LAMÉ SYSTEM OF ELASTICITY THEORY IN A PLANE ORTHOTROPIC MEDIUM

E. A. Abapolova and A. P. Soldatov UDC 517.956.223;517.958:539.3

Abstract. The authors develop a functional-theoretic approach to solving boundary-value problems for the Lamé system of elasticity theory. Special attention is paid to the case of a plane orthotropic medium.

The state of the medium of plane anisotropic elasticity theory [4] is characterized by the stress tensor σ and the deformation tensor ε, which are symmetric 2×2-matrix functions and which can be written in the form

$$\sigma = \begin{pmatrix} \sigma_1 & \sigma_3 \\ \sigma_3 & \sigma_2 \end{pmatrix}, \qquad \varepsilon = \begin{pmatrix} \varepsilon_1 & \varepsilon_3 \\ \varepsilon_3 & \varepsilon_2 \end{pmatrix},$$

where ε_j are expressed through the displacement vector $u = (u_1, u_2)$ according to the formulas

$$\varepsilon_1 = \frac{\partial u_1}{\partial x}, \qquad \varepsilon_2 = \frac{\partial u_2}{\partial y}, \qquad 2\varepsilon_3 = \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}.$$

When there are no mass forces, the matrix σ satisfies the equilibrium equations

$$\frac{\partial \sigma_{(1)}}{\partial x} + \frac{\partial \sigma_{(2)}}{\partial y} = 0,$$

where $\sigma_{(j)}$ denotes the jth column of the matrix σ, which is connected with the deformation tensor ε by Hooke's law:

$$\begin{aligned} \sigma_1 &= \alpha_1\varepsilon_1 + \alpha_4\varepsilon_2 + 2\alpha_6\varepsilon_3, \\ \sigma_2 &= \alpha_4\varepsilon_1 + \alpha_2\varepsilon_2 + 2\alpha_5\varepsilon_3, \\ \sigma_3 &= \alpha_6\varepsilon_1 + \alpha_5\varepsilon_2 + 2\alpha_3\varepsilon_3, \end{aligned} \qquad \alpha = \begin{pmatrix} \alpha_1 & \alpha_4 & \alpha_6 \\ \alpha_4 & \alpha_2 & \alpha_5 \\ \alpha_6 & \alpha_5 & \alpha_3 \end{pmatrix},$$

where the matrix α is positive-definite. The entries α_j of this matrix are called the elasticity modules; by the Sylvester criterion, they satisfy the inequalities $\alpha_j > 0$, $j = 1, 2, 3$, and $\alpha_1\alpha_2 > \alpha_4^2$, $\alpha_1\alpha_3 > \alpha_6^2$, $\alpha_2\alpha_3 > \alpha_5^2$.

Substituting for ε_j their expressions through the displacement vector, we obtain the following form of relations of Hooke's law:

$$\sigma_{(i)} = a_{i1}\frac{\partial u}{\partial x} + a_{i2}\frac{\partial u}{\partial y}, \quad i = 1, 2 \tag{1}$$

with matrix coefficients

$$a_{11} = \begin{pmatrix} \alpha_1 & \alpha_6 \\ \alpha_6 & \alpha_3 \end{pmatrix}, \qquad a_{12} = \begin{pmatrix} \alpha_6 & \alpha_4 \\ \alpha_3 & \alpha_5 \end{pmatrix},$$

$$a_{21} = \begin{pmatrix} \alpha_6 & \alpha_3 \\ \alpha_4 & \alpha_5 \end{pmatrix}, \qquad a_{22} = \begin{pmatrix} \alpha_3 & \alpha_5 \\ \alpha_5 & \alpha_2 \end{pmatrix}.$$

Substituting expressions (1) in the equilibrium equations, we obtain the following second-order elliptic system for the displacement vector $u = (u_1, u_2)$:

$$a_{11}\frac{\partial^2 u}{\partial x^2} + (a_{12} + a_{21})\frac{\partial^2 u}{\partial x \partial y} + a_{22}\frac{\partial^2 u}{\partial y^2} = 0; \tag{2}$$

it is called the Lamé system.

As is known [6], the main boundary conditions for the Lamé system considered in a domain D on the plane are either the assignment of the displacement vector

$$u|_\Gamma = f, \tag{3}$$

on the boundary $\Gamma = \partial D$ or the normal component $\sigma n = \sigma_{(1)} n_1 + \sigma_{(2)} n_2$ of the stress tensor σ, i.e.,

$$(\sigma n)|_\Gamma = g, \tag{4}$$

where $n = (n_1, n_2)$ is the unit exterior normal to Γ. Obviously, (3) corresponds to the Dirichlet problem. According to (1), we can write

$$\sigma n = \sum_{i,j=1,2} a_{ij} \frac{\partial u}{\partial x_i} n_j,$$

where $x_1 = x$ and $x_2 = y$. Therefore, (4) corresponds to the Neumann problem for the Lamé system. These problems are also called the first and second boundary-value problems.

The second boundary-value problem (4) can be written in the form of the first boundary-value problem with respect to the so-called conjugate function v. The latter is defined by the relation

$$\frac{\partial v}{\partial x} = -\left(a_{21} \frac{\partial u}{\partial x} + a_{22} \frac{\partial u}{\partial y}\right), \qquad \frac{\partial v}{\partial y} = a_{11} \frac{\partial u}{\partial x} + a_{12} \frac{\partial u}{\partial y}. \tag{5}$$

Then comparing (1) and (5), we see that $\sigma n = [v[x(s), y(s)]]'$, where $x(s) + iy(s)$ is the natural parametrization of Γ by the arclength parameter s. Hence after integration, the boundary condition (4) takes the form

$$v|_\Gamma = f, \tag{6}$$

where f is the primitive of the function g considered as a function of the arclength s on v.

Let us consider the characteristic polynomial of the elliptic system (1):

$$\chi(z) = \det P(z), \qquad P(z) = a_{11} + (a_{12} + a_{21})z + a_{22} z^2.$$

Two cases are possible when in the upper half-plane, the equation $\chi(z) = 0$ has

(i) two distinct roots $\nu_1 \neq \nu_2$ and
(ii) one multiple root ν.

In accordance with these cases, we set

$$\begin{aligned} &\text{(i)} \quad J = \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix}, \quad \nu_1 \neq \nu_2; \\ &\text{(ii)} \quad J = \begin{pmatrix} \nu & i \\ 0 & \nu \end{pmatrix}. \end{aligned} \tag{7}$$

In each of these cases, there exist linearly independent vectors $x, y \in \mathbb{C}^2$ such that

(i) $P(\nu_1)x = P(\nu_2)y = 0$ and, respectively,
(ii) $P(\nu)x = P(\nu)y + P'(\nu)x = 0$.

Let us compose the matrix $B \in \mathbb{C}^{2\times 2}$ from these vectors considered as column vectors, and in the notation (1), (7), let us set $C = (a_{21}B + a_{22}BJ)$. According to [9, 10], the general solution u of the Lamé system and the function v conjugated to it are described by the following relations in terms of these matrices:

$$u = \operatorname{Re} B\phi, \qquad v = \operatorname{Re} C\phi, \tag{8}$$

where the 2-vector-valued function $\phi = (\phi_1, \phi_2)$ is a solution of the Douglis system

$$\frac{\partial \phi}{\partial y} - J \frac{\partial \phi}{\partial x} = 0. \tag{9}$$

Here, it is assumed that the domain D is simply connected and the functions u, v, and ϕ vanish at a fixed point $z_0 \in D$; under this assumption, ϕ is uniquely determined by u.

In explicit form, the matrices B and C can be described through the roots of the characteristic polynomial χ [9]. For this purpose, let us represent χ in the form

$$\chi(z) = g_(z)g_2(z) - g_3^2(z) = h_1(z) - zh_2(z) + z^2 h_3(z)$$

with quadratic trinomials

$$\begin{aligned} g_1(z) &= \alpha_1 + 2\alpha_6 z + \alpha_3 z^2, & h_1(z) &= \beta_2 - \beta_5 z + \beta_4 z^2, \\ g_2(z) &= \alpha_3 + 2\alpha_5 z + \alpha_2 z^2, & h_2(z) &= \beta_5 - \beta_3 z + \beta_6 z^2, \\ g_3(z) &= \alpha_6 + (\alpha_3 + \alpha_4) z + \alpha_5 z^2, & h_3(z) &= \beta_4 - \beta_6 z + \beta_1 z^2. \end{aligned} \tag{10}$$

Here,

$$\beta = \begin{pmatrix} \beta_1 & \beta_4 & \beta_6 \\ \beta_4 & \beta_2 & \beta_5 \\ \beta_6 & \beta_5 & \beta_3 \end{pmatrix} = (\det \alpha)\alpha^{-1}$$

denotes the matrix associated with the matrix α in Hooke's law; it is also positive-definite. In explicit form,

$$\begin{aligned} \beta_1 &= \alpha_2\alpha_3 - \alpha_5^2, & \beta_2 &= \alpha_1\alpha_3 - \alpha_6^2, & \beta_3 &= \alpha_1\alpha_2 - \alpha_4^2, \\ \beta_4 &= \alpha_5\alpha_6 - \alpha_3\alpha_4, & \beta_5 &= \alpha_4\alpha_6 - \alpha_1\alpha_5, & \beta_6 &= \alpha_4\alpha_5 - \alpha_2\alpha_6. \end{aligned} \tag{11}$$

In this notation, in accordance with two cases in (7), we have the relations

$$B = \begin{pmatrix} -g_3(\nu_1) & g_2(\nu_2) \\ g_1(\nu_1) & -g_3(\nu_2) \end{pmatrix}, \qquad C = \begin{pmatrix} -\nu_1 h_2(\nu_1) & -\nu_2 h_3(\nu_2) \\ h_2(\nu_1) & h_3(\nu_2) \end{pmatrix}, \quad h_3(\nu_1) = 0, \tag{12i$_1$}$$

$$B = \begin{pmatrix} -g_3(\nu_1) & g_2(\nu_2) \\ g_1(\nu_1) & -g_3(\nu_2) \end{pmatrix}, \qquad C = \begin{pmatrix} -\nu_1 h_2(\nu_1) & -\nu_2 h_3(\nu_2) \\ h_2(\nu_1) & h_3(\nu_2) \end{pmatrix}, \quad h_3(\nu_1) = 0, \tag{12i$_2$}$$

$$B = \begin{pmatrix} g_2(\nu) & g_2'(\nu) \\ -g_3(\nu) & -g_3'(\nu) \end{pmatrix}, \qquad C = \begin{pmatrix} -\nu h_3(\nu) & -h_3(\nu) - \nu h_3'(\nu) \\ h_3(\nu) & h_3'(\nu) \end{pmatrix}, \tag{12i$_i$}$$

where, for definiteness, it is assumed that $h_3(\nu_2) \neq 0$ in the case (i). Since h_3 is a quadratic trinomial with real coefficient, at least one of the numbers ν_1 and ν_2 is not its root, and the above assumption does not restrict the generality. As is shown in [9], in all the cases, the matrices B and C are invertible.

Note that a close approach to representation of solutions of anisotropic plane elasticity theory is considered in [5].

The general solution ϕ of system (9) is expressed by the following formulas according to two cases (7) of the matrix J:

$$\begin{aligned} &\text{(i)} \quad \phi_k(x,y) = \psi_k(x + \nu_k y), \quad k = 1,2, \\ &\text{(ii)} \quad \phi_1(x,y) = \psi_1(x + \nu y) + y\psi_2'(x + \nu y), \quad \phi_2(x,y) = \psi_2(x + \nu y), \end{aligned} \tag{13}$$

where the functions ψ_k are analytic in the domain $D(\nu_k) = \{x + \nu_k y,\ (x,y) \in D\}$ in the case (i) and in the domain $D(\nu)$ in the case (ii). In particular, in the isotropic case where

$$\alpha_1 = \alpha_2 = \lambda + 2\mu, \quad \alpha_3 = \mu, \quad \alpha_4 = \lambda \tag{14}$$

with some positive λ and μ, we have the case (ii) with $\nu = i$, and the substitution of (10)–(13) in (8) leads to the known Kolosov–Muskhelishvili formulas [6].

An elastic medium is said to be orthotropic if the coordinate lines are symmetry axes. This case corresponds to vanishing of the elasticity modules α_5 and α_6 of the matrix α. In particular, the matrix

α is block-diagonal, and the matrix β associated to it has a similar property. Therefore, with account for (11), expressions (10) take the form

$$\begin{aligned} g_1(z) &= \alpha_1 + \alpha_3 z^2, & h_1(z) &= \alpha_3(\alpha_1 - \alpha_4 z^2), \\ g_2(z) &= \alpha_3 + \alpha_2 z^2, & h_2(z) &= (\alpha_4^2 - \alpha_1\alpha_2)z, \\ g_3(z) &= (\alpha_3 + \alpha_4)z, & h_3(z) &= \alpha_3(-\alpha_4 + \alpha_2 z^2), \end{aligned} \tag{15}$$

and, respectively, the characteristic polynomial is given by the relation

$$\chi(z) = \alpha_3(\alpha_1 + 2az^2 + \alpha_2 z^4), \qquad a = \frac{\alpha_1\alpha_2 - \alpha_4^2 - 2\alpha_3\alpha_4}{2\alpha_3}. \tag{16}$$

It is easy to verify that $a + \sqrt{\alpha_1\alpha_2} > 0$. Indeed, this inequality is equivalent to the inequality

$$(\sqrt{\alpha_1\alpha_2} + \alpha_3)^2 > (\alpha_3 + \alpha_4)^2,$$

which is obvious because of the condition $\sqrt{\alpha_1\alpha_2} > |\alpha_4|$. Therefore, we have the following expressions for the roots of the bi-quadratic polynomial (16) in the upper half-plane:

$$\nu_\pm = ie^{\pm i\theta}\sqrt[4]{\frac{\alpha_1}{\alpha_2}}, \quad 2\theta = \arctan\frac{a}{\sqrt{\alpha_1\alpha_2 - a^2}}, \quad |a| < \sqrt{\alpha_1\alpha_2}, \tag{17i$_1$}$$

$$\nu_\pm = i\sqrt{\frac{a \pm \sqrt{a^2 - \alpha_1\alpha_2}}{\alpha_2}}, \quad a > \sqrt{\alpha_1\alpha_2}, \tag{17i$_2$}$$

$$\nu = i\sqrt[4]{\frac{\alpha_1}{\alpha_2}}, \quad a = \sqrt{\alpha_1\alpha_2}. \tag{17i$_\mathrm{i}$}$$

In accordance with this, formulas (12) are also improved.

Theorem 1. *Under the assumption $\alpha_5 = \alpha_6 = 0$ of orthotropness, the matrices B and C are described as follows.*

If $(\alpha_3 + \alpha_4)(\sqrt{\alpha_1\alpha_2} - 2\alpha_3 - \alpha_4) \neq 0$, then

$$B = \begin{pmatrix} \alpha_3 + \alpha_2\nu_1^2 & \alpha_3 + \alpha_2\nu_2^2 \\ -(\alpha_3 + \alpha_4)\nu_1 & -(\alpha_3 + \alpha_4)\nu_2 \end{pmatrix}, \qquad C = \alpha_3 \begin{pmatrix} \alpha_4\nu_1 - \alpha_2\nu_1^3 & \alpha_4\nu_2 - \alpha_2\nu_2^3 \\ -\alpha_4 + \alpha_2\nu_1^2 & -\alpha_4 + \alpha_2\nu_2^2 \end{pmatrix}, \tag{18i$_1$}$$

where $\nu_{1,2}$ are defined by (17i$_1$) with an arbitrary choice of signs.

If $\alpha_3 + \alpha_4 = 0$, then

$$B = \begin{pmatrix} 0 & \alpha_3 + \alpha_2\nu_2^2 \\ \alpha_1 + \alpha_3\nu_1^2 & 0 \end{pmatrix}, \qquad C = \alpha_3 \begin{pmatrix} (\alpha_1\alpha_2 - \alpha_4^2)\nu_1^2 & \alpha_4\nu_2 - \alpha_2\nu_2^3 \\ -(\alpha_1\alpha_2 - \alpha_4^2)\nu_1 & -\alpha_4 + \alpha_2\nu_2^2 \end{pmatrix}, \tag{18i$_2$}$$

where $\nu_{1,2}$ are defined by (17i$_2$) and the choice of signs is determined by the condition $\alpha_2\nu_2^2 \neq \alpha_4$.

If $\sqrt{\alpha_1\alpha_2} = 2\alpha_3 + \alpha_4$, then

$$B = \begin{pmatrix} \alpha_3 + \alpha_2\nu^2 & 2\alpha_2\nu \\ -(\alpha_3 + \alpha_4)\nu & -(\alpha_3 + \alpha_4) \end{pmatrix}, \qquad C = \begin{pmatrix} \alpha_4\nu - \alpha_2\nu^3 & \alpha_4 - 3\alpha_2\nu^2 \\ -\alpha_4 + \alpha_2\nu^2 & 2\alpha_2\nu \end{pmatrix}, \tag{18i$_\mathrm{i}$}$$

where ν is defined by (17i$_\mathrm{i}$).

Note that the theorem covers all possible cases, since the double relation $\alpha_3 + \alpha_4 = \sqrt{\alpha_1\alpha_2} - 2\alpha_3 - \alpha_4 = 0$ is impossible. Indeed, it implies the relation $\sqrt{\alpha_1\alpha_2} = \alpha_4^2$, which contradicts the condition $\sqrt{\alpha_1\alpha_2} > \alpha_4^2$ stipulated by the positive definiteness of the matrix α.

Proof. First of all, we note that the condition $a = \sqrt{\alpha_1\alpha_2}$ in (17i$_\mathrm{i}$) is equivalent to $\sqrt{\alpha_1\alpha_2} - 2\alpha_3 - \alpha_4 = 0$. Indeed, according to (16), we can rewrite it in the form $(\sqrt{\alpha_1\alpha_2} + \alpha_4)(\sqrt{\alpha_1\alpha_2} - 2\alpha_3 - \alpha_4) = 0$, and it remains to note that the first factor is positive here.

Furthermore, let us show that the relation $h_3(\nu)=0$ is equivalent to $\alpha_3+\alpha_4=0$ and is possible only in the case (17i$_2$). Indeed, according to (15), it is impossible in case (17i$_1$). Therefore, this relation reduces to $a\pm\sqrt{a^2-\alpha_1\alpha_2}=-\alpha_4$. In particular, it should be $\alpha_4<0$ and $(a+\alpha_4)^2=a^2-\alpha_1\alpha_2$. With account for (16), the latter relation can be rewritten in the form $(\alpha_1\alpha_2-\alpha_3\alpha_4)(\alpha_3+\alpha_4)=0$.

Since $\alpha_4<0$, the first factor is different from zero, which proves the above assertion. □

Therefore, different cases in (12) exactly correspond to the corresponding cases in (18). Therefore, it remains to substitute relations (15) in (12) and use (17).

Note that various theoretic-functional approaches to orthotropic plane elasticity theory were developed by many authors [1–3].

Let us especially dwell on the case of isotropic medium. According to (14), this case corresponds to formulas (17i$_\text{i}$) and (18i$_\text{i}$) with multiple root $\nu=i$. In explicit form,

$$B=\begin{pmatrix}-(\lambda+\mu) & 2(\lambda+2\mu)i\\ -(\lambda+\mu)i & -(\lambda+\mu)\end{pmatrix}=(\lambda+\mu)\begin{pmatrix}-1 & (\varkappa+1)i\\ -i & -1\end{pmatrix},$$

$$C=\mu\begin{pmatrix}2(\lambda+\mu)i & 4\lambda+6\mu\\ -2(\lambda+\mu) & 2(\lambda+2\mu)i\end{pmatrix}=\mu(\lambda+\mu)\begin{pmatrix}2i & \varkappa+3\\ -2 & (\varkappa+1)i\end{pmatrix},$$

where we set $\varkappa=(\lambda+3\mu)/(\lambda+\mu)$.

Obviously, matrices of the form

$$M=\begin{pmatrix}x & y\\ 0 & x\end{pmatrix},\quad x,y\in\mathbb{C},$$

commute with the Jordan block J in (7), and, therefore, the replacement $\phi=M\tilde\phi$ does not derive from the solution class of the Douglis system (9). Hence setting $\tilde B=BM$ and $\tilde C=CM$, we can rewrite relations (8) with respect to $\tilde\phi$ and $\tilde B$, $\tilde C$. In other words, the matrices B and C are defined with accuracy up to postmultiplication by the matrix M.

Setting

$$M=\frac{1}{\lambda+\mu}\begin{pmatrix}-1 & -(\varkappa+1)i\\ 0 & -1\end{pmatrix},$$

in this case, we can replace the previous expression for B and C by

$$B=\begin{pmatrix}1 & 0\\ i & -\varkappa\end{pmatrix},\qquad C=\mu\begin{pmatrix}-2i & \varkappa-1\\ 2 & -(\varkappa+1)i\end{pmatrix}.$$

In the case (i), the substitution of (13) in (8) yields the representation of the general solution of the Lamé system through the pair of analytic functions ψ_1 and ψ_2 described in [4]. However, these functions are defined in distinct domains $D(\nu_k)$, $k=1,2$, which complicate the study of the boundary-value problems (3) and (6) by using analytic functions.

In the case (ii), the domain $D(\nu)$ of the functions ψ_k is the same, but, according to (13), in this case, u and v linearly depend on ψ and the derivative ψ', which also leads to an additional complication. This obstruction in the isotropic case is overcome by using special integral representations of analytic functions, which are suggested by D. I. Sherman, N. I. Muskhelishvili, and others [6]. For this reason [11], it is more convenient to develop a direct approach to studying problems (3) and (6) based on the application of the analytic function tools directly to solutions of the Douglis system (9).

Let the domain D be finite, and let it be bounded by a simple Lyapunov contour Γ. It is convenient to assume that the point $z=0$ belongs to D.

Obviously, we can write problem (3) in the equivalent form

$$u+\xi_1=f,\quad u(0)=0,$$

where $\xi_1\in\mathbb{R}^l$ must be found with respect to the pair (u,ξ), $\xi\in\mathbb{R}^2$.

Analogously, we can proceed for problem (6). In accordance with (8), we can rewrite these problems for solutions of system (9) in the form

$$\operatorname{Re}\; G\phi^{+} + \xi = f, \quad \phi(0) = 0, \tag{19}$$

with respect to the pair (ϕ, ξ), where $G = B$ or $G = C$ in accordance with the second problem and ϕ^{+} is the boundary value of ϕ.

We solve problem (9) in the class of vector-valued functions ϕ Hölder continuous in the closed domain D. According to [11], we can uniquely represent this function in the form

$$\phi(z) = \frac{1}{\pi i} \int_{\Gamma} (t - z)_J^{-1} dt_J \varphi(t) + i\xi_0, \quad \xi_0 \in \mathbb{R}^2, \tag{20}$$

where the vector-valued function $\varphi = (\varphi_1, \varphi_2)$ is real and Hölder continuous on Γ.

Here, for $x, y \in \mathbb{R}$, we use the matrix notation $(x + iy)_J = x \cdot 1 + yJ$, where 1 is the identity 2×2-matrix, and the notation $(dx + idy)_J$ for the matrix differential has an analogous meaning. In this case, the following analog of the Sokhotskii–Plejmel formula holds:

$$\phi^{+}(t_0) = \varphi(t_0) + \frac{1}{\pi i} \int_{\Gamma} (t - t_0)_J^{-1} dt_J \varphi(t) + i\xi_0. \tag{21}$$

Here, it is convenient to denote the singular integral with matrix kernel by $(S_J\varphi)(t_0)$. For $J = i$, it passes to the classical Cauchy integral $(S\varphi)(t_0)$. Substitution of (21) reduces (19) to the equivalent system of singular equations

$$\begin{gathered} \operatorname{Re} G(\varphi + S_J\varphi) - (JmG)\xi_0 + \xi = f, \\ \frac{1}{\pi i} \int_{\Gamma} t_J^{-1} dt_J \varphi(t) + i\xi_0 = 0. \end{gathered}$$

Let us write

$$\frac{1}{\pi} t_J^{-1} dt_J = b(t) ds_t, \tag{22}$$

with the 2×2-matrix function $b(t)$ $(dx + idy)_J$ Hölder continuous on Γ. Then we can rewrite the previous system with respect to the pair (ϕ, ξ):

$$\begin{gathered} \operatorname{Re} G(\varphi + S_J\varphi) - (Jm\; G)(\operatorname{Re} b, \varphi) + \xi = f, \\ (Jm\; b, \varphi) = 0. \end{gathered} \tag{23}$$

Here, (c, φ) is the integral of $c\varphi$ on Γ with respect to the arclength; in the case where the matrix function $c(t)$ is real, this integral belongs to $\mathbb{R}^2$.

Obviously,

$$2 \operatorname{Re} S_J\varphi = S_J\varphi - S_{\overline{J}}\varphi,$$

where we take into account that the vector-valued function φ is real. Therefore, system (23) is equivalent to the system

$$\begin{gathered} N\varphi + \xi = f, \\ (\varphi, g_j) = 0, \quad j = 1, 2, \end{gathered} \tag{24}$$

with respect to the pair (φ, ξ), where g_1 and g_2 are rows of the matrix b and

$$2N\varphi = G(\varphi + S_J\varphi) + \overline{G}(\varphi - S_{\overline{J}}\varphi) - 2(Jm\; G)(\operatorname{Re} b, \varphi).$$

Let us agree to write $N_1 \sim N_2$ if the difference is an integral operator of the form

$$[(N_1 - N_2)\varphi](t_0) = \int_{\Gamma} \frac{k(t_0, t)}{t - t_0} ds_t,$$

where the vector-valued function $k(t_0, t)$ is Hölder continuous on $\Gamma \times \Gamma$ and vanishes for $t = t_0$. By assumption, the contour Γ is Lyapunov. Therefore, a simple verification shows that $S_J \sim S$ and, analogously, $S_{\overline{J}} \sim S$. Hence

$$N \sim GP_+ + \bar{G}P_-, \qquad 2P_\pm = 1 \pm S.$$

Since the matrix G is constant and invertible, the singular operator $GP_+ + \bar{G}P_-$ is invertible, and the operator inverse to it is written by the explicit formula [7] by using the canonical function. As a result, (24) reduces to the system of Fredholm integral equations with the operator $M = (GP_+ + \bar{G}P_-)^{-1}N \sim 1$ in the principal part; the known approximate methods [8] can be applied to its numerical solution.

An analogous approache can also be realized for domains with piecewise-smooth boundary as is shown in [12] for the Dirichlet problem for weakly coupled elliptic systems; however, it is now based on the tools of nonclassical singular equations [13]. This approach can cover the case of multiconnected domains, finite or infinite, in particular the case where the domain D is the upper half-plane, and the solution of the problem (3), (6) is written in explicit form [14].

Precisely, let a function f satisfy the Hölder condition on the extended line $\overline{\mathbb{R}} = \mathbb{R} \cup \infty$ (i.e., $f(t)$ and $f(1/t)$ have this property on any finite closed interval of the line) and vanish at ∞. The solution u, v of problems (3) and (6) is sought for in an analogous class for the closed half-plane $\overline{D}$. Then according to [14],

$$u(z) = \operatorname{Re} \frac{1}{\pi} \int_{\mathbb{R}} B(t - x)_J^{-1} B^{-1} f(t)\, dt,$$

$$v(z) = \operatorname{Re} \frac{1}{\pi} \int_{\mathbb{R}} B(t - z)_{\overline{J}}^{-1} C^{-1} f(t)\, dt.$$

In the case of orthotropic medium, the substitution of formulas (18) in this relation allows us to obtain final solutions.

REFERENCES

1. H. Begehr and Lin Wei, “A mixed-contact problem in orthotropic elasticity,” In: *Partial Diif-ferential Equations with Real Analysis*, (H. Begerhand, A. Jeffrey, eds.), Longman Scientific & Technical (1992), pp. 219–239.
2. R. P. Gilbert, “Plane ellipticity and related problems,” *Amer. Math. Soc.* (1981).
3. R. P. Gilbert and Lin Wei, “Function theoretical solutions to problems of orthotropic elasticity,” *J. Elasticity*, **15**, 143–154 (1985).
4. G. G. Lekhnitskii, *Elasticity Theory of an Anisotropic Body* [in Russian], GITTL, Moscow–Leningrad (1950).
5. S. P. Mitin, “On representation of solutions of anisotropic elasticity theory,” *Differents. Uravn.*, **34**, No. 1, 94–100 (1998).
6. N. I. Muskhelishvili, *Some Basic Problems of Mathematical Elasticity Theory* [in Russian], Nauka, Moscow (1966).
7. N. I. Muskhelishvili, *Singular Integral Equations* [in Russian], Nauka, Moscow (1968).
8. V. Z. Parton and P. I. Perlin, *Integral Equations of Elasticity Theory* [in Russian], Nauka, Moscow (1977).
9. A. P. Soldatov, “Lamé system of plane anisotropic elasticity theory,” *Dokl. Ross. Akad. Nauk*, **385**, No. 2, 163–167 (2002).
10. A. P. Soldatov, “On the first and second boundary-value problems for elliptic systems on the plane,” *Differents. Uravn.*, **39**, No. 5, 674–676 (2003).
11. A. P. Soldatov, “Method of function theory in boundary-value problems on the plane. I. Smooth case,” *Izv. Akad. Nauk SSSR, Ser. Mat.*, **55**, No. 5, 1070–1100 (1991).

12. A. P. Soldatov, “On the index of the Dirichlet problem for elliptic systems on the plane,” *Differents. Uravn.*, **42**, No. 8, 1092–1105 (2006).
13. A. P. Soldatov, *One-Dimensional Singular Operators and Boundary-Value Problems of Function Theory* [in Russian], Vysshaya Shkola, Moscow (1991).
14. A. P. Soldatov, “Second-order elliptic systems on a half-plane,” *Izv. Ross. Acad. Nauk, Ser. Mat.*, **70**, No. 6 (2006).

E. A. Abapolova and A. P. Soldatov
Belgorod State University, Belgorod, Russia
E-mail: soldatov@bsu.edu.ru