

FINAL DISTRIBUTION DENSITY OF MATERIAL FRAGMENT SIZES AT SLOW FRAGMENTATION PROCESS

*R. Ye. Brodskii*¹ and *Yu.P. Virchenko*^{2*}

¹*Single Crystal Institute, 61000, Kharkov, Ukraine*

²*Belgorod State University, 308015, Belgorod, Russia*

The process of slow material fragmentation is studied when the diffusion approximation is applicable. Final distribution density of fragment sizes is calculated in the case of scale-homogeneity of subdivision mechanism.

1. INTRODUCTION

A lot of works are dedicated to theoretical researching of material fragmentation process from the statistical physics viewpoint (see, for example, [1–3]). Importance of such investigations is stipulated by its practical necessity. The complexity of the study of such physical process on the basis of statistical physics is connected with serious difficulties if microscopic representations are used. In such conditions, main investigation approach to study fragmentation dynamics consists in the construction of the most general probabilistic fragmentation models which are phenomenologically reasonable. After that it is necessary to study the most general properties of such models.

2. PROBLEM FORMULATION

Let $N(r, t)$ is the number of fragments which have sizes being not larger than r at the time moment t and $g(r, t) = \partial N(r, t)/\partial r$ is the corresponding distribution density. We use further the following kinetic equation that describes the temporal evolution of the density

$$\dot{g}(r, t) = \int_r^\infty K(r, r'; t)g(r', t)dr' - \mu(r, t)g(r, t). \quad (1)$$

It has been formulated in [3]. Here $K(r, r'; t)$ is a non-negative function. From the physical point of view, it is the average number of fragments which are formed during the small time interval $(t, t + dt)$, $\mu(r, t)$ is the average number of fragments having the size r which are disintegrated during the same temporal interval. It is supposed that the conservation of total volume of all fragments takes place that is formulated in the form $\int_0^\infty g(r, t)r^3 dr = \text{const}$ [2]. It is assumed that

the energy expended on the breaking of intermolecular bonds during the time interval dt , which is proportional to the value $\int_0^\infty g(r, t)r^2 dr$ of total surface area of all fragments [2], is also constant. At the slowness condition of fragmentation process, the diffusion equation is applicable [3]. Using it and also using conditions of volume conservation and the constancy of the expending energy intensity, the equation (1) is transformed to the following [3]

$$\dot{g}(r, t) = \gamma(r, t)g(r, t) + \frac{2}{3} \frac{\partial}{\partial r} [r\gamma(r, t)g(r, t)] + \frac{1}{6} \frac{\partial^2}{\partial r^2} [r^2\gamma(r, t)g(r, t)],$$

where $\gamma(r, s) > 0$ is the intensity of the fragment formation.

At the assumption that the fragment subdivision is steady-state stable at all scales after sufficiently long evolution when $\gamma(r, t) \sim c(r)\gamma(t)$, we have

$$\frac{\partial g}{\partial s} = c(r)g(r, s) + \frac{2}{3} \frac{\partial}{\partial r} [rc(r)g(r, s)] + \frac{1}{6} \frac{\partial^2}{\partial r^2} [r^2c(r)g(r, s)], \quad (2)$$

where we introduce the effective time scale with elementary interval $ds = \gamma(t)dt$. Taking into account that $g(r, s)$ is concentrated near the point $r = 0$, $g(r, s) \sim N(s)\delta(r)$ ($N(s)$ is the total number of fragments) when the time evolution is sufficiently large, it is important to investigate the function $g(r, s)$ by such a way that, firstly, to find its structure near this point. In this situation there are two qualitatively different cases connected with the behavior of the function $c(r)$: the scale invariant (Kolmogorov) case when $c(0) > 0$ and the scale homogeneous case when $c(r) \propto r^\beta$, $\beta > 0$. In this work we calculate the final distribution density in the last case. In such

*Corresponding author E-mail address: virch@bsu.edu.ru

a situation, the equation (2) is transformed to the following

$$\frac{\partial g}{\partial s} = r^\beta g(r, s) + \frac{2}{3} \frac{\partial}{\partial r} [r^{\beta+1} g(r, s)] + \frac{1}{6} \frac{\partial^2}{\partial r^2} [r^{2+\beta} g(r, s)], \quad (3)$$

where the transformation to dimensionless variable $r \Rightarrow r/r^*$, $r^* = c_0^{-1/\beta}$ is done in the density $g(r, s)$.

It is necessary to solve the equation (3) on the semi-axis $(0, \infty)$ taking into account the boundary conditions $rg(r, s) \rightarrow 0$ at $r \rightarrow 0$ and $r^4 g(r, s) \rightarrow 0$ at $r \rightarrow \infty$ which follow from the integrability of density $g(r, s)$ near the zero point and the finiteness of integral $\int_0^\infty r^3 g(r, s) dr < \infty$.

3. FINAL DISTRIBUTION DENSITY

For solving the equation (3), we make the following transformations. At first, we introduce a new function $h(r, s) = r^{\beta+3} g(r, s)$ and after that we pass to a new independent variable $x = r^{-\beta/2}$. In a result, the equation (3) in terms of new values takes the form

$$\frac{\partial h}{\partial s} = \frac{\beta^2}{24} \frac{\partial^2 h}{\partial x^2} + \frac{\beta}{6x} [(1 + \beta/2)/2 - 1] \frac{\partial h}{\partial x}. \quad (4)$$

Here, we essentially simplify in comparison with the work [3] the procedure of the solution building and the calculations connected with it (in particular, it permits to remove the inaccuracy that takes place in the cited work). The initial boundary problem on the axis $x \in (0, \infty)$ for the equation obtained is necessary to solve at the conditions $x^{\beta+2} h(x, s) \rightarrow 0$ at $x \rightarrow 0$ and $x^{\beta-1} h(x, s) \rightarrow 0$ at $x \rightarrow \infty$. Further, in this work we analyze only the case $\beta = 2$, when the equation (4) has the most simple form

$$\frac{\partial h}{\partial s} = \frac{1}{6} \frac{\partial^2 h}{\partial x^2}. \quad (5)$$

The initial boundary problem on the semi-axis is solved on the basis of the Laplace transformation on the time s . Its solution can be represented by the formula

$$h(x, s) = \int_0^\infty G(x, x'; s) h(x', 0) dx', \quad (6)$$

where the Green function $G(x, x'; s)$ has the following form

$$G(x, x'; s) = \sqrt{\frac{3}{2\pi s}} \times \left[\exp\left(-\frac{3(x-x')^2}{2s}\right) - \exp\left(-\frac{3(x+x')^2}{2s}\right) \right]. \quad (7)$$

From (6) one may obtain the following formula for the function $g(r, s)$

$$g(r, s) = \int_0^\infty g(r', 0) \cdot \left[\frac{r_*}{r'}\right]^2 \cdot \left[\frac{r'}{r}\right]^5 \cdot G\left(\frac{r_*}{r}, \frac{r_*}{r'}; s\right) d\left(\frac{r'}{r_*}\right).$$

The expression $g(r, 0) = \delta(r - r_0)$ corresponds to very important special case, when the fragmentation process starts from one fragment having the size r_0 . In this case

$$g(r, s) = \left[\frac{r_*^2 r_0^3}{r^5}\right] \cdot G\left(\frac{r_*}{r}, \frac{r_*}{r_0}; s\right).$$

To obtain the asymptotic expression for this distribution density at $s \rightarrow \infty$, i.e., in probabilistic terminology, the final density, it is sufficient to take into account that, after the long evolution, the density $g(r, s)$ should be concentrated in the region of small values $r \ll r_0$. Assuming also that it takes place by such a way that $r_*^2 \ll rr_0$ (the asymptotic formula obtained in this case is nonuniform on r at $r \rightarrow 0$, i.e. it is not right at very small r), one may find that

$$g(r, s) = \frac{C(s)}{r^6} \exp\left(-\frac{\alpha(s)}{r^2}\right), \quad (8)$$

where $\alpha(s) = 3r_*^2/2s$, $C(s) = 6r_*^4 r_0^2 (3/2\pi s^3)^{1/2}$. Then the average fragment number $N(s)$, correspondingly normalized to unit probability distribution density $f(r, s) = g(r, s)/N(s)$ and the average fragment size $\langle \tilde{r} \rangle$ are determined by formulas

$$N(s) = \frac{3}{8} \sqrt{\pi} \frac{C(s)}{\alpha^{5/2}(s)},$$

$$f(r, s) = \frac{8}{3\sqrt{\pi}} \frac{\alpha^{5/2}(s)}{r^6} \exp\left(-\frac{\alpha(s)}{r^2}\right),$$

$$\langle \tilde{r} \rangle = \frac{3}{4\sqrt{\pi}} \alpha^{1/2}(s).$$

The relative squared variation of the fragment size is constant, since

$$D\tilde{r} = \frac{2}{3} \left(1 - \frac{8}{3\sqrt{\pi}}\right) \alpha(s).$$

4. CONCLUSIONS

The research made in the present work permits essentially simplify the calculation of the final distribution density of fragment sizes when the scale homogeneous mechanism of fragment subdivision takes place. We find general equation (4) defining the final distribution density at any homogeneity parameter β which is reduced to the Schrödinger equation with the imaginary time and with the potential being proportional to $\sim x^{-2}$. We illustrate the calculation of the final distribution density explicitly on the case when $\beta = 2$. It is remarkable that the distribution density in the case under consideration is decreased by the power way when the size r tends to infinity $\sim r^{-6}$. Besides, it is very important that the values $N(s)$, $\langle \tilde{r} \rangle$ vary essentially different from those which take place in the Kolmogorov case, $N(s) \sim s$, $\langle \tilde{r} \rangle \sim s^{-1/2}$.

References

1. A.N. Kolmogorov. On logarithmically normal distribution law of particle sizes at division process // *DAN SSSR*. 1941, v. 31, p. 99-101.
2. R.Z. Sagdeev, A.V. Tur, V.V. Yanovskii. Formation and universal properties of the size distribution in the fragmentation theory // *DAN SSSR*. 1987, v. 294, p. 1105-1110.
3. R.Ye. Brodskii, Yu.P. Virchenko. Final probability distribution of random sizes at the self-similar division mechanism // *Scientific Bulletin of Belgorod State University. Mathematics & Physics*. 2008, v. 13 (53), p. 23-32.

ФИНАЛЬНАЯ ПЛОТНОСТЬ РАСПРЕДЕЛЕНИЯ РАЗМЕРОВ ФРАГМЕНТОВ МАТЕРИАЛА В УСЛОВИЯХ МЕДЛЕННОСТИ ФРАГМЕНТАЦИИ

Р.Е. Бродский, Ю.П. Вирченко

Рассмотрен процесс медленной фрагментации материала в условиях, когда применимо диффузионное приближение. Вычислена финальная плотность распределения размеров фрагментов в случае масштабной однородности механизма дробления.

ФИНАЛЬНА ГУСТИНА РОЗПОДІЛУ РОЗМІРІВ ФРАГМЕНТІВ МАТЕРІАЛУ В УМОВАХ ПОВІЛЬНОСТІ ФРАГМЕНТАЦІЇ

Р.Є. Бродський, Ю.П. Вірченко

Розглянуто процес повільної фрагментації матеріалу в умовах, коли можна застосувати дифузійне наближення. Обчислена фінальна густина розподілу розмірів фрагментів у випадку масштабної однорідності механізму дроблення.