# Quasiaverages and Classification of Equilibrium States of Condensed Media with Spontaneously Broken Symmetry

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**Abstract**—Classification of equilibrium states of condensed media with spontaneously broken symmetry is carried out. Conditions of residual symmetry and spatial symmetry are formulated. The connection between these symmetry conditions and equilibrium states of various media with tensor order parameter is found out. Superfluid $^3$He, liquid crystals, quadrupolar magnetics are considered in detail. Possible homogeneous and heterogeneous states are found out. Discrete and continuous thermodynamic parameters, which define an equilibrium state, allowable form of order parameter, residual symmetry, and spatial symmetry generators are established.

## 1. INTRODUCTION

Second-order phase transitions are followed by a change of symmetry of equilibrium state of a medium when getting over critical temperature. Appropriate description of such states in condensed media below critical temperature needs an introduction of additional thermodynamic parameters, which are not concerned with the conservation laws, but resulting from physical nature of the new phase, into the theory. Under the phenomenological theory of equilibrium states classification [1] it is necessary to know free energy as an order parameter function in obvious form. Considerations of symmetry apply quite strong restrictions on the obvious form of free energy, but the attempts of connecting phenomenological parameters with intermolecular interaction parameters are confronted with considerable difficulties. Another difficulty of the mentioned approach is its correctness only in neighborhood of the point of phase transition, where one can neglect influence of fluctuation. Far from the critical temperature there are difficulties with the choice of obvious form of free energy as order parameter function and with solving corresponding nonlinear equations for it [2, 3]. Theoretical approach [4, 5], which does not depend on any definite mathematical model, uses idea of residual symmetry [6] of degenerate equilibrium state as a symmetry subgroup of a normal state.

In terms of this approach, effective equations for finding the equilibrium structure of order parameter, which do not include assumptions about the form of free energy and do not require closeness of temperature to the point of phase transition are found. Using the conception of quasiaveragues [7], confluent condensed media: superfluid $^3$He, liquid crystals, quadrupolar magnetics are studied, classification of equilibrium states is carried out and physical interpretation of the obtained solutions is given.

## 2. SYMMETRY PROPERTIES OF EQUILIBRIUM STATE OF NORMAL CONDENSED MEDIUM

Many-particle systems theory, which describes normal equilibrium states of condensed medium, in the context of statistical mechanics, can be built on basis of Gibbs operator

$$\hat{w} = \exp\left(\Omega - Y_a \hat{\gamma}_a\right). \qquad (1)$$

Here, $\hat{\gamma}_a \equiv \left(\hat{H}, \hat{P}_k, \hat{N}, \hat{S}_\alpha\right)$ $(a \equiv 0, k, 4, \alpha)$ —additive motion integrals: $\hat{H}$—Hamiltonian, $\hat{P}_k$ $(k = 1,2,3)$—impulse operator, $\hat{N}$ —particle number operator, $\hat{S}_\alpha$ $(\alpha = 1,2,3)$—spin operator. Thermodynamic potential $\Omega$ is determined from the condition of normalization $\mathrm{Sp}\hat{w} = 1$. Generalized thermodynamic forces $Y_a$, adjointed to additive motion integrals $\hat{\gamma}_a$, include: $Y_0^{-1} \equiv T$ is temperature, $-Y_k/Y_0 \equiv v_k$ is velocity, $-Y_4/Y_0 \equiv \mu_k$ is chemical potential and $-Y_\alpha/Y_0 \equiv$

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$h_\alpha$ is effective magnetic field. Symmetry properties of statistic operator (1) are:

$$\left[\hat{w}, \hat{P}_k\right] = 0, \quad \left[\hat{w}, \hat{H}\right] = 0, \quad \left[\hat{w}, \hat{N}\right] = 0, \quad (2)$$
$$\left[\hat{w}, \hat{\Sigma}_\alpha(\mathbf{Y})\right] = 0, \quad \left[\hat{w}, \hat{L}_k(\mathbf{Y})\right] = 0.$$

First three relations in expression (2) represent spatially-temporal translational invariance and phase invariance of equilibrium state. The last two relations in (2) represent invariance of normal equilibrium state relative to the rotations in configurational and spin spaces. The expressions (2) include $\hat{\Sigma}_\alpha(\mathbf{Y})$ and $\hat{L}_i(\mathbf{Y})$ —generalized spin moment and orbital moment generators, which are determined as:

$$\hat{L}_i(\mathbf{Y}) = \hat{L}_i + \hat{L}_i^{\mathbf{Y}} \quad \hat{L}_i^{\mathbf{Y}} \equiv -i\varepsilon_{ikl}Y_k\frac{\partial}{\partial Y_l}, \quad (3)$$
$$\hat{\Sigma}_\alpha(\mathbf{Y}) = \hat{S}_\alpha + \hat{S}_\alpha^{\mathbf{Y}}, \quad \hat{S}_\alpha^{\mathbf{Y}} \equiv -i\varepsilon_{\alpha\beta\gamma}Y_\beta\frac{\partial}{\partial Y_\gamma},$$

they operate in Hilbert space and in space of thermodynamic functions, moreover for the differential operators relations have place: $i\left[\hat{L}_i^{\mathbf{Y}}, Y_j\right] = \varepsilon_{ikj}Y_k$, $i\left[\hat{S}_\alpha^{\mathbf{Y}}, Y_\beta\right] = \varepsilon_{\alpha\beta\gamma}Y_\gamma$. Under the definition (3) operators $\hat{\Sigma}_\alpha(\mathbf{Y})$ and $\hat{L}_j(\mathbf{Y})$ satisfy commutative relations $i\left[\hat{L}_i(\mathbf{Y}), \hat{L}_k(\mathbf{Y})\right] = -\varepsilon_{ikl}\hat{L}_l(\mathbf{Y}), i\left[\hat{\Sigma}_\alpha(\mathbf{Y}), \hat{\Sigma}_\beta(\mathbf{Y})\right] = -\varepsilon_{\alpha\beta\gamma}\hat{\Sigma}_\gamma(\mathbf{Y})$. Symmetry conditions relative to the rotations in spin and configurational spaces (2) mean neglect the weak dipole and spin-orbital interactions for the characteristic of an equilibrium state. Full symmetry group of normal state of condensed medium is given by

$$G = [SO(3)]_S \times [SO(3)]_L \times [U(1)]_\varphi \times [T(3)] \times [T(1)].$$

Here, $[SO(3)]_S$, $[SO(3)]_L$ are symmetry groups relative to the rotations in spin and configurational spaces, $[T(3)]$, $[T(1)]$ are translational groups in space and time, $[U(1)]_\varphi$ are phase symmetry group. Each element of a group is a unitary operator $U \equiv \exp i\hat{G}g$ ($g$ is real transformation parameters), which leaves the Gibbs distribution invariant: $U\hat{w}U^+ = \hat{w}$. Linear combinations of operators $\hat{G} \in \left\{\hat{\Sigma}_\alpha, \hat{L}_i, \hat{N}, \hat{P}\right\}$ are generators of these transformations. This property is right for any voluntary transformation parameters by virtue of symmetry relations (2). Averages like $\mathrm{Sp}\hat{w}\left[\hat{G}, \hat{b}(\mathbf{x})\right]$ turn into zero at voluntary quasilocal operator $\hat{b}(\mathbf{x})$ for $\hat{G} \in \left\{\hat{\Sigma}_\alpha, \hat{L}_i, \hat{N}, \hat{P}\right\}$. This is also right for operators $\hat{b}(\mathbf{x}) \equiv \hat{\Delta}(\mathbf{x})$, which have a physical meaning of order parameter operator and do not commute with motion integrals $\hat{G}$. Since order parameter operators are linear or bilinear on creation and annihilation operators, averages $\mathrm{Sp}\hat{w}\left[\hat{G}, \hat{\Delta}(\mathbf{x})\right]$ are linear on order parameters $\Delta(\mathbf{x})$. This causes equilibrium averages of order parameters to turn into zero in normal state $\mathrm{Sp}\hat{w}\hat{\Delta}_a(\mathbf{x}) = 0$.

## 3. EQUILIBRIUM STATES OF CONFLUENT CONDENSED MEDIA AND THE PROBLEM OF THEIR CLASSIFICATION

Theoretical footing for description of equilibrium states of condensed media with spontaneously broken symmetry is formed by quasiaverage conception of Bogolyubov [7]. Constructive feature of this conception is introduction into the statistic operator of an infinitely small perturbation or a source $\nu\hat{F}$, which decreases the symmetry of statistic equilibrium state in comparison with the symmetry of Hamiltonian and allows to generalize the Gibbs distribution on condensed media in conditions of spontaneously broken symmetry. The Gibbs distribution for degenerate media, according to the conception of quasiaverages, is given by

$$\hat{w}_\nu \equiv \exp\left(\Omega_\nu - Y_a\hat{\gamma}_a - \nu\hat{F}\right).$$

The source $\hat{F}$, which breaks the equilibrium state symmetry, is a linear functional of the order parameter operator $\hat{\Delta}_a(\mathbf{x})$:

$$\hat{F} = \int d^3x\left(f_a(\mathbf{x})\hat{\Delta}_a(\mathbf{x}) + \mathrm{h.c.}\right).$$

In this state the equilibrium average of order parameter does not equal zero $\langle\hat{\Delta}_a(\mathbf{x})\rangle = \mathrm{Sp}\hat{w}\hat{\Delta}_a(\mathbf{x}) \equiv \lim_{\nu\to 0}\lim_{V\to\infty}\mathrm{Sp}\hat{w}_\nu\hat{\Delta}_a(\mathbf{x}) \neq 0$. To obtain effective equations that specify equilibrium values of order parameter it is necessary to know transformation properties of these physical values. The translation invariance condition is given by

$$i\left[\hat{P}_k, \hat{\Delta}_a(\mathbf{x})\right] = -\nabla_k\hat{\Delta}_a(\mathbf{x}).$$

The generator of phase transformation group is the particle number operator $\hat{N}$. Order parameter operator $\hat{\Delta}_a(\mathbf{x})$ is transformed according to the relation

$$\left[\hat{N}, \hat{\Delta}_a(\mathbf{x})\right] = -g\hat{\Delta}_a(\mathbf{x}).$$

Value of constant $g$ depends on tensor dimension and internal structure of order parameter operator. In the case of spin rotations with generators $\hat{S}_\alpha$, operators $\hat{\Delta}_a(\mathbf{x})$ transform according to the relation:

$$i\left[\hat{S}_\alpha, \hat{\Delta}_a(\mathbf{x})\right] = -g_{\alpha ab}\hat{\Delta}_b(\mathbf{x}),$$

where $g_{\alpha ab}$ are certain constants. In case of spatial rotations order parameter operators transform in accordance with formula

$$i\left[\hat{L}_i, \hat{\Delta}_a(\mathbf{x})\right] = -g_{iab}\hat{\Delta}_b(\mathbf{x}) - \varepsilon_{ijk}x_k\nabla_j\hat{\Delta}_a(\mathbf{x}),$$

where $\hat{L}_i$ is orbital moment coefficient and values $g_{iab}$ are constants. Linearity of the written switching relations by order parameter results in the linearity of equations for its equilibrium structure determination.

For translationally invariant equilibrium states, that satisfy the relation $\left[\hat{w}, \hat{P}_k\right] = 0$, let us determine the residual symmetry condition of equilibrium state [8]

$$\left[\hat{w}, \hat{T}(\xi, \mathbf{Y})\right] = 0,$$

where the residual symmetry generator $\hat{T}(\xi, \mathbf{Y}) \equiv a_i\hat{L}_i(\mathbf{Y}) + b_\alpha\hat{\Sigma}_\alpha(\mathbf{Y}) + c\hat{N}$ is a linear combination of motion integrals with real parameters $a_i, b_\alpha, c \equiv \xi$; $Y_i, Y_\alpha$ and reflects presence of residual symmetry in the condensed medium, which is less than in the initial, more symmetric, state.

The conditions of residual symmetry result in a system of linear differential equations in partial derivatives for the order parameter:

$$a_i\left(g_{iab}\Delta_b + \varepsilon_{ikl}Y_k\partial\Delta_a/\partial Y_l\right)$$
$$+ b_\alpha\left(g_{\alpha ab}\Delta_b + \varepsilon_{\alpha\beta\gamma}Y_\beta\partial\Delta_a/\partial Y_\gamma\right) + igc\Delta_a = 0.$$

These equations are considerably simplified in case when vectors $Y_\alpha = Y_k = 0$, and become linear homogeneous algebraic equations

$$T_{ab}(\xi, 0)\,\Delta_b = 0, \qquad (4)$$
$$T_{ab}(\xi, 0) \equiv a_i g_{iab} + b_\alpha g_{\alpha ab} + igc\delta_{ab}.$$

The condition of nontrivial solution $\Delta_a \neq 0$ results in equation on allowable values of residual symmetry generator $\det\left|T_{ab}(\xi, 0)\right| = 0$ and classifies equilibrium states of degenerate condensed media. In this case Gibbs operator depends on thermodynamic parameters and parameters of residual symmetry generator $\hat{w} = \hat{w}(Y, \xi)$.

For spatially-heterogeneous equilibrium states of degenerate condensed media residual and spatial symmetry generators are characterized by equalities [8]

$$\hat{T}(\xi, \mathbf{Y}) \equiv a_i\hat{L}_i(\mathbf{Y}) + b_\alpha\hat{\Sigma}_\alpha(\mathbf{Y}) + c\hat{N} + d_i\hat{P}_i, \quad (5)$$
$$\hat{P}_k(\eta, \mathbf{Y}) \equiv \hat{P}_k - p_k\hat{N} - q_{k\alpha}\hat{\Sigma}_\alpha(\mathbf{Y}) - t_{kj}\hat{L}_j(\mathbf{Y})$$

and lead to the relations

$$i\mathrm{Sp}\left[\hat{w}, \hat{T}(\xi, \mathbf{Y})\right]\hat{\Delta}_a(\mathbf{x}) = 0, \qquad (6)$$
$$i\mathrm{Sp}\left[\hat{w}, \hat{P}_k(\eta, \mathbf{Y})\right]\hat{\Delta}_a(\mathbf{x}) = 0.$$

Here, $a_i, b_\alpha, c, d_i \equiv \xi$ and $\eta \equiv (p_k, q_{k\alpha}, t_{kj})$ are real parameters, which characterize residual and spatial symmetry generators. Relations (6) lead to the order parameter dependence on coordinate and on parameters of residual and spatial symmetry $\xi$, $\eta$. To specify the structure of order parameter and symmetry generators relations (6) should be supplemented with the Jacobi identities for operators $\hat{w}$, $\hat{T}$, $\hat{P}_k$ and $\hat{w}$, $\hat{P}_i$, $\hat{P}_k$:

$$\mathrm{Sp}\left[\hat{w}, \left[\hat{T}(\xi, \mathbf{Y}), \hat{P}_k(\eta, \mathbf{Y})\right]\right]\hat{\Delta}_a(\mathbf{x}) = 0, \qquad (7)$$
$$\mathrm{Sp}\left[\hat{w}, \left[\hat{P}_i(\eta, \mathbf{Y}), \hat{P}_k(\eta, \mathbf{Y})\right]\right]\hat{\Delta}_a(\mathbf{x}) = 0.$$

Conditions (6), (7) lead to connections of parameters of symmetry generators and allow to solve the problem of classification of equilibrium states for heterogeneous condensed media. By virtue of relations (6), (7) in heterogeneous case the statistic Gibbs operator depends on the following thermodynamic parameters $\hat{w} = \hat{w}(Y, \xi, \eta)$.

## 4. CLASSIFICATION OF EQUILIBRIUM STATES OF SUPERFLUID $^3$He

As the first example of application of the being developed approach let us consider superfluid $^3$He. Order parameter operator of triplet coupling of such a fluid is given by [9]: $\hat{\Delta}_{\alpha k}(\mathbf{x}) \equiv \hat{\psi}(\mathbf{x})\sigma_2\sigma_\alpha\nabla_k\hat{\psi}(\mathbf{x}) - \nabla_k\hat{\psi}(\mathbf{x})\sigma_2\sigma_\alpha\hat{\psi}(\mathbf{x})$. Average value of this order parameter is characterized by 18 real values. For this order parameter operator the algebra of quantum Poisson brackets with motion integrals is established:

$$i\left[\hat{S}_\alpha, \hat{\Delta}_{\beta i}(\mathbf{x})\right] = -\varepsilon_{\alpha\beta\gamma}\hat{\Delta}_{\gamma i}(\mathbf{x}), \qquad (8)$$
$$\left[\hat{N}, \hat{\Delta}_{\beta i}(\mathbf{x})\right] = -2\hat{\Delta}_{\beta i}(\mathbf{x}),$$
$$i\left[\hat{P}_k, \hat{\Delta}_{\alpha i}(\mathbf{x})\right] = -\nabla_k\hat{\Delta}_{\alpha i}(\mathbf{x}),$$
$$i\left[\hat{L}_k, \hat{\Delta}_{\alpha i}(\mathbf{x})\right] = -\varepsilon_{kjl}x_j\nabla_l\hat{\Delta}_{\alpha i}(\mathbf{x}) - \varepsilon_{kil}\hat{\Delta}_{\alpha l}(\mathbf{x}).$$

Results of the foregoing approach application for homogeneous equilibrium states of superfluid $^3$He come to the following statements. The variety of possible anisotropic equilibrium states of such fluid is described by four types of residual symmetry generators and can be classified by two discrete quantum numbers $m_l = 0, \pm1$ and $m_s = 0, \pm1$, and continuous parameters, that reflect medium anisotropy. In this superfluid liquid there are 12 possible anisotropic states

**Table 1.** The classification of homogeneous equilibrium states in $^3$He

| Residual symmetry generator | $m_s$ | $m_l$ | Order parameter $\Delta_{\alpha k}$ | Phase |
|---|---|---|---|---|
| $\hat{L}_i + R_{i\alpha}\hat{S}_\alpha$ | – | – | $\Delta R_{\alpha k}$ | $B$ |
| $l_i\hat{L}_i - \frac{m_l}{2}\hat{N}$, $d_\alpha\hat{S}_\alpha - \frac{m_s}{2}\hat{N}$ | 0 | $\pm1$ | $\Delta d_\alpha (m_k \mp i n_k)$ | $A$ |
| | $\pm1$ | 0 | $\Delta (e_\alpha \mp i f_\alpha) l_k$ | $\beta$ |
| | $\pm1$ | $\pm1$ | $\Delta (e_\alpha \mp i f_\alpha)(m_k \mp i n_k)$ | $A_1$ |
| | 0 | 0 | $\Delta d_\alpha l_k$ | Polar |
| $-2m_l\,\lvert m_s\rvert\,\vec{l}\hat{\vec{L}} + \vec{d}\hat{\vec{S}} - \frac{1}{2}m_s\hat{N}$ | 0 | $0, \pm1$ | $d_\alpha (A m_k + B n_k + C l_k)$ | – |
| | $\pm1$ | $\pm1$ | $A (m_k \mp i n_k)(e_\alpha \mp i f_\alpha) + B d_\alpha (m_k \mp i n_k)$ | $A + A_1$ |
| | $0, \pm1$ | 0 | $(e_\alpha \mp i f_\alpha)(A m_k + B n_k + C l_k)$ | – |
| $-2m_s\,\lvert m_l\rvert\, d_\alpha\hat{S}_\alpha + l_i\hat{L}_i - \frac{1}{2}m_l\hat{N}$ | $0, \pm1$ | 0 | $(A e_\alpha + B f_\alpha + C d_\alpha) l_k$ | – |
| | $\pm1$ | $\pm1$ | $A (m_k \mp i n_k)(e_\alpha \mp i f_\alpha) + B l_k (e_\alpha \mp i f_\alpha)$ | $\beta + A_1$ |
| | 0 | $0, \pm1$ | $(A e_\alpha + B f_\alpha + C d_\alpha)(m_k \mp i n_k)$ | – |
| $l_i\hat{L}_i + d_\alpha\hat{S}_\alpha - \frac{m_l + m_s}{2}\hat{N}$ | 0 | 0 | $e_\alpha (A m_k - B n_k) + f_\alpha (B m_k + A n_k) + C d_\alpha l_k$ | $\varsigma$ |
| | 0 | $\pm1$ | $A l_k (e_\alpha \pm i f_\alpha) + B d_\alpha (m_k \pm i n_k)$ | $\varepsilon$ |
| | $\pm1$ | 0 | $\Delta (e_\alpha \mp i f_\alpha)(m_k \mp i n_k)$ | $A_1$ |
| | $\pm1$ | $\pm1$ | | |

and one isotropic state ($B$ phase). In Table 1 the order parameter structure is presented in terms of orthonormal frame in spin and configurational spaces ($e_\alpha$, $f_\alpha$, $d_\alpha$), ($m_i, n_i, l_i$), order parameter amplitudes $\Delta$, $A$, $B$, $C$ and orthogonal rotation matrix $R_{i\alpha}$. The last values are additional continuous thermodynamic parameters of superfluid $^3$He. So, for isotropic equilibrium state we have $\hat{w} = \hat{w}(Y, \varphi, R)$. For anisotropic case we obtain $\hat{w} = \hat{w}(Y, \varphi, \mathbf{e}_\alpha, \mathbf{f}_\alpha, \mathbf{m}, \mathbf{n}, A, B, C, m_l, m_s)$. The form of equilibrium structure of tensorial order parameter and form of residual symmetry generator of superfluid $^3$He are presented in Table 1. The obtained homogeneous equilibrium states coincide with solutions of works [4, 5, 10–13] known before.

Let us consider spatially-heterogeneous states of superfluid $^3$He. From the conditions (6), (7) taking into account transformation properties (8) structure of the parameter $q_{i\alpha}$ is obtained: $q_{i\alpha} = q_i n_\alpha$. Here, $q_k = q l_k$ is vector of magnetic spiral, $n_\alpha$ is anisotropy axis in spin space, and $\mathbf{l}$ is anisotropy axis in configurational space. General heterogeneous solution for order parameter is given by

$$\Delta_{\beta i}(\mathbf{x}) = e^{2i\varphi(\mathbf{x})} a_{\beta\gamma}(\mathbf{n}\theta(\mathbf{x}))\, a_{ik}(\mathbf{l}\psi(\mathbf{x}))\, \Delta_{\gamma k}(0).$$

Here, $\theta(\mathbf{x}) = \theta + \mathbf{q}\mathbf{x}$, $a_{\beta\gamma}(\mathbf{n}\theta(\mathbf{x}))$ is orthogonal matrix of rotation in spin space, $a_{ik}(\mathbf{l}\psi(\mathbf{x}))$ is orthogonal matrix of rotation around axis $\mathbf{l}$ in configurational space with angle $\psi(\mathbf{x}) = \psi + t\mathbf{l}\mathbf{x}$, spiral step is equal to $q = 2\pi/t$. Obviously, structure of the order parameter may be presented in form of product of homogeneous part and spatial-coordinates dependent part. At that the solution of classification of residual symmetry of heterogeneous equilibrium state effectively comes to the problem of translationally-invariant case.

## 5. EQUILIBRIUM STATES OF LIQUID CRYSTALS

In this part equilibrium states of liquid crystals with continuously broken translational and rotational symmetry are classified. Let us determine the order parameter $\hat{Q}_{uv}(\mathbf{x})$ of such condensed media by equality

$$\hat{Q}_{uv}(\mathbf{x}) \equiv \nabla_u\hat{\psi}^+(\mathbf{x})\nabla_v\hat{\psi}(\mathbf{x}) + \nabla_v\hat{\psi}^+(\mathbf{x})\nabla_u\hat{\psi}(\mathbf{x}) - \frac{2}{3}\delta_{uv}\nabla_j\hat{\psi}^+(\mathbf{x})\nabla_j\hat{\psi}(\mathbf{x}),$$

for which quantum brackets with motion integrals are obtained

$$\left[\hat{S}_\alpha, \hat{Q}_{uv}(\mathbf{x})\right] = 0, \quad \left[\hat{N}, \hat{Q}_{uv}(\mathbf{x})\right] = 0, \qquad (9)$$

$$i\left[\hat{P}_k, \hat{Q}_{uv}(\mathbf{x})\right] = -\nabla_k\hat{Q}_{uv}(\mathbf{x}),$$

$$i\left[\hat{L}_i, \hat{Q}_{uv}(\mathbf{x})\right] = -\varepsilon_{ikl}x_k\nabla_l\hat{Q}_{uv}(\mathbf{x})$$

**Table 2.** The classification of equilibrium states of liquid crystals

| Equilibrium state | Residual symmetry generator $\hat{T}$ | Space symmetry generator $\hat{P}_k$ | Order parameter $Q_{ik}(\mathbf{x})$ |
|---|---|---|---|
| Uniaxial nematic | $a_i\hat{L}_i$ | $\hat{P}_k$ | $Q\left(n_i n_k - \frac{1}{3}\delta_{ik}\right)$ |
| Biaxial nematic | $(\alpha n_i + \beta m_i)\,\hat{L}_i(\mathbf{m},\mathbf{n})$ <br> $\alpha^2+\beta^2=1$ | $\hat{P}_k$ | $Q_1\left(e_i^{(1)}e_k^{(1)} - \frac{1}{3}\delta_{ik}\right) + Q_2\left(e_i^{(2)}e_k^{(2)} - \frac{1}{3}\delta_{ik}\right);$ <br> $e_i^{(1)} = m_i\cos\varphi + n_i\sin\varphi,$ <br> $e_i^{(2)} = -m_i\sin\varphi + n_i\cos\varphi$ |
| Cholesteric | $l_i\hat{L}_i + d_i\hat{P}_i$ | $\hat{P}_k + \frac{l_k}{\mathbf{dl}}l_i\hat{L}_i$ | $Q\left(e_i^{(1)}(\mathbf{x})e_k^{(1)}(\mathbf{x}) - \frac{1}{3}\delta_{ik}\right);$ <br> $e_i^{(1)}(\mathbf{x}) = m_i\cos\varphi(\mathbf{x}) + n_i\sin\varphi(\mathbf{x}),$ <br> $\varphi(\mathbf{x}) = \varphi - \mathbf{lx}/\mathbf{dl}$ |
| Dual spiral structure | $l_i\hat{L}_i(\mathbf{m},\mathbf{n}) + d_i\hat{P}_i$ | $\hat{P}_k + \frac{l_k}{\mathbf{dl}}l_i\hat{L}_i(\mathbf{m},\mathbf{n})$ | $Q_1\left(e_i^{(1)}(\mathbf{x})e_k^{(1)}(\mathbf{x}) - \frac{1}{3}\delta_{ik}\right) +$ <br> $Q_2\left(e_i^{(2)}(\mathbf{x})e_k^{(2)}(\mathbf{x}) - \frac{1}{3}\delta_{ik}\right);$ <br> $e_i^{(1)}(\mathbf{x}) = m_i\cos\varphi(\mathbf{x}) + n_i\sin\varphi(\mathbf{x}),$ <br> $e_i^{(2)}(\mathbf{x}) = -m_i\sin\varphi(\mathbf{x}) + n_i\cos\varphi(\mathbf{x})$ |

$$- \varepsilon_{iuj}\hat{Q}_{jv}(\mathbf{x}) - \varepsilon_{ivj}\hat{Q}_{ju}(x)\,.$$

Average value of order parameter is symmetric and traceless. The order parameter contains five independent values and may be written as

$$Q_{ik}(\mathbf{x}) \equiv Q(\mathbf{x})\left[n_i(\mathbf{x})n_k(\mathbf{x}) - \frac{1}{3}\delta_{ik}\right] + Q'(\mathbf{x})\left[m_i(\mathbf{x})m_k(\mathbf{x}) - \frac{1}{3}\delta_{ik}\right].$$

Here, $Q$, $Q'$ are order parameter modules, unit and orthogonal vectors $\mathbf{n}$ and $\mathbf{m}$ define anisotropy directions of liquid crystals. Results of classification of equilibrium states are presented in Table 2. The generalized operator of orbital moment coefficient is defined by equalities $\hat{L}_i(\mathbf{m},\mathbf{n}) \equiv \hat{L}_i + L_i^{\mathbf{m}} + L_i^{\mathbf{n}}$, $L_i^{\mathbf{m}} \equiv -i\varepsilon_{ikl}m_k\partial/\partial m_l$, $L_i^{\mathbf{n}} \equiv -i\varepsilon_{ikl}n_k\partial/\partial m_l$ and vectors $\mathbf{n}$, $\mathbf{m}$, $\mathbf{l}$ form an orthonormal frame in configurational space. The first two homogeneous solutions describe correspondingly uniaxial and biaxial nematics. Heterogeneous uniaxial solution corresponds to the state of cholesteric. Spatial structure of dual spiral resembles structure of a DNA molecule [14]. Complementary Watson–Crick pairs adenine —thymine and guanine—cytosine lie in the plane of vectors $\mathbf{e}^{(1)}(\mathbf{x})$, $\mathbf{e}^{(2)}(\mathbf{x})$ and rotate around anisotropy axis $\mathbf{l}$ when shifting the coordinate. The case when vectors $\mathbf{l}$ and $\mathbf{d}$ are collinear corresponds to B family of DNA. At that complementary pairs settle themselves in a plane, orthogonal to the spiral axis. If vectors $\mathbf{l}$ and $\mathbf{d}$ are canted, this spatial structure is A family, so spatial structure ordering of the splay spiral type have place. Complementary pairs in this case are situated angularly to the spiral axis. The difference in sign of the scalar product of vectors $\mathbf{ld} > 0$ or $\mathbf{ld} < 0$ means a possibility for physical realization of right-side and left-side spirals.

## 6. EQUILIBRIUM STATES OF MAGNETIC MEDIA WITH THE QUADRUPOLAR ORDER PARAMETER

Quadrupolar magnetic operator is described by the tensor order parameter

$$\hat{Q}_{\alpha\beta}(\mathbf{x}) = \frac{1}{2}\Big(\hat{s}_{\alpha n}(\mathbf{x})\hat{s}_{\beta n}(\mathbf{x}) + \hat{s}_{\beta n}(\mathbf{x})\hat{s}_{\alpha n}(\mathbf{x}) - \frac{2}{3}\delta_{\alpha\beta}\hat{s}^2_{\gamma n}(\mathbf{x})\Big).$$

This operator is symmetric and traceless $\hat{Q}_{\alpha\beta}(\mathbf{x}) = \hat{Q}_{\beta\alpha}(\mathbf{x})$, $\hat{Q}_{\alpha\alpha}(\mathbf{x}) = 0$. Quadrupolar order parameter operator satisfies the written out operator algebra

$$i\left[\hat{S}_\alpha, \hat{Q}_{\beta\gamma}(\mathbf{x})\right] = -\varepsilon_{\alpha\beta\rho}\hat{Q}_{\gamma\rho}(\mathbf{x}) - \varepsilon_{\alpha\gamma\rho}\hat{Q}_{\beta\rho}(\mathbf{x}), \qquad (10)$$

$$i\left[\hat{P}_k, \hat{Q}_{\alpha\beta}(\mathbf{x})\right] = -\nabla_k\hat{Q}_{\alpha\beta}(\mathbf{x}).$$

**Table 3.** The classification of equilibrium states of quadrupolar magnetics

| Equilibrium state | Residual symmetry generator $\hat{T}$ | Space symmetry generator | Order parameter |
|---|---|---|---|
| Uniaxial spin nematic | $\hat{T} \equiv b_\alpha \hat{\Sigma}_\alpha(\mathbf{e})$ | $\hat{P}_k$ | $Q_{uv} = Q\left(e_u e_v - \frac{1}{3}\delta_{uv}\right)$ |
| Biaxial spin nematic | $\hat{T} \equiv b_\alpha \hat{\Sigma}_\alpha(\mathbf{e}, \mathbf{f})$ | $\hat{P}_k$ | $Q_{uv} = Q\left(e_u e_v - \frac{1}{3}\delta_{uv}\right) + Q'\left(f_u f_v - \frac{1}{3}\delta_{uv}\right)$ |
| Spin cholesteric | $\hat{T}(\xi) \equiv b_\alpha \hat{\Sigma}_\alpha(\mathbf{e}) + d_i \hat{P}_i$ | $\hat{P}_k(\eta) \equiv \hat{P}_k - q_{k\alpha}\hat{\Sigma}_\alpha(\mathbf{e}, \mathbf{f})$ | $Q\left(e_i^{(1)}(\mathbf{x}) e_k^{(1)}(\mathbf{x}) - \frac{1}{3}\delta_{ik}\right)$; $e_i^{(1)}(\mathbf{x}) = m_i \cos\varphi(\mathbf{x}) + n_i \sin\varphi(\mathbf{x})$, $\varphi(\mathbf{x}) = \varphi - \mathbf{lx}/\mathbf{dl}$ |
| Dual spiral spin structure | $\hat{T}(\xi) \equiv b_\alpha \hat{\Sigma}_\alpha(\mathbf{e}, \mathbf{f}) + d_i \hat{P}_i$ | $\hat{P}_k(\eta) \equiv \hat{P}_k - q_{k\alpha}\hat{\Sigma}_\alpha(\mathbf{e}, \mathbf{f})$ | $Q_1\left(e_i^{(1)}(\mathbf{x}) e_k^{(1)}(\mathbf{x}) - \frac{1}{3}\delta_{ik}\right) + Q_2\left(e_i^{(2)}(\mathbf{x}) e_k^{(2)}(\mathbf{x}) - \frac{1}{3}\delta_{ik}\right)$; $e_i^{(1)}(\mathbf{x}) = m_i \cos\varphi(\mathbf{x}) + n_i \sin\varphi(\mathbf{x})$, $e_i^{(2)}(\mathbf{x}) = -m_i \sin\varphi(\mathbf{x}) + n_i \cos\varphi(\mathbf{x})$, $\varphi(\mathbf{x}) = \varphi - \mathbf{lx}/\mathbf{dl}$ |

One can see that right sides of Poisson brackets (10) are linear by the quadrupolar order parameter operator. This circumstance results in linear equations that classify equilibrium states of this magnetic systems. Five independent components of the quadrupolar order parameter may be parametrized by relation

$$Q_{\alpha\beta}(\mathbf{x}, \hat{\rho}) = Q(\mathbf{x}, \hat{\rho})\left(e_\alpha(\mathbf{x}, \hat{\rho}) e_\beta(\mathbf{x}, \hat{\rho}) - \frac{1}{3}\delta_{\alpha\beta}\right) + Q'(\mathbf{x}, \hat{\rho})\left(f_\alpha(\mathbf{x}, \hat{\rho}) f_\beta(\mathbf{x}, \hat{\rho}) - \frac{1}{3}\delta_{\alpha\beta}\right).$$

Here $Q$ and $Q'$ are modules of the order parameter. Vectors $d_\alpha$, $e_\alpha$, $f_\alpha$ form an orthonormal frame: $\mathbf{de} = 0$, $\mathbf{df} = 0$, $\mathbf{ef} = 0$, $\mathbf{f}^2 = 1$, $\mathbf{d}^2 = 1$, $\mathbf{e}^2 = 1$, $\mathbf{e} \times \mathbf{f} = \mathbf{d}$, $\mathbf{f} \times \mathbf{d} = \mathbf{e}$, $\mathbf{d} \times \mathbf{e} = \mathbf{f}$. Let us define the generator of uniaxial spin symmetry and biaxial spin symmetry by equalities

$$\hat{\Sigma}_\alpha(\mathbf{e}) = \hat{S}_\alpha + \hat{S}_\alpha^{\mathbf{e}}, \quad \hat{\Sigma}_\alpha(\mathbf{e}, \mathbf{f}) = \hat{S}_\alpha + \hat{S}_\alpha^{\mathbf{e}} + \hat{S}_\alpha^{\mathbf{f}}.$$

Let us introduce into consideration residual symmetry generators for uniaxial and biaxial cases:

$$\hat{T} \equiv b_\alpha \hat{\Sigma}_\alpha(\mathbf{e}) = \hat{T}(\mathbf{b}, \mathbf{e}),$$
$$\hat{T} \equiv b_\alpha \hat{\Sigma}_\alpha(\mathbf{e}, \mathbf{f}) = \hat{T}(\mathbf{b}, \mathbf{e}, \mathbf{f}).$$

The residual symmetry condition together with the spatial homogeneity condition according to algebraic relations (10) results in equation for uniaxial case

$$b_\alpha\left\{\varepsilon_{\alpha u\rho} Q_{\rho v} + \varepsilon_{\alpha v\rho} Q_{\rho u} + \varepsilon_{\alpha\beta\gamma} e_\beta \frac{\partial Q_{uv}}{\partial e_\gamma}\right\} = 0, \quad \nabla_k Q_{uv}(\mathbf{x}) = 0. \tag{11}$$

Solution of this equation is given by $Q_{uv} = Q(e_u e_v - 1\frac{1}{3}\delta)_{uv}$ and it identically holds true at every value of vector $\mathbf{b}$. Magnetic states with order parameter of such a form, according to work [15], we call uniaxial spin nematics.

In biaxial case we obtain similar equations

$$b_\alpha\left\{\varepsilon_{\alpha u\rho} Q_{\rho v} + \varepsilon_{\alpha v\rho} Q_{\rho u} + \varepsilon_{\alpha\beta\gamma} e_\beta \frac{\partial Q_{uv}}{\partial e_\gamma} + \varepsilon_{\alpha\beta\gamma} f_\beta \frac{\partial Q_{uv}}{\partial f_\gamma}\right\} = 0, \quad \nabla_k Q_{uv}(\mathbf{x}) = 0. \tag{12}$$

To establish allowable values of vector $\mathbf{b}$ we look for it in form of decomposition by orthonormal frame $b_\lambda = \alpha e_\lambda + \beta f_\lambda + \gamma d_\lambda$, where numbers $\alpha$, $\beta$, $\gamma$ are connected by relation $\alpha^2 + \beta^2 + \gamma^2 = 1$. As a result, we obtain the equality

$$\gamma\left(Q - Q'\right)\left(e_\beta f_\alpha + e_\alpha f_\beta\right) = 0, \tag{13}$$

obviously, if $Q \neq Q'$, the parameter $\gamma = 0$. This solution describes spatially-homogeneous biaxial spin nematic with order parameter being equal to

$$Q_{uv} = Q\left(e_u e_v - \frac{1}{3}\delta_{uv}\right) + Q'\left(f_u f_v - \frac{1}{3}\delta_{uv}\right),$$

and vector $\mathbf{b}$ is given by $b_\lambda = \alpha e_\lambda + \beta f_\lambda$, at that $\alpha^2 + \beta^2 = 1$. Another solution is obtained from (13), if $Q = Q'$. In this case vector $\mathbf{b}$ is voluntary and the order parameter get form of uniaxial spin nematic $Q_{uv} = -Q\left(d_u d_v - \frac{1}{3}\delta_{uv}\right)$.

Let us go now to heterogeneous equilibrium states. Residual and spatial symmetry operators are given by

$$\hat{T}(\xi) \equiv b_\alpha \hat{\Sigma}_\alpha(\mathbf{e},\mathbf{f}) + d_i \hat{P}_i,$$
$$\hat{P}_k(\eta) \equiv \hat{P}_k - q_{k\alpha}\hat{\Sigma}_\alpha(\mathbf{e},\mathbf{f}).$$

The set of parameters of residual symmetry operator includes vectors $(\xi) \equiv \mathbf{b}, \mathbf{e}, \mathbf{f}, \mathbf{d}$. Through $(\eta)$ the parameters of spatial symmetry generator $(\eta) \equiv q_{k\alpha}, \mathbf{e}, \mathbf{f}$ are labeled. The tensor $q_{i\alpha}$ have the structure $q_{i\alpha} = q_i n_\alpha$, where $q_i$ is vector of magnetic spiral and $n_\alpha$ is axis of spin anisotropy. Equations on structure of the quadrupolar order parameter and allowable values of vector $\mathbf{b}$:

$$b_\alpha F_\alpha^{uv}(\mathbf{x}) = 0, \quad (\mathbf{b}\times\mathbf{n})_\alpha F_\alpha^{uv}(\mathbf{x}) = 0,$$
$$\nabla_k Q_{uv}(\mathbf{x}) = q_k n_\alpha F_\alpha^{uv}(\mathbf{x}),$$

where label is introduced

$$F_\alpha^{uv}(x) = \varepsilon_{\alpha u\rho} Q_{\rho v}(x) + \varepsilon_{\alpha v\rho} Q_{\rho u}(x)$$
$$+ \varepsilon_{\alpha\beta\gamma} e_\beta \frac{\partial Q_{uv}(x)}{\partial e_\gamma} + \varepsilon_{\alpha\beta\gamma} f_\beta \frac{\partial Q_{uv}(x)}{\partial f_\gamma}.$$

The solution of the given equation set is the form of quadrupolar order parameter

$$Q_{uv}(\mathbf{x}) = Q\left(m_u(\mathbf{x}) m_v(\mathbf{x}) - \frac{1}{3}\delta_{uv}\right)$$
$$+ Q'\left(l_u(\mathbf{x}) l_v(\mathbf{x}) - \frac{1}{3}\delta_{uv}\right),$$

where dependent on coordinate vectors $\mathbf{m}(\mathbf{x})$ and $\mathbf{l}(\mathbf{x})$ are defined by the equalities

$$\mathbf{m}(\mathbf{x}) = \mathbf{e}\cos\varphi(\mathbf{x}) + \mathbf{f}\sin\varphi(\mathbf{x}),$$
$$\mathbf{l}(\mathbf{x}) = -\mathbf{e}\sin\varphi(\mathbf{x}) + \mathbf{f}\cos\varphi(\mathbf{x}), \quad \varphi(x) = \varphi - \mathbf{q}\mathbf{x}$$

and vector $\mathbf{n}$ is collinear to $\mathbf{d}$. Classification of homogeneous and heterogeneous equilibrium states of quadrupolar magnetic is presented in Table 3. In the uniaxial case when $Q \neq 0$, $Q' = 0$ the solution describes magnetic systems, which are called, according to work [15], spin cholesterics. Vector $\mathbf{b} = \alpha\mathbf{e} + \beta\mathbf{f}$ is orthogonal to spin axis $\mathbf{d}$. In biaxial case, when $Q \neq 0$, $Q' \neq 0$, $Q \neq Q'$ the quadrupolar order parameter describes dual spiral magnetic. Vector $\mathbf{b} = \alpha\mathbf{e} + \beta\mathbf{f}$ is orthogonal to spin axis $\mathbf{d}$. This situation corresponds to uniaxial quadrupolar order parameter with magnetic state of spin cholesteric.

When comparing the obtained equilibrium states of liquid crystals and magnetics with quadrupolar order parameter, it is necessary to note the obvious similarity of the results. We have homogeneous states of uniaxial and biaxial nematic and heterogeneous states of uniaxial cholesteric and dual spiral. But there are some differences, which lie in form of residual and spatial symmetry generators and are result of different transformational properties of order parameter operators of liquid crystals and magnetics. Further let us note the fact that, when applying the being developed approach to the problem of classification of equilibrium states of liquid crystals, we obtain equilibrium states, each of which was observed experimentally, but in case of quadrupolar magnetics a state of dual magnetic spiral was obtained, which was not discovered in experiment.

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