu_0}{p^2_{0,f}}\langle\tilde{P}^2\rangle_{Y_f}.$$

Now the strict positive definiteness of the tensor $\mathbb{A}^f$ follows immediately from the equality above and the geometry of the elementary cell Y_f. Namely, suppose that $(\mathbb{A}^s:\mathbb{Z}):\mathbb{Z}=0$ for some matrix $\mathbb{Z}$ such that $\mathbb{Z}:\mathbb{Z}=1$. Then $(\mathbb{D}(y,\mathbf{Z})+\mathbb{Z})=0$, which is possible if and only if $\boldsymbol{Z}$ is a linear function in $\boldsymbol{y}$. On the other hand, all linear periodic functions on Y_f are constant. Finally, the normalization condition $\langle\boldsymbol{V}^{(ij)}\rangle_{Y_f}=0$ yields that $\boldsymbol{Z}=0$. However, this is impossible because the functions $\boldsymbol{V}^{(ij)}$ are linearly independent.

Finally, Eqs. (7.21) and (7.22) for the pressures follow from Eqs. (7.5), (7.6), (7.8) and equality

$$\begin{aligned}\langle\operatorname{div}_y \boldsymbol{V}\rangle_{Y_f} &= \mathbb{E}^f_0 : \mathbb{D}(x,\boldsymbol{v}) + c^f_0 p_s + b^f_0\operatorname{div}\boldsymbol{v}\\ &\quad+\int_0^t\big(\mathbb{E}^f_1(t-\tau):\mathbb{D}\big(x,\boldsymbol{v}(\boldsymbol{x},\tau)\big)+c^f_1(t-\tau)p_s(\boldsymbol{x},\tau)+b^f_1(t-\tau)\operatorname{div}\boldsymbol{v}(\boldsymbol{x},\tau)\big)\,d\tau\end{aligned}$$

with

$$\left.\begin{aligned}\mathbb{E}^f_0 &= \mu_0\sum_{i,j=1}^{3}\langle\operatorname{div}_y\boldsymbol{V}_0^{(ij)}\rangle_{Y_f}\otimes\mathbb{J}^{ij},\\ \mathbb{E}^f_1(t) &= \mu_0\sum_{i,j=1}^{3}\langle\operatorname{div}_y\boldsymbol{V}^{(ij)}\rangle_{Y_f}\otimes\mathbb{J}^{ij},\\ c^f_0 = \mu_0\langle\operatorname{div}_y\boldsymbol{V}_0^{(0)}\rangle_{Y_f},&\quad c^f_1(t)=\mu_0\langle\operatorname{div}_y\boldsymbol{V}^{(0)}\rangle_{Y_f},\\ b^f_0 = \mu_0\langle\operatorname{div}_y\boldsymbol{V}_0^{(1)}\rangle_{Y_f},&\quad b^f_1(t)=\mu_0\langle\operatorname{div}_y\boldsymbol{V}^{(1)}\rangle_{Y_f}.\end{aligned}\right\}\tag{7.31}$$

The lemma is proved. □

7.5. Homogenized Equations II. We complete the proof of Theorem 5 with homogenized equations for the solid component.

Let $\lambda_1<\infty$. In the same manner as above, we verify that the limit $\boldsymbol{v}$ of the sequence $\{\boldsymbol{v}^\varepsilon\}$ satisfies the initial-boundary-value problem like (7.20)–(7.23). The main difference here is that, in general, the weak limit $\partial\boldsymbol{w}/\partial t$ of the sequence $\{\partial\boldsymbol{w}^\varepsilon/\partial t\}$ differs from $\boldsymbol{v}$. More precisely, the following statement is true.

Lemma 7.7. *Let $\lambda_1 < \infty$. Then the weak limits $\boldsymbol{v}$, $\boldsymbol{w}^s$, p, q, and π of the sequences $\{\boldsymbol{v}^\varepsilon\}$, $\{(1-\chi^\varepsilon)\boldsymbol{w}^\varepsilon\}$, $\{p^\varepsilon\}$, $\{q^\varepsilon\}$, and $\{\pi^\varepsilon\}$ satisfy the initial-boundary-value problem in Ω_T, consisting of the balance of momentum equation*

$$\rho_f m\frac{\partial \boldsymbol{v}}{\partial t} + \rho_s\frac{\partial^2 \boldsymbol{w}^s}{\partial t^2} + \nabla(q+\pi) - \hat{\rho}\boldsymbol{F} \\ = \operatorname{div}_x\Big\{\mu_0 A_0^f : D(x,\boldsymbol{v}) + B_0^f\pi + B_1^f \operatorname{div}_x \boldsymbol{v}\} + \int_0^t B_2^f(t-\tau)\operatorname{div}_x \boldsymbol{v}(\boldsymbol{x},\tau)\,d\tau\Big\}, \quad (7.32)$$

the continuity equation (7.21) *and first state equation in* (7.22) *for the liquid component, where A_0^f, B_0^f–B_2^f are the same as in* (7.20)*, the continuity equation*

$$\frac{1}{p_*}\frac{\partial p}{\partial t} + \frac{1}{\eta_0}\frac{\partial \pi}{\partial t} + \operatorname{div}_x\frac{\partial \boldsymbol{w}^s}{\partial t} + m\operatorname{div}_x \boldsymbol{v} = 0, \quad (7.33)$$

the relation

$$\frac{\partial \boldsymbol{w}^s}{\partial t} = (1-m)\boldsymbol{v}(\boldsymbol{x},t) + \int_0^t B_1^s(t-\tau)\cdot\tilde{\boldsymbol{z}}(\boldsymbol{x},\tau)\,d\tau, \quad (7.34)$$

$$\tilde{\boldsymbol{z}}(\boldsymbol{x},t) = -\frac{1}{1-m}\nabla_x\pi(\boldsymbol{x},t) + \rho_s\boldsymbol{F}(\boldsymbol{x},t) - \rho_s\frac{\partial \boldsymbol{v}}{\partial t}(\boldsymbol{x},t)$$

in the case $\lambda_1 > 0$, or the balance of momentum equation in the form

$$\rho_s\frac{\partial^2 \boldsymbol{w}^s}{\partial t^2} = \rho_s B_2^s\cdot\frac{\partial \boldsymbol{v}}{\partial t} + \big((1-m)I - B_2^s\big)\cdot\Big(-\frac{1}{1-m}\nabla_x\pi + \rho_s\boldsymbol{F}\Big) \quad (7.35)$$

in the case $\lambda_1 = 0$ for the solid component. The problem is supplemented by boundary and initial conditions (7.23) *for the velocity $\boldsymbol{v}$ of the liquid component and by the homogeneous initial conditions and the boundary condition*

$$\boldsymbol{w}^s(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x}) = 0, \quad (\boldsymbol{x},t)\in S, \quad t>0, \quad (7.36)$$

for the displacement $\boldsymbol{w}^s$ of the solid component. In Eqs. (7.34)–(7.36) *$\boldsymbol{n}(\boldsymbol{x})$ is the unit normal vector to S at a point $\boldsymbol{x}\in S$, and matrices $B_1^s(t)$ and B_2^s are given below by Eqs.* (7.38) *and* (7.40)*.*

Proof. The boundary condition (7.36) follows from Eq. (7.9), the equality

$$\frac{\partial \boldsymbol{w}}{\partial t} = \frac{\partial \boldsymbol{w}^s}{\partial t} + m\boldsymbol{v},$$

and the homogeneous boundary condition for $\boldsymbol{v}$.

The same equality and Eq. (7.8) imply (7.33). The homogenized equations of balance of momentum (7.32) are derived exactly as before. Therefore, we omit the relevant proofs now and focus only on the derivation of the homogenized equation of the balance of momentum for the solid displacements $\boldsymbol{w}^s$.

(a) If $\lambda_1 > 0$, then the solution of the system of microscopic equations (7.10), (7.16), and (7.17), provided with the homogeneous initial data, is given by formula

$$\boldsymbol{W} = \int_0^t \big(\boldsymbol{v}(\boldsymbol{x},\tau) + \mathbf{B}_1^s(\boldsymbol{y},t-\tau)\cdot\tilde{\boldsymbol{z}}(\boldsymbol{x},\tau)\big)\,d\tau, \quad R = \int_0^t \boldsymbol{R}_f(\boldsymbol{y},t-\tau)\cdot\tilde{\boldsymbol{z}}(\boldsymbol{x},\tau)\,d\tau,$$

in which

$$\mathbf{B}_1^s(\boldsymbol{y},t) = \sum_{i=1}^3 \boldsymbol{W}^i(\boldsymbol{y},t)\otimes\boldsymbol{e}_i, \quad \boldsymbol{R}_f(\boldsymbol{y},t) = \sum_{i=1}^3 R^i(\boldsymbol{y},t)\boldsymbol{e}_i$$

and the functions $\boldsymbol{W}^i(\boldsymbol{y},t)$ and $R^i(\boldsymbol{y},t)$ are defined by virtue of the periodic initial-boundary-value problem

$$\left.\begin{array}{c}\rho_s\dfrac{\partial^2\boldsymbol{W}^i}{\partial t^2}-\lambda_1\Delta\boldsymbol{W}^i+\nabla R^i=0,\quad \operatorname{div}_y\boldsymbol{W}^i=0,\quad \boldsymbol{y}\in Y_s,\quad t>0,\\ \boldsymbol{W}^i=0,\quad \boldsymbol{y}\in\gamma,\quad t>0;\\ \boldsymbol{W}^i(y,0)=0,\quad \rho_s\dfrac{\partial\boldsymbol{W}^i}{\partial t}(y,0)=\boldsymbol{e}_i,\quad \boldsymbol{y}\in Y_s.\end{array}\right\} \tag{7.37}$$

In Eq. (7.37) $\boldsymbol{e}_i$ is the standard Cartesian basis vector.

Therefore,

$$B_1^s(t)=\left\langle\frac{\partial\mathbf{B}_1^s}{\partial t}\right\rangle_{Y_s}(t). \tag{7.38}$$

Note that due to restrictions on the geometry of the elementary cell Y, the problem (7.37) has a unique weak solution, which cannot be a classical one in view of unmatched boundary and initial conditions. That is why the function $B_1^s(t)$ has no time derivative at $t=0$.

(b) If $\lambda_1=0$, then in the process of solving the system (7.10), (7.18), and (7.19) we firstly find the pressure $R(\boldsymbol{x},t,\boldsymbol{y})$ by virtue of solving the Neumann problem for Laplace's equation in Y_s in the form

$$R(\boldsymbol{x},t,\boldsymbol{y})=\sum_{i=1}^{3}R_i(\boldsymbol{y})\boldsymbol{e}_i\cdot\tilde{\boldsymbol{z}}(\boldsymbol{x},t),$$

where $R^i(\boldsymbol{y})$ is the solution of the problem

$$\Delta_y R_i=0,\quad \boldsymbol{y}\in Y_s;\quad \nabla_y R_i\cdot\boldsymbol{n}=\boldsymbol{n}\cdot\boldsymbol{e}_i,\quad \boldsymbol{y}\in\gamma. \tag{7.39}$$

Formula (7.35) appears as the result of homogenization of Eq. (7.18) and

$$B_2^s=\sum_{i=1}^{3}\langle\nabla R_i(\boldsymbol{y})\rangle_{Y_s}\otimes\boldsymbol{e}_i, \tag{7.40}$$

where the matrix $(1-m)I-B_2^s$ is symmetric and positively definite. In fact, let

$$\tilde{R}=\sum_{i=1}^{3}R_i\xi_i$$

for any unit vector ξ. Then

$$(B\cdot\xi)\cdot\xi=\langle(\xi-\nabla\tilde{R})^2\rangle_{Y_f}>0$$

due to the same reasons as in Lemma 7.6. □

8. Proof of Theorem 6

8.1. Weak and Two-Scale Limits of Sequences of Displacement and Pressures. Let $\mu_0=0$. We use again Lemma 2.1 and conclude that there are functions $\boldsymbol{w}_f^\varepsilon$, $\boldsymbol{w}_s^\varepsilon\in L^\infty\big(0,T;W_2^1(\Omega)\big)$ such that

$$\boldsymbol{w}_f^\varepsilon=\boldsymbol{w}^\varepsilon\ \text{ in }\ \Omega_f\times(0,T),\quad \boldsymbol{w}_s^\varepsilon=\boldsymbol{w}^\varepsilon\ \text{ in }\ \Omega_s\times(0,T).$$

On the strength of Theorem 1, the sequences $\{p^\varepsilon\}$, $\{q^\varepsilon\}$, $\{\pi^\varepsilon\}$, $\{\boldsymbol{w}^\varepsilon\}$, $\{\boldsymbol{w}_f^\varepsilon\}$, $\{\sqrt{\alpha_\mu}\nabla\boldsymbol{w}_f^\varepsilon\}$, $\{\boldsymbol{w}_s^\varepsilon\}$, and $\{\sqrt{\alpha_\lambda}\nabla\boldsymbol{w}_s^\varepsilon\}$ are uniformly in ε bounded in $L^2(\Omega_T)$. Hence there exist a subsequence of small parameters $\{\varepsilon>0\}$ and functions p, q, π, $\boldsymbol{w}$, $\boldsymbol{w}_f$, and $\boldsymbol{w}_s$ such that

$$p^\varepsilon\to p,\quad q^\varepsilon\to q,\quad \pi^\varepsilon\to\pi,\quad \boldsymbol{w}^\varepsilon\to\boldsymbol{w},\quad \boldsymbol{w}_f^\varepsilon\to\boldsymbol{w}_f,\quad \boldsymbol{w}_s^\varepsilon\to\boldsymbol{w}_s \tag{8.1}$$

weakly in $L^2(\Omega_T)$ as $\varepsilon\searrow 0$.

Note also that

$$(1-\chi^\varepsilon)\alpha_\lambda D(x,\boldsymbol{w}_s^\varepsilon)\to 0,\quad \chi^\varepsilon\alpha_\mu D(x,\boldsymbol{w}_f^\varepsilon)\to 0 \tag{8.2}$$

strongly in $L^2(\Omega_T)$ as $\varepsilon\searrow 0$.

Relabeling if necessary, we assume that the sequences converge themselves.

Now, taking into account Nguetseng's theorem, we conclude that there exist 1-periodic in $\boldsymbol{y}$ functions $P(\boldsymbol{x},t,\boldsymbol{y})$, $\Pi(\boldsymbol{x},t,\boldsymbol{y})$, $Q(\boldsymbol{x},t,\boldsymbol{y})$, $\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$, $\boldsymbol{W}_f(\boldsymbol{x},t,\boldsymbol{y})$, and $\boldsymbol{W}_s(\boldsymbol{x},t,\boldsymbol{y})$ such that the sequences $\{p^\varepsilon\}$, $\{\pi^\varepsilon\}$, $\{q^\varepsilon\}$, $\{\boldsymbol{w}^\varepsilon\}$, $\{\boldsymbol{w}^\varepsilon_f\}$, and $\{\boldsymbol{w}^\varepsilon_s\}$ two-scale converge to $P(\boldsymbol{x},t,\boldsymbol{y})$, $\Pi(\boldsymbol{x},t,\boldsymbol{y})$, $Q(\boldsymbol{x},t,\boldsymbol{y})$, $\mathbf{W}(\boldsymbol{x},t,\boldsymbol{y})$, $\boldsymbol{W}_f(\boldsymbol{x},t,\boldsymbol{y})$, and $\boldsymbol{W}_s(\boldsymbol{x},t,\boldsymbol{y})$, respectively.

Finally note that if $\mu_1=\infty$ ($\lambda_1=\infty$), then due to Lemma 7.4 the sequence $\{\boldsymbol{w}^\varepsilon_f\}$ ($\{\boldsymbol{w}^\varepsilon_s\}$) converges strongly to $\boldsymbol{w}_f$ ($\boldsymbol{w}_s$) and $\boldsymbol{w}^f=\langle\mathbf{W}\rangle_{Y_f}=m\boldsymbol{w}_f$ ($\boldsymbol{w}^s=\langle\mathbf{W}\rangle_{Y_s}=(1-m)\boldsymbol{w}_s$).

8.2. Micro- and Macroscopic Equations. As before we begin the proof of the theorem with the macro- and microscopic equations connected with the continuity equations.

Lemma 8.1. *For all $\boldsymbol{x}\in\Omega$ and $\boldsymbol{y}\in Y$, weak and two-scale limits of the sequences $\{p^\varepsilon\}$, $\{\pi^\varepsilon\}$, $\{q^\varepsilon\}$, $\{\boldsymbol{w}^\varepsilon\}$, $\{\boldsymbol{w}^\varepsilon_f\}$, and $\{\boldsymbol{w}^\varepsilon_s\}$ satisfy the relations*

$$Q=q\frac{\chi}{m},\quad P=p\frac{\chi}{m},\quad \Pi=\pi\frac{1-\chi}{1-m}; \tag{8.3}$$

$$\frac{q}{m}=\frac{\pi}{1-m},\quad q=p+\nu_0p_*^{-1}\frac{\partial p}{\partial t}; \tag{8.4}$$

$$\frac{p}{p_*}+\frac{\pi}{\eta_0}+\operatorname{div}_x\boldsymbol{w}=0; \tag{8.5}$$

$$\boldsymbol{w}(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x})=0,\quad \boldsymbol{x}\in S,\quad t>0; \tag{8.6}$$

$$\operatorname{div}_y\boldsymbol{W}=0; \tag{8.7}$$

$$\boldsymbol{W}=\chi\boldsymbol{W}_f+(1-\chi)\boldsymbol{W}_s. \tag{8.8}$$

Proof. The derivation of Eqs. (8.3)–(8.8) is the same as the derivation of Eqs. (7.4), (7.5), (7.8) and (7.10), (7.11) in Lemma 7.1. Thus, for example, the first relation in Eq. (8.4) follows from Eq. (8.3) and from the strong convergence of the sequence $\{q^\varepsilon+\pi^\varepsilon\}$ to $q+\pi$, which implies the equality $Q+\Pi=q+\pi$. ☐

Lemma 8.2. *For all $(\boldsymbol{x},t)\in\Omega_T$ the relation*

$$\rho_f\frac{\partial^2\boldsymbol{w}^f}{\partial t^2}+\rho_s\frac{\partial^2\boldsymbol{w}^s}{\partial t^2}=-\frac{1}{1-m}\nabla_x\pi+\hat{\rho}\boldsymbol{F} \tag{8.9}$$

holds true.

Proof. Substituting a test function of the form $\boldsymbol{\psi}=\boldsymbol{\psi}(\boldsymbol{x},t)$ into integral identity (1.3) and passing to the limit as $\varepsilon\searrow 0$, we arrive at Eq. (8.9). ☐

Lemma 8.3. *Let $\mu_1=\infty$ and $\lambda_1<\infty$. Then functions $\{\boldsymbol{W},\,\boldsymbol{w}_f,\,\pi\}$ satisfy in Y_s the system of microscopic equations*

$$\rho_s\frac{\partial^2\boldsymbol{W}}{\partial t^2}=\lambda_1\Delta_y\boldsymbol{W}-\nabla_yR^s-\frac{1}{1-m}\nabla_x\pi+\rho_s\boldsymbol{F},\quad \boldsymbol{y}\in Y_s, \tag{8.10}$$

$$\mathbf{W}=\boldsymbol{w}_f,\quad \boldsymbol{y}\in\gamma, \tag{8.11}$$

in the case $\lambda_1>0$, and relations

$$\rho_s\frac{\partial^2\boldsymbol{W}}{\partial t^2}=-\nabla_yR^s-\frac{1}{1-m}\nabla_x\pi+\rho_s\boldsymbol{F},\quad \boldsymbol{y}\in Y_s, \tag{8.12}$$

$$(\boldsymbol{W}-\boldsymbol{w}_f)\cdot\boldsymbol{n}=0,\quad \boldsymbol{y}\in\gamma, \tag{8.13}$$

in the case $\lambda_1=0$.

In Eq. (8.13) $\boldsymbol{n}$ *is the unit normal to* γ.

The proof of this lemma repeats the proof of Lemma 7.5.

In the same way one can prove the following lemma.

Lemma 8.4. *Let $\mu_1 < \infty$ and $\lambda_1 = \infty$. Then functions $\{\boldsymbol{W}, \boldsymbol{w}_s, q\}$ satisfy in Y_f the system of microscopic equations*

$$\rho_f \frac{\partial^2 \boldsymbol{W}}{\partial t^2} = \mu_1 \Delta_y \frac{\partial \boldsymbol{W}}{\partial t} - \nabla_y R^f - \frac{1}{m}\nabla_x q + \rho_f \boldsymbol{F}, \quad \boldsymbol{y} \in Y_f, \tag{8.14}$$

$$\boldsymbol{W} = \boldsymbol{w}_s, \quad \boldsymbol{y} \in \gamma, \tag{8.15}$$

in the case $\mu_1 > 0$, and relations

$$\rho_f \frac{\partial^2 \boldsymbol{W}}{\partial t^2} = -\nabla_y R^f - \frac{1}{m}\nabla_x q + \rho_f \boldsymbol{F}, \quad \boldsymbol{y} \in Y_f, \tag{8.16}$$

$$(\boldsymbol{W} - \boldsymbol{w}_s) \cdot \boldsymbol{n} = 0, \quad \boldsymbol{y} \in \gamma, \tag{8.17}$$

in the case $\mu_1 = 0$.

Lemma 8.5. *Let $\mu_1 < \infty$ and $\lambda_1 < \infty$ and $\tilde{\rho} = \rho_f \chi + \rho_s(1-\chi)$. Then functions $\{\boldsymbol{W}, \pi\}$ satisfy in Y the system of microscopic equations*

$$\tilde{\rho}\frac{\partial^2 \boldsymbol{W}}{\partial t^2} + \frac{1}{1-m}\nabla_x \pi - \tilde{\rho}\boldsymbol{F} = \operatorname{div}_y \left\{ \mu_1 \chi D\left(y, \frac{\partial \boldsymbol{W}}{\partial t}\right) + \lambda_1 (1-\chi) D(y, \boldsymbol{W}) - RI \right\}. \tag{8.18}$$

In the proof of the last lemma we additionally use Nguetseng's theorem, which states that the sequence $\{\varepsilon D(x, \partial \boldsymbol{w}^\varepsilon/\partial t)\}$ ($\{\varepsilon D(x, \boldsymbol{w}^\varepsilon)\}$) two-scale converges to $D(y, \partial \boldsymbol{W}/\partial t)$ ($D(y, \boldsymbol{W})$).

8.3. Homogenized Equations. Lemmas 8.1 and 8.2 imply the following statement.

Lemma 8.6. *Let $\mu_1 = \lambda_1 = \infty$; then $\boldsymbol{w}_f = \boldsymbol{w}_s = \boldsymbol{w}$ and functions $\boldsymbol{w}$, p, q, and π satisfy in Ω_T the system of acoustic equations*

$$\hat{\rho}\frac{\partial^2 \boldsymbol{w}}{\partial t^2} = -\frac{1}{(1-m)}\nabla_x \pi + \hat{\rho}\boldsymbol{F}, \tag{8.19}$$

$$\frac{1}{p_*}p + \frac{1}{\eta_0}\pi + \operatorname{div}_x \boldsymbol{w} = 0, \tag{8.20}$$

$$q = p + \frac{\nu_0}{p_*}\frac{\partial p}{\partial t}, \quad \frac{1}{m}q = \frac{1}{1-m}\pi, \tag{8.21}$$

the homogeneous initial conditions

$$\boldsymbol{w}(\boldsymbol{x}, 0) = \frac{\partial \boldsymbol{w}}{\partial t}(\boldsymbol{x}, 0) = 0, \quad \boldsymbol{x} \in \Omega, \tag{8.22}$$

and the homogeneous boundary condition

$$\boldsymbol{w}(\boldsymbol{x}, t) \cdot \boldsymbol{n}(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in S, \quad t > 0. \tag{8.23}$$

Now we pass to more complicated cases.

Lemma 8.7. *Let $\mu_1 = \infty$ and $\lambda_1 < \infty$. Then functions $\boldsymbol{w}_f$, $\boldsymbol{w}^s$, p, q, and π satisfy in Ω_T the system of acoustic equations, which consists of the state equations* (8.21)*, balance of momentum equation for the liquid component*

$$\rho_f m \frac{\partial^2 \boldsymbol{w}_f}{\partial t^2} + \rho_s \frac{\partial^2 \boldsymbol{w}^s}{\partial t^2} = -\frac{1}{m}\nabla_x q + \hat{\rho}\boldsymbol{F}, \tag{8.24}$$

the continuity equation

$$\frac{1}{p_*}p + \frac{1}{\eta_0}\pi + m \operatorname{div}_x \boldsymbol{w}_f + \operatorname{div}_x \boldsymbol{w}^s = 0, \tag{8.25}$$

and the relation

$$\frac{\partial \boldsymbol{w}^s}{\partial t} = (1-m)\frac{\partial \boldsymbol{w}_f}{\partial t} + \int\limits_0^t B_1^s(t-\tau)\cdot \boldsymbol{z}^s(\boldsymbol{x},\tau)\,d\tau, \tag{8.26}$$

$$\boldsymbol{z}^s(\boldsymbol{x},t) = -\frac{1}{1-m}\nabla_x \pi(\boldsymbol{x},t) + \rho_s \boldsymbol{F}(\boldsymbol{x},t) - \rho_s \frac{\partial^2 \boldsymbol{w}_f}{\partial t^2}(\boldsymbol{x},t),$$

in the case $\lambda_1 > 0$, or the balance of momentum equation for the solid component in the form

$$\rho_s \frac{\partial^2 \boldsymbol{w}^s}{\partial t^2} = \rho_s B_2^s \cdot \frac{\partial^2 \boldsymbol{w}_f}{\partial t^2} + \big((1-m)I - B_2^s\big)\cdot\Big(-\frac{1}{1-m}\nabla_x \pi + \rho_s \boldsymbol{F}\Big) \tag{8.27}$$

in the case $\lambda_1 = 0$. The problem (8.21), (8.24)–(8.27) *is supplemented by homogeneous initial conditions* (8.22) *for the displacements in the liquid and solid components and homogeneous boundary condition* (8.23) *for the displacements $\boldsymbol{w} = m\boldsymbol{w}_f + \boldsymbol{w}^s$.*

In Eqs. (8.26), (8.27) *matrices $B_1^s(t)$ and B_2^s are the same as in Theorem* 5.

Proof. Equation (8.24) follows directly from Eq. (8.9). The continuity equation (8.25) follows from Eq. (8.5) if we take into account the equality

$$\boldsymbol{w} = m\boldsymbol{w}_f + \boldsymbol{w}^s.$$

The derivation of Eqs. (8.26), (8.27) is exactly the same as in Lemma 7.7. □

Lemma 8.8. *Let $\mu_1 < \infty$ and $\lambda_1 = \infty$. Then functions $\boldsymbol{w}^f$, $\boldsymbol{w}_s$, p, q, and π satisfy in Ω_T the system of acoustic equations, which consists of the state equations* (8.21), *the balance of momentum equation for the solid component*

$$\rho_f \frac{\partial^2 \boldsymbol{w}^f}{\partial t^2} + \rho_s(1-m)\frac{\partial^2 \boldsymbol{w}_s}{\partial t^2} = -\frac{1}{(1-m)}\nabla_x \pi + \hat{\rho}\boldsymbol{F}, \tag{8.28}$$

the continuity equation

$$\frac{1}{p_*}p + \frac{1}{\eta_0}\pi + \operatorname{div}_x \boldsymbol{w}^f + (1-m)\operatorname{div}_x \boldsymbol{w}_s = 0, \tag{8.29}$$

and the relation

$$\frac{\partial \boldsymbol{w}^f}{\partial t} = m\frac{\partial \boldsymbol{w}_s}{\partial t} + \int\limits_0^t B_1^f(t-\tau)\cdot \boldsymbol{z}^f(\boldsymbol{x},\tau)\,d\tau, \tag{8.30}$$

$$\boldsymbol{z}^f(\boldsymbol{x},t) = -\frac{1}{m}\nabla_x q(\boldsymbol{x},t) + \rho_f \boldsymbol{F}(\boldsymbol{x},t) - \rho_f \frac{\partial^2 \boldsymbol{w}_s}{\partial t^2}(\boldsymbol{x},t),$$

in the case $\mu_1 > 0$, or the balance of momentum equation for the liquid component in the form

$$\rho_f \frac{\partial^2 \boldsymbol{w}^f}{\partial t^2} = \rho_f B_2^f \cdot \frac{\partial^2 \boldsymbol{w}_s}{\partial t^2} + (mI - B_2^f)\cdot\Big(-\frac{1}{m}\nabla_x q + \rho_f \boldsymbol{F}\Big) \tag{8.31}$$

in the case $\mu_1 = 0$. The problem (8.21), (8.28)–(8.31) *is supplemented by homogeneous initial conditions* (8.22) *for displacements in the liquid and solid components and homogeneous boundary condition* (8.23) *for the displacements $\boldsymbol{w} = \boldsymbol{w}^f + (1-m)\boldsymbol{w}_s$.*

In Eqs. (8.30), (8.31) *matrices $B_1^f(t)$ and B_2^f are given below by formulas* (8.32), (8.33) *and the symmetric matrix $mI - B_2^f$ is strictly positively definite.*

Proof. The proof of this lemma repeats the proofs of the previous lemmas and

$$B_1^f(t) = \left\langle \sum_{i=1}^{3} \boldsymbol{V}^i(\boldsymbol{y},t) \right\rangle_{Y_f} \otimes \boldsymbol{e}_i, \tag{8.32}$$

$$B_2^f = \sum_{i=1}^{3} \langle \nabla R_i^f(\boldsymbol{y}) \rangle_{Y_f} \otimes \boldsymbol{e}_i, \tag{8.33}$$

where functions $\boldsymbol{V}^i(\boldsymbol{y},t)$ solve the periodic initial-boundary-value problem

$$\left.\begin{aligned} \rho_f \frac{\partial \boldsymbol{V}^i}{\partial t} - \mu_1 \Delta \boldsymbol{V}^i + \nabla R^i = 0, \quad \operatorname{div}_y \boldsymbol{V}^i = 0, \quad \boldsymbol{y} \in Y_f, \quad t > 0, \\ \boldsymbol{V}^i = 0, \quad \boldsymbol{y} \in \gamma, \quad t > 0; \quad \rho_f \boldsymbol{V}^i(y,0) = \boldsymbol{e}_i, \quad \boldsymbol{y} \in Y_f, \end{aligned}\right\} \tag{8.34}$$

and functions $R_i^f(\boldsymbol{y})$ solve the periodic boundary-value problem

$$\Delta_y R_i^f = 0, \quad \boldsymbol{y} \in Y_f; \quad \nabla_y R_i^f \cdot \boldsymbol{n} = \boldsymbol{n} \cdot \boldsymbol{e}_i, \quad \boldsymbol{y} \in \gamma. \tag{8.35}$$

In fact, by definition

$$\boldsymbol{w}^f(\boldsymbol{x},t) = \langle \boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y}) \rangle_{Y_f},$$

where functions $\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$ and $R^f(\boldsymbol{x},t,\boldsymbol{y})$ for $\mu_1 > 0$ are the solution to the system of microscopic equations (8.7), (8.14), (8.15). We look for the solution of this system in the form

$$\frac{\partial \boldsymbol{W}}{\partial t}(\boldsymbol{x},t,\boldsymbol{y}) = \frac{\partial \boldsymbol{w}_s}{\partial t}(\boldsymbol{x},t) + \int_0^t \mathbf{B}_1^f(\boldsymbol{y},t-\tau) \cdot \boldsymbol{z}^f(\boldsymbol{x},\tau)\, d\tau,$$

$$R^f(\boldsymbol{x},t,\boldsymbol{y}) = \int_0^t \boldsymbol{R}_f(\boldsymbol{y},t-\tau) \cdot \boldsymbol{z}^f(\boldsymbol{x},\tau)\, d\tau.$$

In turn,

$$\mathbf{B}_1^f(\boldsymbol{y},t) = \sum_{i=1}^{3} \boldsymbol{V}^i(\boldsymbol{y},t) \otimes \boldsymbol{e}_i, \quad \boldsymbol{R}_f(\boldsymbol{y},t) = \sum_{i=1}^{3} R^i(\boldsymbol{y},t)\boldsymbol{e}_i.$$

If $\mu_1 = 0$, then functions $\mathbf{W}(\boldsymbol{x},t,\boldsymbol{y})$ and $R^f(\boldsymbol{x},t,\boldsymbol{y})$ solve the system (8.7), (8.16), (8.17), where

$$R^f(\boldsymbol{x},t,\boldsymbol{y}) = \sum_{i=1}^{3} R_i^f(\boldsymbol{y})\boldsymbol{e}_i \cdot \boldsymbol{z}^f(\boldsymbol{x},t).$$

Note that, as before, for the case of the matrix $(1-m)I - B_2^s$ in Lemma 8.7, the matrix $mI - B_2^f$ is symmetric and strictly positively definite. □

The proof of Theorem 6 is completed by

Lemma 8.9. *Let $\mu_1 < \infty$ and $\lambda_1 < \infty$. Then functions $\boldsymbol{w}$, p, q, and π satisfy in Ω_T the system of acoustic equations, which consists of the continuity and the state equations (8.20) and (8.21) and the relation*

$$\frac{\partial \boldsymbol{w}}{\partial t} = \int_0^t B^\pi(t-\tau) \cdot \nabla \pi(\boldsymbol{x},\tau)\, d\tau + \boldsymbol{f}(\boldsymbol{x},t), \tag{8.36}$$

where $B^\pi(t)$ and $\boldsymbol{f}(\boldsymbol{x},t)$ are given below by Eqs. (8.40) and (8.41).

The problem (8.20), (8.21), and (8.36) is supplemented by homogeneous initial and boundary conditions (8.22) and (8.23).

Proof. Let

$$\boldsymbol{W} = \int_0^t \sum_{i=1}^3 \Big\{ \boldsymbol{W}_i^\pi(\boldsymbol{y}, t-\tau) \frac{\partial \pi}{\partial x_i}(\boldsymbol{x}, \tau) + \boldsymbol{W}_i^F(\boldsymbol{y}, t-\tau) F_i(\boldsymbol{x}, \tau) \Big\} d\tau,$$

$$R = \int_0^t \sum_{i=1}^3 \Big\{ R_i^\pi(\boldsymbol{y}, t-\tau) \frac{\partial \pi}{\partial x_i}(\boldsymbol{x}, \tau) + R_i^F(\boldsymbol{y}, t-\tau) F_i(\boldsymbol{x}, \tau) \Big\} d\tau,$$

where $\boldsymbol{F} = \sum_{i=1}^3 F_i \boldsymbol{e}_i$ and functions $\{\boldsymbol{W}_i^\pi(\boldsymbol{y}, t), R_i^\pi(\boldsymbol{y}, t)\}$ and $\{\boldsymbol{W}_i^F(\boldsymbol{y}, t), R_i^F(\boldsymbol{y}, t)\}$ are periodic in $\boldsymbol{y}$ solutions of the system

$$\left.\begin{array}{c} \operatorname{div}_y \left\{ \mu_1 \chi D\left(y, \dfrac{\partial \boldsymbol{W}_i^j}{\partial t}\right) + \lambda_1 (1-\chi) D(y, \boldsymbol{W}_i^j) - R_i^j I \right\} = \tilde{\rho} \dfrac{\partial^2 \boldsymbol{W}_i^j}{\partial t^2}, \\ \operatorname{div}_y \boldsymbol{W}_i^j = 0, \quad \boldsymbol{y} \in Y, \quad t > 0, \quad j = \pi, F, \end{array}\right\} \tag{8.37}$$

which satisfy the following initial conditions:

$$\boldsymbol{W}_i^\pi(\boldsymbol{y}, 0) = 0, \quad \tilde{\rho} \frac{\partial \boldsymbol{W}_i^\pi}{\partial t}(\boldsymbol{y}, 0) = -\frac{1}{1-m} \boldsymbol{e}_i, \quad \boldsymbol{x} \in Y, \tag{8.38}$$

$$\boldsymbol{W}_i^F(\boldsymbol{y}, 0) = 0, \quad \frac{\partial \boldsymbol{W}_i^F}{\partial t}(\boldsymbol{y}, 0) = \boldsymbol{e}_i, \quad \boldsymbol{x} \in Y. \tag{8.39}$$

Then the functions $\boldsymbol{W}$ and R solve the system of microscopic equations (8.7) and (8.18) and by definition $\boldsymbol{w} = \langle \mathbf{W} \rangle_Y$. Therefore,

$$B^\pi(t) = \sum_{i=1}^3 \left\langle \frac{\partial \boldsymbol{W}_i^\pi}{\partial t}(\boldsymbol{y}, t) \right\rangle_Y \otimes \boldsymbol{e}_i, \tag{8.40}$$

$$\mathbf{f}(\boldsymbol{x}, t) = \int_0^t \sum_{i=1}^3 \left\langle \frac{\partial \boldsymbol{W}_i^F}{\partial t} \right\rangle_Y (t-\tau) F_i(\boldsymbol{x}, \tau)\, d\tau. \tag{8.41}$$

The solvability and uniqueness of problems (8.37), (8.38) or (8.37), (8.39) follow directly from the energy identity

$$\frac{1}{2}\left\langle \tilde{\rho}\left(\frac{\partial \boldsymbol{W}_i^j}{\partial t}\right)^2 \right\rangle_Y (t) + \frac{1}{2}\langle \lambda_1 D(y, \boldsymbol{W}_i^j) : D(y, \boldsymbol{W}_i^j) \rangle_{Y_s}(t)$$
$$+ \int_0^t \left\langle \mu_1 D\left(y, \frac{\partial \boldsymbol{W}_i^j}{\partial \tau}\right) : D\left(y, \frac{\partial \boldsymbol{W}_i^j}{\partial \tau}\right) \right\rangle_{Y_s} (\tau)\, d\tau = \frac{1}{2} \beta^j$$

for $i = 1, 2, 3$ and $j = \pi, F$. Here

$$\beta^\pi = \left\langle \frac{1}{\tilde{\rho}} \right\rangle_Y, \quad \beta^F = \langle \tilde{\rho} \rangle_Y.$$

As before, Eqs. (8.37) are understood in the sense of distributions and the function $B^\pi(t)$ has no time derivative at $t = 0$. That is why we cannot represent relation (8.36) in the form of the balance of momentum equation, like (8.19) or (8.27). □

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