ON WEAKLY SPECIAL RADICAL OF SEMIGROUPS

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Abstract. The coincidence of the special radical of semigroups and the Baer radical is established. The characterization of semisimple semigroups in the sense of the Baer radical is found.

In this paper, the concept of weakly separative semigroups is defined and it is proved that semisimple semigroups in the sense of the (left, right) Baer radical are (left, right) weakly separative semigroups. Hence, multiplicative semigroups of semiprime rings are semiprime semigroups.

Let $\mathcal{K}$ be a class of semigroups closed under homomorphisms and ideals. A radical on $\mathcal{K}$ is a function ρ such that for all S from $\mathcal{K}$ ρS is a congruence on S and the following properties hold:

(1) $[\rho S = 0,\ T \cong S] \Longrightarrow \rho T = 0$;
(2) $\rho(S/\theta) = 0 \Longrightarrow \rho S \subseteq \theta$;
(3) $\rho(S/\rho S) = 0$.

Classes of ρ-semisimple and ρ-radical semigroups are denoted as follows:

$$\mathcal{P}_\rho = \{S \in \mathcal{K} \mid \rho S = 0_S\}, \quad \mathcal{R}_\rho = \{S \in \mathcal{K} \mid \rho S = 1_S\}.$$

A class of semigroups $\mathcal{A}$ is called a type if $\mathcal{A}$ is closed under homomorphisms and contains a one-element semigroup [1, 11.6]. Let $\mathcal{B}$ be a subtype of $\mathcal{A}$. For every semigroup $S \in \mathcal{B}$ define $\mathcal{B}\text{-}\operatorname{rad} S$ as follows:

$$\mathcal{B}\text{-}\operatorname{rad} S = \bigcap\{\beta \mid \beta \in \mathcal{B}(S),\ S/\beta \in \mathcal{B}\}.$$

Semisimple semigroups under $\mathcal{B}\text{-}\operatorname{rad}$ are precisely those that are isomorphic to a subdirect product of semigroups from $\mathcal{B}$. Therefore, a class of semisimple semigroups uniquely defines a radical of semigroups. For example, the class of separative semigroups defines the compressive radical of semigroups.

A semigroup S is called weakly right regular (left regular) if

$$(\forall s \in S^1)\,(asx = asy) \Longrightarrow a = 0 \vee x = y \quad ((\forall s \in S^1)\,(xsa = ysa) \Longrightarrow a = 0 \vee x = y).$$

If a semigroup S is weakly right regular and weakly left regular, then S is called a prime semigroup. A congruence π on S is called a prime congruence if S/π is a prime semigroup.

A semigroup S is called a weakly left (right) separative semigroup if

$$(\forall s \in S)\,(asa = asb\ \&\ bsb = bsa) \Longrightarrow a = b \quad ((\forall s \in S)\,(asa = bsa\ \&\ bsb = asb) \Longrightarrow a = b).$$

A semigroup S is called a weakly separative semigroup if S is a left and right weakly separative semigroup.

It is well known that on semigroups, as a rule, left and right analogs of a radicals do not coincide. For example, the left Baer radical is equal to the intersection of all left regular congruences and the right Baer radical is equal to the intersection of all right regular congruences.

A semigroup S is called left (right) reductive if

$$(\forall s \in S)\,(sa = sb) \Longrightarrow a = b \quad ((\forall s \in S)\,(as = bs) \Longrightarrow a = b).$$

A semigroup S is called left (right) hereditarily reductive if any ideal of S is left (right) reductive. If a semigroup S is left and right hereditarily reductive, then S is called hereditarily reductive.

The class of left (right) hereditarily reductive semigroups $\mathcal{Q}_\mathrm{l}$ ($\mathcal{Q}_\mathrm{r}$) defines the left (right) Baer radical, and such semigroups are semisimple semigroups under corresponding radicals of semigroups. The class

$\mathcal{Q}_\mathrm{l} \cap \mathcal{Q}_\mathrm{r}$ defines the radical of semigroups l_0, which is the least special and weakly special radical of semigroups (see [3]).

Semisimple semigroups under l_0 are described by the following theorem.

Theorem 1. *The following conditions are equivalent*:

(1) *S is a weakly separative semigroup*;
(2) *S is a hereditarily reductive semigroup*;
(3) *S is a subdirect product of prime semigroups*;
(4) *S is a semisimple semigroup under l_0.*

Proof. The equivalence of (2), (3), and (4) is proved in [2, Proposition 8] and [3, Proposition 2].

(1) $\Longrightarrow$ (4). Suppose that S is not semisimple under l_0; then $S \notin \mathcal{Q}_\mathrm{l} \cap \mathcal{Q}_\mathrm{r}$. If $S \notin \mathcal{Q}_\mathrm{r}$, then $V_\mathrm{r} \neq 0_S$ and, therefore, there exists a pair of different elements $a, b \in S$ for which there exists a least natural number n such that for all $s_1, s_2, \dots, s_n$ from S we have

$$as_1a \cdots as_na = as_1a \cdots as_nb, \quad bs_1b \cdots bs_nb = bs_1b \cdots bs_na.$$

It is easy to see that for the elements

$$a_1 = as_1a \cdots as_{n-1}a \neq b_1 = as_1a \cdots as_{n-1}b$$

we have

$$a_1sa_1 = a_1sb_1, \quad b_1sb_1 = b_1sa_1$$

for all $s \in S$. Now $a = b$ follows from weak separability. The case $S \notin \mathcal{Q}_l$ can be proved similarly.

(3) $\Longrightarrow$ (1). Let S be a subdirect product of right regular semigroups S_α, $\alpha \in Y$, possibly with zeroes. Suppose that for $x, y \in S$ we have $xsx = xsy$ and $ysy = ysx$ for all $s \in S$. Then for any $\alpha \in Y$,

$$x_\alpha s_\alpha x_\alpha = x_\alpha s_\alpha y_\alpha, \quad y_\alpha s_\alpha y_\alpha = y_\alpha s_\alpha y_\alpha$$

for all $s_\alpha \in S_\alpha$. If S_α has no zero, then from weak right regularity it follows that $x_\alpha = y_\alpha$. If S_α has a zero 0, then it follows that $x_\alpha = 0$ if and only if $y_\alpha = 0$. In the case $x_\alpha \neq 0$ we have $y_\alpha \neq 0$, which yields $x = y$, i.e., S is a weakly right separative semigroup. One verifies similarly that S is a weakly left separative semigroup. □

Corollary 1. *The class of weakly separative semigroups is closed under ideals.*

This follows from the fact that the class of hereditarily reductive semigroups is closed under ideals [3, Corollary 6].

Corollary 2. *The radical l_0 is the least weakly separative congruence.*

The statement follows from (2) of the definition of a radical and Theorem 1.

Obviously, left and right analogs of Theorem 1 are true.

Now let us consider the congruences V_l and V_r from [2]: $(a, b) \in V_\mathrm{l}$ ($(a, b) \in V_\mathrm{r}$) if there exists some natural number k such that for all $s_1, s_2, \dots, s_k \in S$ we have

$$as_1a \cdots as_ka = bs_1a \cdots as_ka \quad (as_1a \cdots as_ka = as_1a \cdots as_kb)$$

and

$$bs_1b \cdots bs_kb = as_1b \cdots bs_kb \quad (bs_1b \cdots bs_kb = bs_1b \cdots bs_ka).$$

For each of these congruences one can construct the lower radical in the sense of Hoehnke [2]. For the V_l-radical β_l (V_r-radical β_r) the semisimple class coincides with the class $\mathcal{Q}_\mathrm{l}$ ($\mathcal{Q}_\mathrm{r}$). The radical l_0 is determined by the class $\mathcal{Q}_\mathrm{l} \cap \mathcal{Q}_\mathrm{r}$, whence $l_0 = \beta_\mathrm{l} \vee \beta_\mathrm{r}$. This means that the radical l_0 is the lower $(V_\mathrm{l} \vee V_\mathrm{r})$-radical, whence we have the next theorem.

Theorem 2. *The lower $(V_\mathrm{l} \vee V_\mathrm{r})$-radical coincides with the least (weak) special radical of semigroups.*

From this theorem it follows that the radical l_0 may be called the prime radical or the Baer radical of semigroups.

REFERENCES

1. A. Clifford and G. Preston, *Algebraic Theory of Semigroups*, Vols. 1, 2, Amer. Math. Soc., Providence (1961).
2. H.-J. Hoehnke, "Über das untere und obere Radikal einer Halbgruppe," *Math. Z.*, **89**, 300–311 (1965).
3. A. V. Tishchenko, "Weak special and special radicals in semigroups," *Mat. Sb.*, **94 (136)**, No. 4 (8), 551–566 (1974).

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