Classification of the equilibrium states of magnets with vector and quadrupole order parameters

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A classification of the equilibrium states of magnetic condensed media whose symmetry is broken spontaneously relative to rotations in spin space and translations in configuration space is constructed. The cases of vector and tensor order parameters, which possess different transformation properties under time reversal, are examined. The conditions for unbroken symmetry and spatial symmetry of the equilibrium states for such media are formulated. In the case of a vector order parameter, the connection between these symmetry conditions and para-, ferro-, antiferro-, ferri-, and spiral magnetic states is determined. The relation of the indicated symmetry conditions with quadrupole magnetic states is determined—one- and two-axis magnetic nonmagnetics, magnetic cholesterics, and a double magnetic spiral.

INTRODUCTION

Second-order phase transitions are accompanied by a change of symmetry of the equilibrium state of a system at a transition through a critical temperature. An adequate description of the equilibrium state and condensed media with broken symmetry below a critical temperature requires introducing into the theory additional thermodynamic parameters which are not associated with conservation laws but are due to the physical nature of the new phase. In accordance with the phenomenology of the classification of the equilibrium states of degenerate condensed media,[1] it is necessary to know the free energy in an explicit form as a function of the order parameter. Minimizing the free energy with respect to the order parameter and the equation arising in the process for the equilibrium structure of the order parameter is, in general, nonlinear. The group-theoretical approach[2,3] which does not depend on a specific mathematical model uses the idea of the unbroken symmetry of a degenerate equilibrium state as a subgroup of the symmetry of the normal phase.

In the present work the classification of the equilibrium states of the magnetic states is constructed on the basis of the conditions of unbroken and spatial symmetry with nonzero values of the order parameter. Magnets with vector and quadrupole order parameters are examined in detail. These parameters possess different parity under time reversal. In our approach the equations for the equilibrium values of the order parameter are found to be linear. A physical interpretation of the solutions obtained for the vector and quadrupole order parameters is given.

I. EQUILIBRIUM STATES OF NORMAL CONDENSED MEDIA

The equilibrium Gibbs statistical operator for magnetic media has the form

$$\hat{w} = \exp(\Omega(Y) - Y_a \hat{\gamma}_a), \tag{1}$$

where Y_a are in the thermodynamic forces, conjugate to the additive integrals of motion, $\hat{\gamma}_a \equiv (\hat{H}, \hat{P}_k, \hat{S}_\alpha)$, $a=0,k,\alpha$. Here $\hat{H}=\int d^3x \hat{\varepsilon}(x)$ is the Hamiltonian, $\hat{P}_k=\int d^3x \hat{\pi}_k(x)$, is the momentum, and $\hat{S}_\alpha=\int d^3x \hat{s}_\alpha(x)$ is the spin. The remaining integrals of motion are immaterial for studying the problem of the classifying the equilibrium states of magnetic media.

The operators appearing in the integrands are the densities of the additive integrals of motion. The thermodynamic potential $\Omega(Y)$ is determined from the normalization condition $\operatorname{Tr} \hat{w}=1$. The set of thermodynamic forces includes $Y_0^{-1} \equiv T$—the temperature, $-Y_k/Y_0 \equiv v_k$—the normal velocity, and $-Y_\alpha/Y_0 \equiv h_\alpha$—the effective magnetic field.

We shall now formulate the symmetry properties of the equilibrium state. The condition of spatial uniformity of the normal state has the form

$$[\hat{w}_n, \hat{P}_k] = 0. \tag{2}$$

The presence of an integral of motion—the spin moment operator—makes it possible to formulate a corresponding symmetry property. To this end we introduce a generalized spin moment operator:

$$\hat{\Sigma}_\alpha(Y) = \hat{S}_\alpha + \hat{S}^Y_\alpha, \quad \hat{S}^Y_\alpha \equiv -i\varepsilon_{\alpha\beta\gamma} Y_\beta \frac{\partial}{\partial Y_\gamma}. \tag{3}$$

This operator acts in Hilbert space and in the space of thermodynamic functions. The differential operator acts as follows on the vector Y_α: $i[\hat{S}^Y_\alpha, Y_\rho] = \varepsilon_{\alpha\beta\rho} Y_\beta$. On the basis of the known commutation relations

$$i[\hat{S}_\alpha, \hat{S}_\beta] = -\varepsilon_{\alpha\beta\gamma} \hat{S}_\gamma$$

and the explicit form of the operators (3) we obtain the quantum Poisson brackets for the operators (3):

$$i[\hat{\Sigma}_\alpha(Y),\hat{\Sigma}_\beta(Y)] = -\varepsilon_{\alpha\beta\gamma}\hat{\Sigma}_\gamma(Y).$$

Using this operator it is convenient to formulate the symmetry properties of the equilibrium statistical operator with respect to spin rotations

$$[\hat{w}_n,\hat{\Sigma}_\alpha(Y)] = 0. \tag{4}$$

The symmetry conditions with respect to rotations in spin space signify that the weak dipole and spin-orbital interactions are neglected in the Hamiltonian of the magnetic system. The symmetry group of the normal equilibrium state of the medium has the form

$$G = [SO(3)]_S \times [T(3)].$$

Here $[SO(3)]_S$ is the symmetry group with respect to rotations in spin space and $[T(3)]$ is the translation group in space. Each element of the group is a unitary operator $U \equiv \exp i\hat{G}g$ (g are real parameters of the transformation), leaving the Gibbs distribution invariant:

$$U\hat{w}_n U^+ = \hat{w}_n. \tag{5}$$

The generators of the transformations (5) are linear combinations of the operators $\hat{G} \in \{\hat{\Sigma}_\alpha, \hat{P}_k\}$. We call attention to the fact that the property of invariance (5) holds for arbitrary transformation parameters, conjugate to the integrals of motion by virtue of the symmetry conditions (2) and (4). For this reason, the averages $\mathrm{Tr}\,\hat{w}[\hat{G},\hat{b}(x)]$ vanish for any arbitrary quasilocal operator $\hat{b}(x)$. This is valid, in particular, for the operators $\hat{b}(x) \equiv \hat{\Delta}_a(x)$, which are order parameter operators and do not commute with the integrals motion $\hat{G}$. Index a reflects tensor dimension of the order parameter. The transformation properties of the order parameter of the magnets under study will be formulated in subsequent sections. Here it is sufficient to note that commutators of the type $[\hat{G},\hat{\Delta}_a(x)]$ that arise are linear and homogeneous in the order parameter operators $\hat{\Delta}_a(x)$. As a result, their equilibrium averages vanish:

$$\mathrm{Tr}\,\hat{w}_n\hat{\Delta}_a(x) = 0$$

in the normal state, i.e. in a state described by the Gibbs statistical operator (1).

The magnetic properties of media in the normal equilibrium state are characterized only by the value of the spin $s_\alpha = \mathrm{Tr}\,\hat{w}_n\hat{s}_\alpha(x)$. The equation $\varepsilon_{\alpha\beta\gamma}s_\gamma + \varepsilon_{\alpha\lambda\rho}Y_\lambda\,\partial s_\beta/\partial Y_\rho = 0$, whence follows $s_\alpha = sY_\alpha/Y$, $Y \equiv |\mathbf{Y}|$, which is valid on the basis of Eqs. (2) and (4). Thus, in this state the spin is directed along the vector $\mathbf{Y}$ and its magnitude in terms of the thermodynamic potential density

$$\omega(Y) = \lim_{V\to\infty}\Omega(Y)/V$$

has, evidently, the form $s(Y) = 2Y\,\partial\omega(Y)/\partial Y^2$. Here V is the volume of the system. The case considered here is the paramagnetic state of equilibrium of a condensed medium.

II. EQUILIBRIUM STATES OF DEGENERATE MAGNETIC MEDIA AND THE PROBLEM OF THEIR CLASSIFICATION

We shall now examine equilibrium states with spontaneously broken symmetry relative to rotations in spin space. In this case the statistical operator (1) does not describe correctly the equilibrium states of the condensed media under study, for which the equilibrium order parameter is different from zero. We shall use the concept of quasiaverages to construct a statistical theory of the equilibrium state of a condensed medium on the basis of the statistical theory.[4] The constructive aspect of this concept is the introduction into the equilibrium statistical operator of an infinitesimal source $\upsilon\hat{F}$, which decreases the symmetry of the state of statistical equilibrium as compared with the symmetry of the Hamiltonian and makes it possible to extend the Gibbs distribution to condensed media under conditions of spontaneous breaking of symmetry. According to Ref. 4, the quasiaverage quantity $a(x)$ in the state of statistical equilibrium with broken symmetry is given by the formula

$$a(x) = \langle\hat{a}(x)\rangle \equiv \lim_{\upsilon\to 0}\lim_{V\to\infty}\mathrm{Tr}\,\hat{w}_\upsilon\hat{a}(x),$$

where

$$\hat{w}_\upsilon = \exp(\Omega_\upsilon - Y_a\hat{\gamma}_a - \upsilon Y_0\hat{F}). \tag{6}$$

The source $\hat{F}$ possesses the symmetry of the phase of the condensed medium under study and lifts the degeneracy of the equilibrium state. Spontaneous breaking of the symmetry of the equilibrium state $[\hat{w},\hat{G}_0] \neq 0$, with $[\hat{H},\hat{G}_0] = 0$, occurs and therefore the equilibrium average of the order parameter is different from zero

$$\Delta_a(x) = \mathrm{Tr}\,\hat{w}\hat{\Delta}_a(x) \neq 0,$$

where $\hat{G}_0$ is a subset of the generators of the group $\hat{G}$ with respect to which symmetry is broken. The operator relation written out above should be understood in the sense of quasiaverages, i.e.

$$\lim_{\upsilon\to 0}\lim_{V\to\infty}\mathrm{Tr}\,\hat{w}_\upsilon[\hat{G}_0,\,\hat{a}(x)] \neq 0.$$

Here and below we employ the same notation for averages and quasiaverages. In accordance with the concept of quasiaverages, we choose the source $\hat{F}$, which breaks the symmetry of the equilibrium state, as a linear functional of the order parameter operator $\hat{\Delta}_a(x)$:

$$\hat{F} = \int d^3x(f_a(x)\hat{\Delta}_a(x) + \text{h.c.}), \tag{7}$$

where $f_a(x)$ is a function of the coordinates which is conjugate to the order parameter operator and gives its equilibrium value $\Delta_a(x) = \langle\hat{\Delta}_a(x)\rangle$. This modification of the Gibbs statistical operator makes it possible to introduce additional thermodynamic parameters.[5–8] We note that introducing a source $\hat{F}$ in general destroys the invariance of the equilibrium statistical operator under translations, i.e. $[\hat{w},\hat{P}_k] \neq 0$, and consequently a coordinate dependence of the order parameter can arise in this case. The specific spatial dependence of the order parameters will be found below when we examine the nonuniform magnetic states of equilibrium.

The order parameter operators $\hat{\Delta}_a(x)$ are definite local functions of the creation and annihilation field operators. We shall formulate the transformation properties of the order parameter operators. The condition of translational invariance has the form

$$i[\hat{P}_k, \hat{\Delta}_a(x)] = -\nabla_k \hat{\Delta}_a(x). \tag{8}$$

Under transformations associated with the group of spin rotations, whose generators are spin operators $\hat{S}_\alpha(\alpha=x,y,z)$, the order parameter operators $\hat{\Delta}_a(x)$ transform according to the representations of this group

$$i[\hat{S}_\alpha, \hat{\Delta}_a(x)] = -g_{\alpha ab}\hat{\Delta}_b(x), \tag{9}$$

where $(\hat{g}_\alpha)_{ab} \equiv g_{\alpha ab}$ are constants which depend on the tensor dimension of the order parameter operator. The generators $\hat{S}_\alpha$ of the spin symmetry group satisfy the relations

$$i[\hat{S}_\alpha, \hat{S}_\beta] = -\varepsilon_{\alpha\beta\gamma}\hat{S}_\gamma,$$

where $\varepsilon_{\alpha\beta\gamma}$ is the antisymmetric Levi–Civita tensor. Using Jacobi's identity for the operators $\hat{S}_\alpha$, $\hat{S}_\beta$, and $\hat{\Delta}(x)$ we obtain from Eqs. (9) the relation

$$[\hat{g}_\alpha, \hat{g}_\beta] = -\varepsilon_{\alpha\beta\gamma}\hat{g}_\gamma. \tag{10}$$

We shall now formulate the symmetry properties of the equilibrium state and introduce additional thermodynamic parameters for degenerate condensed media. First, we shall examine the translationally invariant subgroups of unbroken symmetry H of the complete symmetry group G. Translational invariance means that the equilibrium statistical operator satisfies the symmetry relation

$$[\hat{w}, \hat{P}_k] = 0. \tag{11}$$

We shall analyze the translationally invariant subgroups of unbroken symmetry of equilibrium states following Ref. 7 on the basis of the relations

$$[\hat{w}, \hat{T}(\mathbf{b}, Y)] = 0, \tag{12}$$

where the generator of unbroken (remaining) symmetry $\hat{T}(\mathbf{b}, Y)$ is a linear combination of the integrals of motion (generators of a subgroup H)

$$\hat{T}(b, Y) \equiv b_\alpha \hat{\Sigma}_\alpha(Y) \tag{13}$$

with a real parameter. We obtain from the equalities

$$i\,\mathrm{Tr}[\hat{w}, \hat{T}(\mathbf{b}, Y)]\hat{\Delta}_a(x) = 0, \quad i\,\mathrm{Tr}[\hat{w}, \hat{P}_k]\hat{\Delta}_a(x) = 0, \tag{14}$$

using the algebraic relations (8)–(11) and the definition (13), a system of linear differential equations which determine the structure of the order parameter in the equilibrium state:

$$b_\alpha\left(g_{\alpha ab}\Delta_b + \varepsilon_{\alpha\beta\gamma}Y_\beta \frac{\partial \Delta_a(x)}{\partial Y_\gamma}\right) = 0,$$

$$\nabla_k \Delta_a(x) = 0. \tag{15}$$

These equations simplify substantially if the vector $Y_\alpha = 0$. Then the first of the differential equations (15) becomes a system of linear algebraic equations for the order parameter

$$T_{ab}(\mathbf{b})\Delta_b = 0, \quad T_{ab}(\mathbf{b}) \equiv b_\alpha g_{\alpha ab}. \tag{16}$$

The condition for the existence of a nontrivial solution $\Delta_a \neq 0$ for the system of linear equations (16) leads to the equality

$$\det|T_{ab}(\mathbf{b})| = 0, \tag{17}$$

which imposes restrictions on the admissible values of the vector $\mathbf{b}$. Thus, in the translationally invariant case the equilibrium Gibbs statistical operator is a function of the thermodynamic parameters and the parameters of the generator of unbroken symmetry, which satisfy the relation (17): $\hat{w} = \hat{w}(Y, \mathbf{b})$.

We shall now examine the equilibrium states that do not possess the property of translational invariance (11). We shall define the spatial symmetry of such nonuniform equilibrium states by the relation

$$[\hat{w}, \hat{P}_k(q, Y)] = 0, \quad \hat{P}_k(q, Y) \equiv \hat{P}_k - q_{k\alpha}\hat{\Sigma}_\alpha(Y), \tag{18}$$

where $q_{k\alpha}$ are real parameters. The generator of unbroken symmetry of these states now includes the momentum operator

$$\hat{T}(\mathbf{b}, \mathbf{d}, Y) \equiv b_\alpha \hat{\Sigma}_\alpha(Y) + d_i \hat{P}_i. \tag{19}$$

The relations

$$i\,\mathrm{Tr}[\hat{w}, \hat{T}(\mathbf{b}, \mathbf{d}, Y)]\hat{\Delta}_a(x) = 0,$$

$$i\,\mathrm{Tr}[\hat{w}, P_k(q, Y)]\hat{\Delta}_a(x) = 0 \tag{20}$$

in accordance with Eqs. (8)–(13) lead to couplings of the parameters appearing in the definition of the generators of unbroken and spatial symmetry. These relations must be supplemented by two conditions on the parameters of the unbroken and spatial symmetries, which are a consequence of the Jacobi identities for the operators $\hat{w}, \hat{T}, \hat{P}_k(q, Y)$ and $\hat{w}, \hat{P}_i(q, Y)$, $\hat{P}_k(q, Y)$:

$$\mathrm{Tr}[\hat{w}, [\hat{T}(\mathbf{b}, \mathbf{d}, Y), \hat{P}_k(q, Y)]]\hat{\Delta}_a(x) = 0,$$

$$\mathrm{Tr}[\hat{w}, [\hat{P}_i(q, Y), \hat{P}_k(q, Y)]]\hat{\Delta}_a(x) = 0. \tag{21}$$

The equations (20) and (21) permit solving this problem of the classification of the equilibrium states of condensed media in the spatially nonuniform case. Thus, the equilibrium statistical operator is now a function of the thermodynamic parameters and the parameters of the generators of unbroken and spatial symmetry: $\hat{w} = \hat{w}(Y, \mathbf{b}, \mathbf{d}, q)$. In addition, the admissible values of the parameters $\mathbf{b}$, $\mathbf{d}$, and q are found from the relations (20) and (21).

III. MAGNETIC CONDENSED MEDIA WITH A VECTOR ORDER PARAMETER. UNIFORM STATES OF EQUILIBRIUM

We shall now examine condensed media with broken symmetry with respect to the spin operators characterized by a vector order parameter

$$\Delta_\alpha(x, \hat{w}) = \mathrm{Tr}\,\hat{w}\hat{\Delta}_\alpha(x) = \hat{\Delta}_\alpha(x, \hat{w}).$$

Here $\hat{\Delta}_\alpha(x)$ is a Hermitian order-parameter operator constructed in a linear manner from the spin operators of the sublattices $\hat{\Delta}_\alpha(x)=\hat{\Delta}_\alpha(x,\hat{\mathbf{s}}_n(x))$. This vector order parameter operator satisfies the operator equations

$$i[\hat{S}_\alpha,\hat{\Delta}_\beta(x)]=-\varepsilon_{\alpha\beta\gamma}\hat{\Delta}_\gamma(x),$$

$$i[\hat{P}_k,\hat{\Delta}_\alpha(x)]=-\nabla_k\hat{\Delta}_\alpha(x). \qquad (22)$$

Examples of such systems are multiple sublattice magnets and spin glasses.[9–15] In accordance with the concept of quasiaverages the equilibrium statistical operator of the magnetic systems studied here has the form

$$\hat{w}_v=\exp(\Omega(Y)-Y_0\hat{H}-Y_\alpha\hat{S}_\alpha-vY_0\hat{F})\equiv\hat{w}_\Delta,$$

$$\hat{F}=\int d^3x(f_\alpha(x)\hat{\Delta}_\alpha(x)+\text{h.c.}), \qquad (23)$$

where $\hat{H}$ is the Hamiltonian of the exchange interaction operator ($[\hat{H},\hat{S}_\alpha]=0$).

We now introduce discrete transformations of spatial reflection and time reversal and write out the corresponding transformation relations for the intervals of motion and the order parameter relative to these transformations:

$$1)\ x_i\to x_i'=-x_i,\quad t\to t'=t,$$

$$2)\ x_i\to x_i'=x_i,\quad t\to t'=-t.$$

The spatial reflection transformation and is defined by the quality

$$\hat{\psi}'(\mathbf{x},t)=\hat{\psi}(-\mathbf{x},t)=\hat{\mathrm{P}}\hat{\psi}(\mathbf{x},t)\hat{\mathrm{P}}^{+}. \qquad (24)$$

Hence follows $[\hat{\mathrm{P}}^2,\hat{\psi}(\mathbf{x})]=0$. Since the operator $\hat{\mathrm{P}}$ is determined to within a phase factor, and noting that an arbitrary operator which commutes with the operators $\hat{\psi}(\mathbf{x},t)$ and $\hat{\psi}^{+}(\mathbf{x},t)$ is a multiple of the unit to operator, it can be assumed that $\hat{\mathrm{P}}^2=1$ and therefore the eigenvalues of the operator $\hat{\mathrm{P}}$ are ± 1. The operator $\hat{\mathrm{P}}$ is called the spatial parity operator. In accordance with the expressions for the densities of the additive integrals of motion $\hat{\pi}_k(x)$ and $\hat{s}_\alpha(x)$

$$\hat{s}_i(x)=\hat{\psi}^{+}_\sigma(x)(s_i)_{\sigma\sigma'}\hat{\psi}_{\sigma'}(x),$$

$$\hat{\pi}_i(x)=-i\{\hat{\psi}^{+}_\sigma(x)\nabla_i\hat{\psi}_\sigma(x)-\nabla_i\hat{\psi}^{+}_\sigma(x)\hat{\psi}_\sigma(x)\}/2$$

in terms of the field operators and the relation (24) we obtain the transformation relations

$$\hat{\mathrm{P}}\hat{\zeta}_a(\mathbf{x})\hat{\mathrm{P}}^{+}=\underline{\varepsilon}_a\hat{\zeta}_a(-\mathbf{x}), \qquad (25)$$

where the factor

$$\underline{\varepsilon}_a\equiv+\delta_{a\alpha}-\delta_{ak}$$

is the spatial signature of the operator $\hat{\zeta}_a(\mathbf{x})$. Since the order parameter operator is linear in the spin operators, and on the basis of the relations (25), we obtain for the vector order parameter operator of the transformation relation

$$\hat{\mathrm{P}}\hat{\Delta}_\alpha(\mathbf{x})\hat{\mathrm{P}}^{+}=\hat{\Delta}_\alpha(-\mathbf{x}). \qquad (26)$$

We shall now examine the discrete time reversal transformation

$$t\to t'=-t.$$

This operation corresponds to the following transformation of the field operator $\hat{\psi}(\mathbf{x},t)$:

$$\hat{\psi}(\mathbf{x},t)\to\hat{\psi}'(\mathbf{x},t')=\mathrm{T}\hat{\psi}^{*}(\mathbf{x},t), \qquad (27)$$

where T is a unitary matrix ($\mathrm{TT}^{+}=1$), acting on the spin indices of $\hat{\psi}(\mathbf{x},t)$ and the asterisk "*" denotes complex conjugation. This operation depends on the choice of basis in Hilbert space. If a definite basis is chosen in Hilbert space, then in this basis the operation of complex conjugation is given by the relation

$$\langle n|\psi^{*}|n'\rangle=\langle n|\psi|n'\rangle^{*}.$$

Since the operators $\hat{\psi}(\mathbf{x},t)$ and $\hat{\psi}'(\mathbf{x},t)$ satisfy the same transposition relations, they are related with the unitary operator $\hat{\mathrm{T}}$ acting in Hilbert space:

$$\hat{\psi}'(\mathbf{x},t)=\hat{\mathrm{T}}\hat{\psi}(\mathbf{x},t)\hat{\mathrm{T}}^{+}=\mathrm{T}\hat{\psi}^{*}(\mathbf{x},-t). \qquad (28)$$

Knowing the expression for the operators of the additive integrals of motion in terms of the creation and annihilation field operators and using the quality (27) with $t=0$, it is easy to establish the validity of the relations

$$\hat{\mathrm{T}}\hat{\zeta}_a(\mathbf{x})\hat{\mathrm{T}}^{+}=\varepsilon_a\hat{\zeta}^{*}_a(-\mathbf{x}). \qquad (29)$$

Here

$$\varepsilon_a\equiv-\delta_{a\alpha}-\delta_{ak}$$

is the time signature on the Schrödinger operator $\hat{\zeta}_a(\mathbf{x})$. It is easy to see that $[\mathrm{T}^2,\hat{\psi}(\mathbf{x},t)]=0$.

Using the relations (28) and (29) and to take account of the linearity of the vector order parameter operator with respect to the spin density operators, we find the transformation law for the order-parameter operator under time reversal

$$\hat{\mathrm{T}}\hat{\Delta}_\alpha(\mathbf{x})\hat{\mathrm{T}}^{+}=-\hat{\Delta}^{*}_\alpha(\mathbf{x}). \qquad (30)$$

The relations (24)–(26) and (29) and (30) make it possible to obtain the transformation properties of the equilibrium Gibbs statistical operator (23) under discrete spatial reflection and time reversal operations

$$\hat{\mathrm{P}}\hat{\mathrm{T}}\hat{w}(Y_0,Y_\alpha,f_{\alpha\beta})(\hat{\mathrm{P}}\hat{\mathrm{T}})^{+}=\hat{w}^{*}(Y_0,-Y_\alpha,-f_{\alpha\beta}). \qquad (31)$$

For the spatially uniform states (11) the generator (13) of unbroken symmetry of magnets has the form

$$\hat{T}(\mathbf{b},Y)\equiv b_\alpha\hat{\Sigma}_\alpha(Y).$$

(To avoid any misunderstandings we call attention to the different notation used for the unbroken symmetry operator $\hat{T}(\mathbf{b},Y)$ and the unitary time reversal operator $\hat{\mathrm{T}}$.) It can be assumed with no loss of generality that the vector $\mathbf{b}$ is a unit vector: $b^2_\alpha=1$. On the basis of the unbroken symmetry condition (12) and the operator algebra (22) we obtain equations which determine the equilibrium structure of the spin and order parameter

$$b_\alpha\left(\varepsilon_{\alpha\beta\gamma}\Delta_\gamma+\varepsilon_{\alpha\lambda\rho}Y_\lambda\frac{\partial\Delta_\beta}{\partial Y_\rho}\right)=0,\quad \nabla_k\Delta_a(x)=0. \tag{32}$$

Since the equilibrium Gibbs statistical operator depends on the vectors $\mathbf{b}$ and $\mathbf{Y}$, $\hat{w}=\hat{w}(\mathbf{Y},\mathbf{b})$, the density of the thermodynamic potential is a function of two scalar invariants $\omega=\omega(\mathbf{Y},\mathbf{b})=\omega(Y^2,\mathbf{Yb})$. Consequently, the following equality holds:

$$s_\alpha=\frac{\partial\omega}{\partial Y^2}2Y_\alpha+\frac{\partial\omega}{\partial\mathbf{Yb}}b_\alpha. \tag{33}$$

Substituting the expression (33) into the relation (32), we see that Eqs. (32) hold identically for any directions of the vector $\mathbf{b}$.

1. If $\lim_{\mathbf{Y}\to 0}\partial\omega/\partial\mathbf{Yb}\neq 0$, then ferromagnetic ordering obtains for $\mathbf{b}\neq 0$, $\mathbf{Y}=0$.

2. The particular case where $\mathbf{b}\neq 0$, $\mathbf{Y}=0$, and $\lim_{\mathbf{Y}\to 0}\partial\omega/\partial\mathbf{Yb}\to 0$ corresponds to antiferromagnetic ordering.

3. The general situation where $\mathbf{b}\neq 0$ and $\mathbf{Y}\neq 0$ corresponds to ferrimagnetic ordering.

IV. NONUNIFORM EQUILIBRIUM STATES OF MAGNETIC MEDIA WITH A VECTOR ORDER PARAMETER

We shall now study the case where the spatial symmetry of the equilibrium state has a complicated spatial structure and is determined by the quality (18). Correspondingly, the generator of unbroken symmetry has the form (19). According to the symmetry conditions, we write the relations

$$i\,\mathrm{Tr}[\hat{w},\hat{T}(\mathbf{b},\mathbf{d},Y)]\hat{\Delta}_\beta(x)=0,\quad i\,\mathrm{Tr}[\hat{w},\hat{P}_k(q,Y)]\hat{\Delta}_\alpha(x)=0. \tag{34}$$

Hence follow equations which establish the equilibrium structure of the order parameter, and restrictions appear on the parameters of the unbroken symmetry and spatial symmetry generators. The condition for spatial symmetry, the form of the generator $\hat{P}_k$ (18), and the algebra (22) lead to the equations

$$\nabla_k\Delta_\beta(x)=\varepsilon_{\alpha\beta\gamma}q_{k\alpha}\Delta_\gamma(x),\quad t_{kj}\varepsilon_{j\mu\upsilon}q_{\upsilon\lambda}\varepsilon_{\lambda\beta\gamma}\Delta_\gamma(x)=0. \tag{35}$$

The second relation in Eq. (35) is a consequence of the requirement that there be no term in the spatial symmetry condition (18) and the first relation in (35) that is linear in the coordinate. The condition for unbroken symmetry in (34) yields, keeping in mind the form of the generator (19), the equations

$$(b_\alpha+d_iq_{i\alpha})\varepsilon_{\alpha\beta\gamma}\Delta_\gamma(x)=0,\quad a_i\varepsilon_{ikl}q_{l\alpha}\varepsilon_{\alpha\beta\gamma}\Delta_\gamma(x)=0. \tag{36}$$

Using the Jacob the identity, we shall establish additional relations between the parameters of the symmetry generators introduced by the relations (18) and (19). We arrive at the following equality for the operators $\hat{w}$, $\hat{T}$, $\hat{P}(q,Y)$, take account of the symmetry properties:

$$\mathrm{Tr}[\hat{w},[\hat{T}(\mathbf{b},\mathbf{d},Y),\hat{P}_k(q,Y)]]\hat{\Delta}_\beta(x)=0.$$

Hence we obtain, on the basis of the relations (18), (19), and (22), the relations

$$(b_\beta q_{i\rho}+b_\rho q_{i\beta})\varepsilon_{\rho\alpha\gamma}\Delta_\gamma(x)=0,$$

$$(a_it_{kl}-a_lt_{ki})q_{l\alpha}\varepsilon_{\alpha\beta\gamma}\Delta_\gamma(x)=0. \tag{37}$$

Using Jacobi's identity for the operators $\hat{w}$, $\hat{P}_i$, and $\hat{P}_k$ and the spatial symmetry property (18), we have the relation

$$\mathrm{Tr}[\hat{w},[\hat{P}_i(q,Y),\hat{P}_k(q,Y)]]\hat{\Delta}_\beta(x)=0.$$

Hence follow the equations

$$(t_{ij}t_{kl}-t_{il}t_{kj})q_{l\alpha}\varepsilon_{\alpha\beta\gamma}\Delta_\gamma(x)=0,$$

$$(q_{l\alpha}q_{i\beta}+q_{l\beta}q_{i\alpha})\Delta_\alpha(x)=0. \tag{38}$$

The system of equations (35)–(38) completely determines the admissible structure of the symmetry generator parameters and the form of the order parameter in equilibrium. It is not difficult to see that the solution of the system of equations can be represented in the form

$$\Delta_\beta(x)=a_{\beta\gamma}(\mathbf{n}\,\theta(x))\underline{\Delta}_\gamma(0),\quad \theta(x)=\theta+\mathbf{qx},$$

$$q_{i\alpha}=q_in_\alpha.$$

Here q_k is the vector of a magnetic spiral, n_α is the anisotropy axis in spin space, and the matrix t_{ik} is arbitrary. The orthogonal rotation matrix is related with the spin rotation angle by the relation

$$a_{\alpha\beta}(\theta(x))\equiv(\exp(\varepsilon\theta(x)))_{\alpha\beta}=\delta_{\alpha\beta}\cos\theta(x)+n_\alpha n_\beta(1-\cos\theta(x))+\varepsilon_{\alpha\beta\gamma}n_\gamma\sin\theta(x),$$

$$\theta_\alpha(x)=n_\alpha\theta(x),\quad n_\alpha^2=1. \tag{39}$$

The equilibrium values of the spin density $s_\alpha(x)$ depend on the coordinates: $s_\alpha(x)=a_{\alpha\beta}(x)s_\beta$. In addition, the entire spatial dependence is contained in the orthogonal rotation matrix $a_{\alpha\beta}(x)$, determined by Eqs. (39). Thus the state of equilibrium of a nonuniform magnetic system with spontaneous breaking of symmetry with respect to spin rotations and spatial translations is characterized by the thermodynamic forces Y_α, the magnetic spiral vector $\mathbf{q}$, and the spin rotation angle θ: $\hat{w}=\hat{w}(Y,\mathbf{q},\theta)$. Such nonuniform magnetic states are said to be spiral magnets.[16–20]

V. MAGNETIC CONDENSED MEDIA WITH A QUADRUPOLE ORDER PARAMETER. UNIFORM STATES OF EQUILIBRIUM

Quadrupole magnets are described by tensor order parameters

$$Q_{\alpha\beta}(x,\hat{w})=\mathrm{Tr}\,\hat{w}\hat{Q}_{\alpha\beta}(x).$$

Here $\hat{Q}_{\alpha\beta}(x)$ is a Hermitian order-parameter operator, which we have derived in a bilinear manner from the operators of the spin sublattices $\hat{\Delta}_a(x)=\hat{\Delta}_\alpha(x,\hat{s}_n(x))$:

$$\hat{Q}_{\alpha\beta}(x)=\frac{1}{2}\left(\hat{s}_{\alpha n}(x)\hat{s}_{\beta n}(x)+\hat{s}_{\beta n}(x)\hat{s}_{\alpha n}(x)-\frac{2}{3}\delta_{\alpha\beta}\hat{s}^2_{\gamma n}(x)\right). \tag{40}$$

This operator is symmetric and traceless

$$\hat{Q}_{\alpha\beta}(x)=\hat{Q}_{\beta\alpha}(x), \quad \hat{Q}_{\alpha\alpha}(x)=0.$$

According to the definition (40) the quadrupole order parameter operator satisfies the operator relations

$$i[\hat{S}_\alpha,\hat{Q}_{\beta\gamma}(x)]=-\varepsilon_{\alpha\beta\rho}\hat{Q}_{\gamma\rho}(x)-\varepsilon_{\alpha\gamma\rho}\hat{Q}_{\beta\rho}(x),$$

$$i[\hat{P}_k,\hat{Q}_{\alpha\beta}(x)]=-\nabla_k\hat{Q}_{\alpha\beta}(x). \tag{41}$$

We can see that the right-hand sides of the quantum Poisson brackets (41) are linear in the quadrupole order parameter operator. This has the consequence, as we shall see below, that the equations classifying the equilibrium states of these magnetic systems are linear.

The state of equilibrium of magnetic condensed medium is described by the Gibbs statistical operator (6). But now the source (7) in the Gibbs exponent has the form

$$\hat{F}=\int d^3x f_{\alpha\beta}(x)\hat{Q}_{\alpha\beta}(x).$$

We now shall examine the transformation property of the quadrupole order-parameter operator under a spatial reflection transformation (24). According to the formula (25) and the definition (40), we have

$$\hat{\mathrm{P}}\hat{Q}_{\alpha\beta}(\mathbf{x})\hat{\mathrm{P}}^{+}=\hat{Q}_{\alpha\beta}(-\mathbf{x}). \tag{42}$$

For a discrete time reversal transformation we obtain, by virtue of Eqs. (28), (29), and (40), the relation

$$\hat{\mathrm{T}}\hat{Q}_{\alpha\beta}(\mathbf{x})\hat{\mathrm{T}}^{+}=\hat{Q}^{*}_{\alpha\beta}(\mathbf{x}). \tag{43}$$

The relations (42) and (43) together with the relations (25) and (29) make it possible to find the transformation relations for reflection of the coordinates and time reversal for the Gibbs statistical operator:

$$\hat{\mathrm{P}}\hat{\mathrm{T}}\hat{w}(Y_0,Y_\alpha,f_{\alpha\beta})(\hat{\mathrm{P}}\hat{\mathrm{T}})^{+}=\hat{w}^{*}(Y_0,-Y_\alpha,f_{\alpha\beta}).$$

We can see that the character of the transformations of the equilibrium statistical operator depends strongly on the transformation properties of the symmetry breaking source. In particular, for $Y_\alpha=0$ the statistical operator is even under *TP* transformations:

$$\hat{\mathrm{P}}\hat{\mathrm{T}}\hat{w}(Y_0,f_{\alpha\beta})(\hat{\mathrm{P}}\hat{\mathrm{T}})^{+}=\hat{w}^{*}(Y_0,f_{\alpha\beta}). \tag{44}$$

The relations (44), (24), (25), (28), and (29) give the equalities

$$\mathrm{Tr}\,\hat{w}(Y_0,f_{\alpha\beta})\hat{s}_\alpha(x)=0, \quad \mathrm{Tr}\,\hat{w}(Y_0,f_{\alpha\beta})\hat{Q}_{\alpha\beta}(x)\neq 0.$$

The five independent components of the quadrupole order parameter can be parameterized by the relation

$$Q_{\alpha\beta}(x,\hat{w})=Q(x,\hat{w})\left(e_\alpha(x,\hat{w})e_\beta(x,\hat{w})-\frac{1}{3}\delta_{\alpha\beta}\right)+Q'(x,\hat{w})\left(f_\alpha(x,\hat{w})f_\beta(x,\hat{w})-\frac{1}{3}\delta_{\alpha\beta}\right).$$

Here Q and Q' are the moduli of the order parameter, and the vectors d_α, e_α, and f_α form an orthonormal basis:

$$\mathbf{de}=0, \quad \mathbf{df}=0, \quad \mathbf{ef}=0, \quad \mathbf{f}^2=1, \quad \mathbf{d}^2=1, \quad \mathbf{e}^2=1,$$
$$\mathbf{e}\times\mathbf{f}=\mathbf{d}, \quad \mathbf{f}\times\mathbf{d}=\mathbf{e}, \quad \mathbf{d}\times\mathbf{e}=\mathbf{f}. \tag{45}$$

It is easy to see that the vectors introduced above satisfy

$$e_\alpha e_\beta+f_\alpha f_\beta+d_\alpha d_\beta=\delta_{\alpha\beta}. \tag{46}$$

We shall now determine the generator of uniaxial spin symmetry and biaxial spin symmetry by the equalities

$$\hat{\Sigma}_\alpha(\mathbf{e})=\hat{S}_\alpha+\hat{S}^{\mathbf{e}}_\alpha, \tag{47}$$

$$\hat{\Sigma}_\alpha(\mathbf{e},\mathbf{f})=\hat{S}_\alpha+\hat{S}^{\mathbf{e}}_\alpha+\hat{S}^{\mathbf{f}}_\alpha. \tag{48}$$

It is easy to see that these operators satisfy

$$i[\hat{\Sigma}_\alpha(\mathbf{e}),\hat{\Sigma}_\beta(\mathbf{e})]=-\varepsilon_{\alpha\beta\gamma}\hat{\Sigma}_\gamma(\mathbf{e}),$$

$$i[\hat{\Sigma}_\alpha(\mathbf{e},\mathbf{f}),\hat{\Sigma}_\beta(\mathbf{e},\mathbf{f})]=-\varepsilon_{\alpha\beta\gamma}\hat{\Sigma}_\gamma(\mathbf{e},\mathbf{f}).$$

We now introduce the broken symmetry generators in accordance with the relations (18), (53), and (54) for the uniaxial and biaxial cases

$$\hat{T}\equiv b_\alpha\hat{\Sigma}_\alpha(\mathbf{e})=\hat{T}(\mathbf{b},\mathbf{e}),$$

$$\hat{T}\equiv b_\alpha\hat{\Sigma}_\alpha(\mathbf{e},\mathbf{f})=\hat{T}(\mathbf{b},\mathbf{e},\mathbf{f}). \tag{49}$$

In addition, we note that the derivatives related with the unit vectors e_α and f_α satisfy

$$\frac{\partial e_u}{\partial e_v}=\frac{\partial f_u}{\partial f_v}\equiv\delta_{uv}-e_ue_v-f_uf_v=d_ud_v. \tag{50}$$

We note that the relations (50) take account of the relations (45) for the vectors e_α and f_α. The relations (50) hold, as we shall see below, for biaxial magnetic systems. However, for analyzing uniaxial magnetic media which are characterized by a single spin vector come there are no relations of the type (45), and in this case the relations (50) are replaced by the relation

$$\frac{\partial e_u}{\partial e_v}=\delta_{uv}-e_ue_v\equiv\delta^{\perp}_{uv}(e). \tag{51}$$

The unbroken symmetry condition (12) together with the spatial uniformity condition (11) and the definitions (49) and the algebraic relations (41) lead to equations for the uniaxial case:

$$b_\alpha\left\{\varepsilon_{\alpha u\rho}Q_{\rho v}+\varepsilon_{\alpha v\rho}Q_{\rho u}+\varepsilon_{\alpha\beta\gamma}e_\beta\frac{\partial Q_{uv}}{\partial e_\gamma}\right\}=0,$$

$$\nabla_kQ_{uv}(x)=0. \tag{52}$$

We obtain similarly the equations for the biaxial case

$$b_\alpha\left\{\varepsilon_{\alpha u\rho}Q_{\rho v}+\varepsilon_{\alpha v\rho}Q_{\rho u}+\varepsilon_{\alpha\beta\gamma}e_\beta\frac{\partial Q_{uv}}{\partial e_\gamma}+\varepsilon_{\alpha\beta\gamma}f_\beta\frac{\partial Q_{uv}}{\partial f_\gamma}\right\}=0,$$

$$\nabla_kQ_{uv}(x)=0. \tag{53}$$

It is obvious that the solutions of Eqs. (52) and (53) are independent of the coordinates $Q_{uv}(x)=Q_{uv}(0)\equiv Q_{uv}$. We seek the solution of Eq. (53) in the form

$$Q_{uv}=Q\left(e_u e_v-\frac{1}{3}\delta_{uv}\right). \tag{54}$$

Using the relation (51), we find

$$\frac{\partial Q_{\alpha\beta}}{\partial e_v}=Q(\delta^{\perp}_{\alpha v}(\mathbf{e})e_\beta+e_\alpha\delta^{\perp}_{\beta v}(\mathbf{e})).$$

Substituting this expression into Eq. (52) we see that it holds identically for all values of the vector **b**. Following Ref. 21, magnetic states whose order parameter is of the form (54) will be termed uniaxial spin nematics.

We shall now find the solutions of Eqs. (53), which we shall seek in the form

$$Q_{uv}=Q\left(e_u e_v-\frac{1}{3}\delta_{uv}\right)+Q'\left(f_u f_v-\frac{1}{3}\delta_{uv}\right). \tag{55}$$

We note that according to (50)

$$\frac{\partial Q_{\alpha\beta}}{\partial e_v}=Qd_v(d_\alpha e_\beta+e_\alpha d_\beta),$$

$$\frac{\partial Q_{\alpha\beta}}{\partial f_v}=Q'd_v(d_\alpha f_\beta+d_\beta f_\alpha). \tag{56}$$

Substituting the expressions (56) and (57) into Eq. (54) we obtain the equation

$$b_\lambda\{Q[\varepsilon_{\lambda\alpha\rho}e_\beta e_\rho+\varepsilon_{\lambda\beta\rho}e_\rho e_\alpha-f_\lambda(d_\alpha e_\beta+d_\beta e_\alpha)]+Q'[\varepsilon_{\lambda\alpha\rho}f_\beta f_\rho+\varepsilon_{\lambda\beta\rho}f_\rho f_\alpha-e_\lambda(d_\beta f_\alpha+d_\alpha f_\beta)]\}=0. \tag{57}$$

To establish the admissible values of the vector **b** we seek it as an expansion in the orthonormal basis

$$\mathbf{b}=\alpha\mathbf{e}+\beta\mathbf{f}+\gamma\mathbf{d}, \tag{58}$$

where the numbers α, β, and γ are related by the relation $\alpha^2+\beta^2+\gamma^2=1$. Substituting the expression (58) into Eq. (57) gives

$$\gamma(Q-Q')(e_\beta f_\alpha+e_\alpha f_\beta)=0. \tag{59}$$

Hence follows that $\gamma=0$ for $Q\neq Q'$. This solution describes a biaxial spin nematic

$$Q_{uv}=Q\left(e_u e_v-\frac{1}{3}\delta_{uv}\right)+Q'\left(f_u f_v-\frac{1}{3}\delta_{uv}\right),$$

and the vector **b** has a form $b_\lambda=\alpha e_\lambda+\beta f_\lambda$, where $\alpha^2+\beta^2=1$.

Another solution is obtained from Eq. (59) if $Q=Q'$. In this case the vector **b** is arbitrary. On the basis up the relation (46) the order parameter acquires the form of a uniaxial spin of the nematic

$$Q_{uv}=-Q\left(d_u d_v-\frac{1}{3}\delta_{uv}\right).$$

Nonlinear dynamical equations were obtained in Ref. 22 for a magnetic system with a quadrupole order parameter:

$$\begin{cases}\dot{s}_\alpha=\varepsilon_{\alpha\beta\gamma}\left(\dfrac{\delta H}{\delta s_\beta}s_\gamma+2Q_{\mu\gamma}\dfrac{\delta H}{\delta f_{\mu\beta}}\right),\\ \dot{f}_{\alpha\beta}=\dfrac{1}{2}\dfrac{\delta H}{\delta f_{\gamma\nu}}(\delta_{\gamma\beta}\varepsilon_{\nu\mu\alpha}+\delta_{\gamma\beta}\varepsilon_{\gamma\beta\mu})s_\mu+(\varepsilon_{\beta\gamma\rho}Q_{\alpha\rho}+\varepsilon_{\alpha\gamma\rho}Q_{\beta\rho})\dfrac{\delta H}{\delta s_\gamma}.\end{cases} \tag{60}$$

Without loss of generality the energy density can be chosen in the following form:

$$\varepsilon=\frac{1}{2\chi}s_\alpha^2+\lambda s_\alpha Q_{\alpha\beta}s_\beta+\frac{1}{2}\mu Q_{\alpha\beta}^2. \tag{61}$$

We shall show that the equilibrium values found for the quadrupole order parameter and spin correspond to a stationary solution of the dynamical equations (60). Indeed, knowing the form of the energy density (61), we obtain the following equalities:

$$\left.\frac{\delta H}{\delta s_\alpha}\right|_{\mathbf{s}=0}=0,\qquad \left.\frac{\delta H}{\delta f_{\alpha\beta}}\right|_{\mathbf{s}=0}=\mu f_{\alpha\beta}. \tag{62}$$

Substituting the expressions (62) into Eq. (60) and taking account of the form of the quadrupole order parameter, it is easily shown that $\dot{s}_\alpha=0$ and $\dot{f}_{\alpha\beta}=0$, which correspond to the stationary case.

VI. NONUNIFORM EQUILIBRIUM STATES OF MAGNETIC MEDIA WITH A QUADRUPOLE ORDER PARAMETER

We shall now turn to nonuniform states of equilibrium (see formula (18)). The operators for unbroken and spatial symmetry have the form

$$\hat{T}(\mathbf{b},\mathbf{d},\mathbf{e},\mathbf{f})\equiv b_\alpha\hat{\Sigma}_\alpha(\mathbf{e},\mathbf{f})+d_i\hat{P}_i,$$

$$\hat{P}_k(q,\mathbf{e},\mathbf{f})\equiv\hat{P}_k-q_{k\alpha}\hat{\Sigma}_\alpha(\mathbf{e},\mathbf{f}). \tag{63}$$

The set of parameters of the unbroken symmetry generator includes the vectors **b**, **d**, **e**, and **f**. Since $[\hat{P}_i,\hat{P}_k]=0$ holds for the components of the momentum operator, we shall require that a similar relation hold for the components of the spatial symmetry generator:

$$[\hat{P}_k(q,\mathbf{e},\mathbf{f}),\hat{P}_k(q,\mathbf{e},\mathbf{f})]=0. \tag{64}$$

Taking account of the explicit form of the expression (63), we arrive at the relation

$$\varepsilon_{\alpha\beta\gamma}q_{i\alpha}q_{k\beta}=0,$$

which holds if the tensor $q_{i\alpha}$ possesses the structure

$$q_{i\alpha}=q_i n_\alpha,$$

where q_i is the magnetic spiral vector and n_α is the spin anisotropy axis. Substituting the latter expression into Eqs. (63), we obtain, according to Eq. (64), relations which determine the admissible form of the quadrupole order parameter and the vector **b**:

$$i\,\mathrm{Tr}[\hat{w},[\hat{T}(\mathbf{b},\mathbf{d},Y),\hat{P}_k(q,\mathbf{e},\mathbf{f})]]\hat{Q}_{\alpha\beta}(x)=0,$$

$$i\,\mathrm{Tr}[\hat{w},\hat{T}(\mathbf{b},\mathbf{d},Y)]\hat{Q}_{\alpha\beta}(x)=0,$$

$$i\,\mathrm{Tr}[\hat{w},\hat{P}_k(q,\mathbf{e},\mathbf{f})]\hat{Q}_{\alpha\beta}(x)=0.$$

Using the quantum brackets (41) we arrive at equations for the structure of the quadrupole order parameter and the admissible values of the vector:

$$b_\alpha F^{uv}_\alpha(x)=0,\quad (\mathbf{b}\times\mathbf{n})_\alpha F^{uv}_\alpha(x)=0,$$

$$\nabla_k Q_{uv}(x)=q_k n_\alpha F^{uv}_\alpha(x), \tag{65}$$

where

$$F^{uv}_\alpha(x)=\varepsilon_{\alpha u\rho}Q_{\rho v}(x)+\varepsilon_{\alpha v\rho}Q_{\rho u}(x)+\varepsilon_{\alpha\beta\gamma}e_\beta\frac{\partial Q_{uv}(x)}{\partial e_\gamma}+\varepsilon_{\alpha\beta\gamma}f_\beta\frac{\partial Q_{uv}(x)}{\partial f_\gamma}.$$

We shall show that the solution of the system of equations (65) is the following form of the quadrupole order parameter:

$$Q_{uv}(x)=Q\left(m_u(x)m_v(x)-\frac{1}{3}\delta_{uv}\right)+Q'\left(l_u(x)l_v(x)-\frac{1}{3}\delta_{uv}\right), \tag{66}$$

where the coordinate dependent vectors $\mathbf{m}(x)$ and $\mathbf{l}(x)$ are given by the equalities

$$\mathbf{m}(x)=\mathbf{e}\cos\varphi(x)+\mathbf{f}\sin\varphi(x),$$

$$\mathbf{l}(x)=-\mathbf{e}\sin\varphi(x)+\mathbf{f}\cos\varphi(x),$$

$$\varphi(x)=\varphi-\mathbf{qx} \tag{67}$$

and the vector **n** is collinear to **d**. First, we shall check the third relation in Eqs. (65). By virtue of the definition (67) the vectors $\mathbf{m}(x)$ and $\mathbf{l}(x)$ satisfy

$$\nabla_k m_u(x)=-q_k l_u(x),\quad \nabla_k l_u(x)=q_k m_u(x).$$

Consequently

$$\nabla_k Q_{uv}(x)=(Q'-Q)q_k(l_u(x)m_v(x)+l_v(x)m_u(x)). \tag{68}$$

On the other hand, since

$$\frac{\partial Q_{uv}(x)}{\partial e_\gamma}=Qd_\gamma(d_u m_v(x)+d_v m_u(x))\cos\varphi(x)-Q'd_\gamma(d_u l_v(x)+d_v l_u(x))\sin\varphi(x),$$

$$\frac{\partial Q_{uv}(x)}{\partial f_\gamma}=Qd_\gamma(d_u m_v(x)+d_v m_u(x))\sin\varphi(x)-Q'd_\gamma(d_u l_v(x)+d_v l_u(x))\cos\varphi(x),$$

we have

$$\varepsilon_{\alpha\beta\gamma}\left(e_\beta\frac{\partial Q_{uv}(x)}{\partial e_\gamma}+f_\beta\frac{\partial Q_{uv}(x)}{\partial f_\gamma}\right)=-Ql_\alpha(x)(d_u m_v(x)+d_v m_u(x))+Q'm_\alpha(x)(d_u l_v(x)+d_v l_u(x)). \tag{69}$$

In addition, we obtain on the basis of Eq. (66)

$$\varepsilon_{\alpha u\rho}Q_{v\rho}(x)+\varepsilon_{\alpha v\rho}Q_{u\rho}(x)=Q(\varepsilon_{\alpha u\rho}m_v(x)+\varepsilon_{\alpha v\rho}m_u(x))m_\rho(x)+Q'l_\rho(x)\times(\varepsilon_{\alpha u\rho}l_v(x)+\varepsilon_{\alpha v\rho}l_u(x)). \tag{70}$$

We find from the relations (69) and (70)

$$F^{uv}_\alpha(x)=Q(\varepsilon_{\alpha u\rho}m_v(x)+\varepsilon_{\alpha v\rho}m_u(x))m_\rho(x)+Q'l_\rho(x)\times(\varepsilon_{\alpha u\rho}l_v(x)+(\varepsilon_{\alpha u\rho}l_v(x))-Ql_\alpha(x)(d_u m_v(x)+d_v m_u(x))-Q'm_\alpha(x)(d_u l_v(x)+d_v l_v(x)).$$

Let the decomposition of the vector **b** in an orthonormal basis have the form

$$\mathbf{b}=\alpha\mathbf{e}+\beta\mathbf{f}+\gamma\mathbf{d}.$$

It is easy to see that

$$F^{uv}_\alpha(x)e_\alpha=F^{uv}_\alpha(x)f_\alpha=0,$$

$$F^{uv}_\alpha(x)d_\alpha=(Q'-Q)(l_u(x)m_v(x)+l_v(x)m_u(x)). \tag{71}$$

We seen from the relations formulas (68) and (71) that all three equations (65) are satisfied if $\gamma=0$ and $Q\neq Q'$.

In the uniaxial case, when, for example, $Q\neq 0$, $Q'=0$, the solution describes magnetic states which, according to Ref. 21, we shall call spin cholesterics. The vector $\mathbf{b}=\alpha\mathbf{e}+\beta\mathbf{f}$ is also orthogonal to the spin axis **d**.

In the two-axis case where $Q\neq 0$, $Q'\neq 0$, and $Q\neq Q'$ the quadrupole order parameter describes a magnetic state which is a double magnetic spiral. The vector $\mathbf{b}=\alpha\mathbf{e}+\beta\mathbf{f}$ is also orthogonal to the spin axis **d**.

If $Q=Q'$, then vector **b** is arbitrary, and this situation corresponds to a uniaxial quadrupole order parameter with the magnetic state of a spin cholesteric.

VII. CONCLUSIONS

In conclusion, we note that the problem of second-order phase transitions, ordinarily studied on the basis of phenomenological approaches, is initially model-dependent. The idea of unbroken and spatial symmetry of the equilibrium state within the framework of the Gibbs approach combined with the concept of quasiaverages makes it possible to formulate an alternative approach which is free of any model assumptions. A classification of the equilibrium states of magnetic media with vector and tensor order parameters was given above. The admissible structure of the order parameters in the equilibrium state and the form of the generators of unbroken and spatial symmetry were determined.

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[1]L. D. Landau, *Collected Works*, Nauka, Moscow (1969), Vol. 1.
[2]V. P. Mineev, Sov. Sci. Rev., Sect. A **2**, 173 (1985).
[3]F. W. Nijhoff, H. W. Capel, and A. den Breems, Physica A **130**, 375 (1985).
[4]N. N. Bogoliubov, Physica (Amsterdam) **26**, 51 (1960).
[5]A. I. Akhiezer and S. V. Peletminskiĭ, *Methods of Statistical Physics*, Nauka, Moscow (1977).
[6]N. N. Bogoliubov Jr., M. Yu. Kovalevskiĭ, A. M. Kurbatov, S. V. Peletminskiĭ, and A. N. Tarasov, Usp. Fiz. Nauk **159**, No. 4, 585 (1989).
[7]M. Yu. Kovalevskiĭ and S. V. Peletminskiĭ, Élement. Chastitsy Atom. Yad. **33**, 1357 (2002).
[8]A. P. Ivashin, M. Yu. Kovalevskiĭ, and N. N. Chekanova, Fiz. Nizk. Temp. **30**, 920 (2004) [Low Temp. Phys. **30**, 691 (2004)]
[9]B. I. Halperin and P. C. Hohenberg, Phys. Rev. **188**, 898 (1969).
[10]D. V. Volkov, A. A. Zheltukhin, and Yu. P. Bliokh, Fiz. Tverd. Tela (Leningrad) **13**, 1668 (1971) [Sov. Phys. Solid State **13**, 1396 (1971)].
[11]B. I. Halperin and W. M. Saslow, Phys. Rev. B **16**, 2154 (1977).
[12]A. F. Andreev and V. I. Marchenko, Usp. Fiz. Nauk **130**, 39 (1980).
[13]I. E. Dzyaloshinkii and G. E. Volovick, Ann. Phys. **125**, 67 (1980).
[14]V. G. Bar'yakhtar, V. G. Belykh, and T. K. Soboleva, Teor. Mat. Fiz. **77**, 311 (1988).
[15]M. Yu. Kovalevskiĭ and S. V. Peletminskiĭ, Teor. Mat. Fiz. **100**, 59 (1994).
[16]T. A. Kaplan, Phys. Rev. **124**, 329 (1961).
[17]V. G. Bar'yakhtar, M. A. Savchenko, and L. A. Shishkin, Fiz. Tverd. Tela (Leningrad) **6**, 1435 (1964).
[18]S. V. Tyablikov, *Methods of the Quantum Theory of Magnetism*, Nauka, Moscow (1975).
[19]I. E. Dzyaloshinskiĭ, Zh. Éksp. Teor. Fiz. **46**, 1420 (1964); I. E. Dzyaloshinskiĭ, *ibid.* **47**, 336 (1964).
[20]Yu. A. Izyumov, *Neutron Diffraction by Long-Period Structures*, Énergoatomizdat, Moscow (1987).
[21]A. F. Andreev and I. A. Grishchuk, Zh. Éksp. Teor. Fiz. **87**, 467 (1984) [Sov. Phys. JETP **87**, 267 (1984)].
[22]A. A. Isaev, M. Yu. Kovalevskiĭ, and S. V. Peletminskiĭ, Élement. Chastitsy Atom. Yad. **27**, 431 (1996).

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