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LINEAR CONJUGATION PROBLEMS

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Abstract. We investigate the linear conjugation problem for polyanalytic functions using function theory and Cauchy-type integrals. We explicitly construct a canonical matrix-function by using the recurrence procedure and use it to study the linear conjugation problem. We found a solutions of the linear conjugation problem and given a formula for its index by using Cauchy type integrals. We got a representation of the solution of the linear conjugation problem through the canonical matrix-function, which is constructed explicitly.

Key words: Linear Conjugation Problems, the Goursat Formula, Cauchy Singular Integral, Functions of Canonic Matrices, Singular Integral Equations.

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ЗАДАЧА ЛИНЕЙНОГО СОПРЯЖЕНИЯ

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Аннотация. Опираясь на теорию функций и интегралы тина Коши в работе рассматривается задача линейного сопряжения для полианалптпческих функций. Применяя процедуру рекуррентности, строится каноническая матричная функция, которая используется для изучения задачи линейного сопряжения. Мы нашли решение задачи о линейном сопряжении п дали формулу для ее индекса с номощью интегралов типа Коши. Получено представление решения задачи линейного сопряжения через каноническую матрицу-функцию, которая построена явно.

Ключевые слова: Задачи линейного сопряжения, формула Гурса, сингулярный интеграл Коши, канонические матрицы-функции, сингулярные интегральные уравнения.

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1. The Goursat Formula. Let D be a subset of $\mathbb{C}$, and u be a C^n function on D, $u(z) = u(x, y)$ in a complex variable $z = x + iy$. This function is called poly-analytic if it is a solution of the equation

$$\frac{\partial^n u}{\partial \bar{z}^n} = 0, \tag{1.1}$$

where

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

To emphasize on the dependency of n, these functions are also called n analytic (bi-analytic when $n = 2$). It is clear that when $n = 1$ the equation (1.1) is the Cauchy-Riemann condition and its solutions are analytic functions.

It is well known that any $n-$ analytic function u is represented in the form

$$u(z) = \phi_1(z) + \bar{z}\phi_2(z) + \frac{\bar{z}^2}{2!}\phi_3(z) + \ldots + \frac{\bar{z}^{n-1}}{(n-1)!}\phi_n(z), \tag{1.2}$$

where $\phi_j(z)$ are analytic functions on D. When n=2 the formula takes the name Goursat, that we conserve also in the general case for any n. In particular, from this formula, it follows that the analytic functions are infinitely differentiable in D.

In (1.2) it is easy to put an induction on n, if it is used on the relation

$$\frac{\partial}{\partial \bar{z}}[\bar{z}^k\phi(z)] = k\bar{z}^{k-1}\phi(z), \tag{1.3}$$

for any natural k and analytical function ϕ. In fact, the Goursat formula is true for $(n-1)$ analytic functions and function $u \in C^n(D)$ satisfying the equation (1.1). So

$$\frac{\partial^{n-1}u}{\partial \bar{z}^{n-1}} = \phi_n(z),$$

where $\phi_n(z)$ is an analytic function, and from (1.3)

$$\frac{\partial^{n-1}}{\partial \bar{z}^{n-1}}\left[u(z) - \frac{\bar{z}^{n-1}}{(n-1)!}\phi_n(z)\right] = 0.$$

According to the induction, hence the validity of the formula (1.2) holds for all n.

From (1.3) and (1.2), we have

$$\frac{\partial^{j-1}u}{\partial \bar{z}^{j-1}} = \phi_j(z) + \bar{z}\phi_{j+1}(z) + \ldots + \frac{\bar{z}^{n-j}}{(n-j)!}\phi_n(z), 1 \le j \le n. \tag{1.4}$$

Put

$$\begin{aligned} &U = (U_1, \ldots, U_n), \quad U_j = \partial^{j-1}u/\partial\bar{z}^{j-1}, \\ &\phi = (\phi_1, \ldots, \phi_n), \qquad P = (P_{ij})_1^n, \end{aligned} \tag{1.5}$$

where $P(z)$ is a upper triangle matrix determined by

$$P_{ij}(z) = \frac{\bar{z}^{j-i}}{(j-i)!}, \quad j \ge i.$$

So the relationship (1.4) can be written in the matrix form

$$U = P\phi. \tag{1.6}$$

It is easy to check that the determinant of P is equal to 1. Therefore, relation (1.6) can be transformed to $\phi = P^{-1}U$. In other words, in Goursat formula (1.2), the set of analytic functions ϕ_j in the same way it is determined by n-analytic function u.

For the upper triangle elements of the inverse matrix P^{-1}, we have the following expression

$$(P^{-1})_{ij}(z) = \frac{(-1)^{j-i}\bar{z}^{j-i}}{(j-i)!}, \quad j \ge i. \tag{1.7}$$

In fact, let Δ be the matrix with elements

$$\Delta_{ij} = \begin{cases} 1, & j-i=1, \\ 0, & j-i \ne 1. \end{cases}$$

Then, we have the identical expression

$$(\Delta^k)_{ij} = \begin{cases} 1, & j-i=k, \\ 0, & j-i \ne k, \end{cases} \qquad 0 \le k \le n-1,$$

clearly, $\Delta^n = 0$. From this notation, we can write

$$P(z) = \sum_{k=0}^{n-1}\frac{\bar{z}^k}{k!}\Delta^k.$$

So, this sum coincides with the series in all $k \geq 0$, $P(z) = \exp(\bar{z}\Delta)$. Therefore,

$$P^{-1}(z) = \exp(-\bar{z}\Delta) = \sum_{k=0}^{n-1} \frac{\bar{z}^k}{k!}(-1)^k \Delta^k,$$

this coincides with (1.7).

Let D be a neighborhood domain of the infinitely distant point ∞, this means, it contains the exterior of $\{|z| \geq R\}$. Suppose, in the notation (1.5), the poly-analytic function $u(z)$, with $|z| \geq R$, satisfies the following inequalities

$$|U_j(z)| \leq C|z|^{l-j}, \quad j = 1, \dots, n, \tag{1.8}$$

with some integer l or, equivalently, $U_j(z) = O(|z|^{l-j})$ when $z \to \infty$.

Due to (1.6), (1.7), we have the following expressions for the components ϕ_k of ϕ

$$\phi_k(z) = \sum_{j=k}^{n} \frac{(-\bar{z})^{j-k}}{(j-k)!} U_j(z).$$

Therefore, the similar inequalities (1.8) are also valid for these components. We also have

$$\phi_j(z) = O(|z|^{l-j}) \text{ when } z \to \infty, \quad j = 1, \dots, n, \tag{1.9}$$

implies (1.8) with some other constant C.

2. Linear Conjugation Problems. Let Γ be a smooth oriented contour on the complex plan, which is composed of simple contours $\Gamma_1, \dots, \Gamma_m$. Therefore, the complement is the open set $D = \mathbb{C} \backslash \Gamma$ composed of connected components $D_0, D_1, \dots, D_m$, where D_0 is unbounded and contains a neighborhood of ∞, the others are bounded. There is no lost of generality, we can assume that

$$\partial D_0 = \Gamma_1 \cup \ldots \cup \Gamma_{m_0}, \quad 1 \leq m_0 \leq m. \tag{2.1}$$

Let's designate $C(\widehat{D})$ denote the class $\varphi \in C(D)$, that in every domain D_j is continuously extensible to the boundary. Obviously, we can define the unilateral boundary values of $\varphi(t)$ by $\varphi^{\pm}(t) = \lim \varphi(z)$ at points $tin\Gamma$, when the point $z \to t$ belongs to the left (right) of Γ with a superior signal (inferior). It is clear that this function is continuous.

Together with this class, we also consider the Hölder class. Let $C^{\mu}(G)$ be the class of functions φ satisfying Hölder condition in domain G, i.e.

$$|\varphi(z_1) - \varphi(z_2)| \leq C|z_1 - z_2|^{\mu}, \quad z_j \in G,$$

with some exponent $0 < \mu \leq 1$. It is clear that the conditions $\varphi \in C^{\mu}(G)$ and $\varphi \in C^{\mu}(\overline{G})$ are equivalent. In this notations, $\varphi \in C^{\mu}(\widehat{D})$ by definition, means that $\varphi \in C^{\mu}(D_0)$ for each bounded sub-domain $D_0 \subseteq D$. Therefore, $\varphi \in C^{\mu}(\overline{D}_j)$, $1 \leq j \leq m$, and $\varphi \in C^{\mu}(\overline{D}_0 \cap \{|z| \leq R\})$, for any $R > 0$.

Given $n \times n$ matrix function $B(t) = (B_{ij}(t))_1^n$ on the contour Γ of the class C^{μ}, whose determinant is different from zero. Consider poly-analytic function u satisfying

$$\begin{gathered} U_j = \frac{\partial^{j-1} u}{\partial \bar{z}^{j-1}} \in C^{\mu}(\widehat{D}), \quad 1 \leq j \leq n, \\ U_j(z) = O(|z|^{l-j}) \text{ when } z \to \infty, \quad j = 1, \dots, n, \end{gathered} \tag{2.2}$$

Consider the linear conjugation problem:

$$\left(\frac{\partial^{i-1} u}{\partial \bar{z}^{i-1}}\right)^{+} - \sum_{j=1}^{n} B_{ij} \left(\frac{\partial^{j-1} u}{\partial \bar{z}^{j-1}}\right)^{-} = f_i, \quad 1 \leq i \leq n. \tag{2.3}$$

With the substitution $U = P\phi$, this problem is the linear conjugation problem

$$\phi^{+} - G\phi^{-} = g, \tag{2.4}$$

For a analytic vector function $\phi \in C^{\mu}(\widehat{D})$ with the matrix coefficient $G = P^{-1}BP$ and the right side $g = P^{-1}f$. Due to (2.2), we have

$$\deg \phi_j \leq l - j, \text{ when } z \to \infty, \quad j = 1, \dots, n. \tag{2.5}$$

With the help of the Cauchy type integral

$$(I\varphi)(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\varphi(t)dt}{t-z}, \tag{2.6}$$

this problem, by the usual manner, may be reduced to a equivalent system of the singular integral equation (see, for example: «Singular integral equations», [N. I. Muskhelishvili, 1946]).

Theorem 2.1. *If $\varphi \in C^\mu(\Gamma)$, then analytic function $\phi = I\varphi$ disappears on the unbounded domain and belongs to the class $C^\mu(\widehat{D})$, and its contour values satisfies the Sokhoski-Plemelj formulas*

$$2\phi^\pm = \pm\varphi + S\varphi, \tag{2.7}$$

with

$$(S\varphi)(t_0) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(t)dt}{t-t_0}, \quad t_0 \in \Gamma \tag{2.8}$$

Cauchy singular integral. Where, $I\varphi$ as a linear operator, is limited by $C^\mu(\Gamma) \to C^\mu(\widehat{D})$.

The inverse is also true: any analytic function $\phi \in C^\mu(\widehat{D})$ that satisfies the condition $\deg\phi \le æ-1$ in unbounded domain with some integer number æ, is inclusively represent-able as $\phi = I\varphi + p$ with density $\varphi \in C^\mu(\Gamma)$ and polynomial $p(z)$, subjected to the conditions

$$\deg p \le æ - 1, \quad \int_\Gamma \varphi(t)q(t)dt = 0, \ \deg q \le -æ - 1,$$

where the last condition of orthogonality is understood in the relation to the polynomials $q(z)$. Where, the polynomials of negative degree are assumed as equal to zero.

The last affirmation of the theorem occurs in the fact that, in the neighborhood of ∞, the function $I\varphi$ possesses the decomposition in Laurent series:

$$(I\varphi)(z) = \sum_{k=0}^{\infty} c_k z^{-k-1}, \quad c_k = -\frac{1}{2\pi i}\int_\Gamma \varphi(t)t^k dt. \tag{2.9}$$

In particularly, for an integer number $æ \le -1$, the condition deg $I\varphi \le æ - 1$ can be expressed in zero equality form

$$\int_\Gamma \varphi(t)q(t)dt = 0,$$

for polynomial q has deg q $\le -æ - 1$.

In particularly, from the theorem, it follows that the singular operator $S\varphi$ is limited on the space $C^\mu(\Gamma)$.

3. Functions of canonic matrices. Suppose that the matrix-function $G \in C^\mu(\Gamma)$ is invertible. By definition, an analytic matrix function $X(z)$ out of Γ is called canonic in relation to G if it belongs to the class $C^\mu(\widehat{D})$, has finite order in the unbounded domain, satisfies the relation

$$X^+ = GX^-, \tag{3.1}$$

and the condition

$$A = \lim_{z\to\infty} X(z)\mathrm{diag}(z^{æ_1}, \ldots, z^{æ_n}), \quad \det A \ne 0, \tag{3.2}$$

in the unbounded domain with some integer number $æ_j$.

By the theory of singular equations, there exists a matrix G such that the integer numbers $æ_1, \ldots, æ_n$ uniquely determined by permutation, and called partial index of G, and

$$æ_1 + \ldots + æ_n = \mathrm{Ind}\, G, \quad \mathrm{Ind}\, G = \frac{1}{2\pi i}\ln\det G(t)\big|_\Gamma. \tag{3.3}$$

For the case $n = 1$, the condition (3.2) and the equality (3.3) are

$$A = \lim_{z\to\infty} z^{æ}X(z) \ne 0, \quad æ = \mathrm{Ind}\, G. \tag{3.4}$$

In this case, the canonic function is built directly. Suppose that $m = 1$, i.e the Γ contour is simple, the conjunction D_0 (D_1) stays inside (outside) of this contour and the point $z_0 \in D_0$ is fixed. Consider $G_0(t) = (t - z_0)^{\pm æ}$, $t \in \Gamma$, where the superior (inferior) is selected if contour Γ is oriented counter-clockwise. Obviously, the Cauchy index of G and G_0 coincide and the function

$$X_0(z) = \begin{cases} 1, & z \in D_0, \\ (z - z_0)^{-æ}, & z \in D_1, \end{cases}$$

is G_0 canonic, or satisfies the conditions (3.1), (3.4) in relation to G_0.

Observe that the Cauchy index of $G_1 = G_0^{-1}G$ is equal to zero, therefore $\ln G_1 \in C^{\mu}(\Gamma)$. Consider one integral Cauchy type $Y = I(\ln G_1)$, this function belongs to a $C^{\mu}(\widehat{D})$, disappears in the infinite, and according to (2.7), satisfies the condition $Y^+ - Y^- = \ln G_1$. Therefore, $X = e^Y X_0$, and G are canonical. The general case when $\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_m$, G_j is the restriction of G on Γ_j and X_j is the G_j canonic function. So the product $X = X_1 \cdots X_m$ is a G canonic function.

From this, the canonic matrix function $X(z)$ corresponding to G, the solution of the problem (2.4), where

$$\deg \phi \le l - 1, \tag{3.5}$$

can be constructed explicitly.

In fact, due to (3.1), vector function $\psi = (\psi_1, \ldots, \psi_n) = X^{-1}\phi$ satisfies the condition of contour $\psi^+ - \psi^- = (X^+)^{-1}g$ due to (3.2), condition (3.5) becomes to

$$\deg \psi_j \le l + æ_j - 1, \quad 1 \le j \le n.$$

As consequence, theorem 2.1 can be applied to ψ, and we have

$$\psi(z) = \int_{\Gamma} \frac{(X^+)^{-1}(t)g(t)dt}{t - z} + p(z).$$

As a observed above, this function in the neighborhood of ∞ possesses the decomposition in Laurent series of the form (2.9) with coefficient

$$a_j = -\frac{1}{2\pi i}\int_{\Gamma} [(X^+)^{-1}g](t)t^{-j-1}dt, \quad j \le -1,$$

and $a_0 + \ldots + a_s z^s = p(z)$. That why the condition $\deg \psi_k \le l + æ_k - 1$ reduces in the fact that $\deg p_k \le l + æ_k - 1$ and

$$\int_{\Gamma} [(X^+)^{-1}g]_k(t)q_k(t)dt = 0, \quad 1 \le k \le n,$$

where the polynomials q_k has $\deg q_k \le -(l + æ_k) - 1$. Obviously, this conditions guarantees that the order in the unbounded domain of vector function $\operatorname{diag}(z^{-æ_1}, \ldots, z^{-æ_n})\psi(z)$ doesn't exceed $l - 1$.

In this way, all solutions of the original problem (2.4), (3.5) are described by the formula

$$\phi = X(I\widetilde{g} + p), \quad \widetilde{g} = (X^+)^{-1}g,$$

where polynomial vector $p = (p_1, \ldots, p_n)$ has $\deg p_k \le l + æ_k - 1$, and the density $\widetilde{g}$ satisfies the conditions of orthogonality

$$\int_{\Gamma} \widetilde{g}_k(t)q_k(t)dt = 0, \quad 1 \le k \le n,$$

where the polynomial q_k has $\deg q_k \le -(l + æ_k) - 1$.

In particularly, the index $æ = \operatorname{Ind} G + nl$.

In the case of the problems (2.4), (2.5), the order at infinity of function ϕ_j has to be aligned and is reduced to the form (3.5), this can be made with the help of the diagonal matrix function

$$Q(z) = \begin{cases} 1, & z \in D \setminus D_0, \\ \operatorname{diag}(1, (z - z_0)^{-1}, \ldots, (z - z_0)^{1-n}), & z \in D_0, \end{cases} \tag{3.6}$$

where $z_0 \in D \setminus D_0$ is fixed.

Remember that, in accordance with (2.1) the boundary of the unbounded domain D_0 is composed of components Γ_j, $1 \le j \le m_0$, of contour Γ. The problem $\phi = Q\widetilde{\phi}$ (2.4) is replaced to the linear conjugation problem

$$\widetilde{\phi}^+ - \widetilde{G}\widetilde{\phi}^- = \widetilde{g}, \tag{3.7}$$

where $\widetilde{G} = (Q^+)^{-1}GQ^-$, and the right side $\widetilde{g} = (Q^+)^{-1}g$. The condition (2.5) at infinity become to (3.5). Due to (1.2), we have the following result.

Theorem 3.1. *Let $\widetilde{X}(z)$ be a function of canonic matrix corresponding to coefficient matrix $\widetilde{G} = (Q^+)^{-1}P^{-1}BPQ^-$, and $\widetilde{æ}_j$, $1 \le j \le n$, be their partial index.*

To solve the problem (2.2), (2.3) it is necessary and sufficient that condition $\widetilde{f} = (\widetilde{X}^+)^{-1}(Q^+)^{-1}P^{-1}f$ satisfies the following orthogonality

$$\int_{\Gamma} \widetilde{f}_k(t)\widetilde{q}_k(t)dt = 0, \quad 1 \le k \le n, \tag{3.8}$$

where the polynomials $\widetilde{q}_k$ *has* $\deg \widetilde{q}_k \leq -(l + \widetilde{æ}_k) - 1$ *(polynomials with negative degree are assumed as zero).*

Under these conditions, the general solution of this problem is given by the formula

$$u(z) = \sum_1^n \phi_j(z) \frac{\bar{z}^{j-1}}{(j-1)!},$$

$$\phi(z) = Q(z)\widetilde{X}(z)\left[\frac{1}{2\pi i}\int_\Gamma \frac{\widetilde{f}(t)dt}{t-z} + \widetilde{p}(z)\right], \tag{3.9}$$

where the polynomial vector $\widetilde{p} = (\widetilde{p}_1, \ldots, \widetilde{p}_n)$ *satisfies the condition* $\deg \widetilde{p}_k \leq l + \widetilde{æ}_k - 1,\ 1 \leq k \leq n.$

From the theorem, the space of the solution of homogeneous system has the same dimension with the class of the polynomial vector $\widetilde{p} = (\widetilde{p}_1, \ldots, \widetilde{p}_n)$ where $\deg \widetilde{p}_k \leq l + \widetilde{æ}_k - 1$. So this dimension is equal to

$$s^+ = (l + \widetilde{æ}_1)^+ + \ldots + (l + \widetilde{æ}_n)^+.$$

In the same manner, the number of conditions that is solved linearly independently is equal to

$$s^- = (-l - \widetilde{æ}_1)^- + \ldots + (-l - \widetilde{æ}_n)^-,$$

where for an integer s put $s^\pm = (|s| \pm s)/2$. In particular, the index $s^+ - s^-$ of the problem is equal.

$$s^+ - s^- = nl + \operatorname{Ind}\widetilde{G}. \tag{3.10}$$

We know that

$$\det \widetilde{G} = \frac{\det Q^-(t)}{\det Q^+(t)} \det B.$$

First suppose that all contours Γ_j in (2.1) are oriented negatively with respect to the D_0, i.e counterclockwise. Then $Q^\pm(t) = 1,\ t \in \Gamma \setminus \partial D_0$ and

$$\det Q^+(t) = 1,$$

$$\det Q^-(t) = (t - z_0)^{-n(n-1)/2}, t \in \Gamma_j, 1 \leq j \leq m_0, \tag{3.11}$$

therefore

$$\sum_{j=1}^m \frac{1}{2\pi i} \ln \left.\frac{\det Q^-(t)}{\det Q^+(t)}\right|_{\Gamma_j} = \frac{-n(n-1)}{2}, \tag{3.12}$$

from (3.10) the index $s^+ - s^-$ of the problem is equal

$$s^+ - s^- = \operatorname{Ind}B + nl - \frac{n(n-1)}{2}.$$

For arbitrary n, let B in (2.3) be an upper triangular matrix, i.e $B_{ij} = 0$ when $i > j$. We have matrix P in (1.5) is upper triangular. Therefore, this property possessed also the matrix $\widetilde{G} = (Q^+)^{-1}P^{-1}BPQ^-$. Then, the canonic matrix - function G can be explicitly constructed from a recursive procedure.

Theorem 3.2. *Let* $G \in C^\mu(\Gamma)$ *be a upper triangular matrix, i.e* $G_{ij} = 0$ *to* $i > j$*, and* $G_{ii}(t) \neq 0$*,* $t \in \Gamma,\ 1 \leq i \leq n$*. Then the canonic matrix* X *is also upper triangular and its partial index* $æ_i = \operatorname{Ind}G_{ii}$*.*

Prove. First, suppose that all diagonal elements of G_{ii} equal 1. Write matrix X in the form $X = 1 + Y$, where $Y(z)$ disappears in ∞ and its element $Y_{ij} = 0$ to $i \geq j$. So (3.1) turns into $Y^+ = GY^- + G - 1$ or $Y^+ - Y^- = (G-1) + (G-1)Y^-$. Write this relation coordinately

$$Y_{ij}^+ - Y_{ij}^- = G_{ij} + \sum\nolimits_{i<l<j} G_{il}Y_{lj}^-, \quad i < j. \tag{3.13}$$

From this, we have the following equalities

$$Y_{n-1,n}^+ - Y_{n-1,n}^- = G_{n-1,n}, \tag{3.14a}$$

$$\begin{gathered} Y_{n-2,n-1}^+ - Y_{n-2,n-1}^- = G_{n-2,n-1}, \\ Y_{n-2,n}^+ - Y_{n-2,n}^- = G_{n-2,n} + G_{n-2,n-1}Y_{n-1,n}^-, \end{gathered} \tag{3.14b}$$

$$\begin{gathered} Y_{n-3,n-2}^+ - Y_{n-3,n-2}^- = G_{n-3,n-2}, \\ Y_{n-3,n-1}^+ - Y_{n-3,n-1}^- = G_{n-3,n-1} + G_{n-3,n-2}Y_{n-2,n-1}^-, \\ Y_{n-3,n}^+ - Y_{n-3,n}^- = G_{n-3,n} + G_{n-3,n-2}Y_{n-2,n}^- + G_{n-3,n-1}Y_{n-1,n}^-, \end{gathered} \tag{3.14c}$$

and so on.

Therefore, using theorem 2.1, we have

$$Y_{n-1,n} = IG_{n-1,n}, \tag{3.15a}$$

$$\begin{gathered} Y_{n-2,n-1} = IG_{n-2,n-1}, \\ Y_{n-2,n} = I(G_{n-2,n} + G_{n-2,n-1}Y^{-}_{n-1,n}), \end{gathered} \tag{3.15b}$$

$$\begin{gathered} Y_{n-3,n-2} = IG_{n-3,n-2}, \\ Y_{n-3,n-1} = I(G_{n-3,n-1} + G_{n-3,n-2}Y^{-}_{n-2,n-1}), \\ Y_{n-3,n} = I(G_{n-3,n} + G_{n-3,n-2}Y^{-}_{n-2,n} + G_{n-3,n-1}Y^{-}_{n-1,n}), \end{gathered} \tag{3.15c}$$

and so on. As a consequence, Y is completely determined and $X = 1 + Y$ is canonic with $æ_j = 0$ in relation to a triangular matrix G with diagonal elements $G_{ii} = 1,\ 1 \le i \le n$.

For the general case, where a triangular matrix G with arbitrary diagonal elements, the problem is reduced to the case considered above by presentation G in the product form

$$G = G_{(1)}G_{(2)}, \quad G_{(1)} = \operatorname{diag}(G_{11}, \dots, G_{nn}), \tag{3.16}$$

where the diagonal elements of the triangular matrix $G_{(2)}$ are equal to 1. Let $X_{(1)i}$ be a canonic function corresponding to the coefficient G_{ii}.

In other words, by (3.1), (3.2), $X^{+}_{(1)i} = G_{ii}X^{-}_{(1)i}$ and $X_{(1)i}(z)z^{æ_i} \to 1$ as $z \to \infty$, where $æ_i = \operatorname{Ind} G_{ii}$. So

$$X_{(1)} = \operatorname{diag}(X_{(1)1}, \dots, X_{(1)n}) \tag{3.17}$$

$G_{(1)}$ is canonic, this is, it satisfies (3.1), (3.2) in relation to the correlation $G_{(1)}$. Let us consider the triangular matrix in Γ

$$\widetilde{G}_{(2)} = (X^{-}_{(1)})^{-1}G_{(2)}X^{-}_{(1)}, \tag{3.18}$$

where the diagonal elements of $\widetilde{G}_{(2)}$ are equal to 1.

So, by what it has been proven above, there exists

$$\lim_{z\to\infty} X_{(2)}(z) = 1.$$

We will affirm that the canonic matrix X to the original coefficient G is an $X = X_{(1)}X_{(2)}$. In fact, the equality

$$X^{+}_{(1)}X^{+}_{(2)} = G_{(1)}G_{(2)}X^{-}_{(1)}X^{-}_{(2)}$$

having in mind the equality $X^{+}_{(1)} = G_{(1)}X^{-}_{(1)}$ passes to $X^{+}_{(2)} = \widetilde{G}_{(2)}X^{-}_{(2)}$.

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