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Uniqueness Criterion for the Solution of Boundary-Value Problems for the Abstract Euler–Poisson–Darboux Equation on a Finite Interval

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Abstract—For the abstract Euler–Poisson–Darboux equation, boundary-value problems with Dirichlet and Neumann conditions are considered. The criterion for the uniqueness of the solution is established.

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1. INTRODUCTION AND STATEMENT OF THE PROBLEM

Let E be a complex Banach space, and let A be a linear closed operator in E whose domain $D(A) \subset E$ is not necessarily dense in E. We will study boundary-value problems on a finite interval $0 < t < 1$, because the general domain $0 < t < T$ can be reduced to the one above by replacing the variable t by t/T. Due to the singularity of the equation at the point $t = 0$, the boundary conditions for the Euler–Poisson–Darboux equation

$$u''(t) + \frac{k}{t} u'(t) = Au(t), \qquad 0 < t < 1, \tag{1.1}$$

are posed depending on the parameter $k \in \mathbb{R}$; these conditions will be given below. The boundary conditions at $t = 1$ will be the same, namely,

$$\alpha u(1) + \beta u'(1) = u_1, \qquad u_1 \in E. \tag{1.2}$$

where $\alpha, \beta \in \mathbb{R}$, $\alpha^2 + \beta^2 > 0$, for all cases of the variation of the parameter k considered in what follows.

Boundary-value problems for Eq. (1.1) are generally ill-posed, but the need to solve ill-posed problems is now generally accepted (see the introduction to [1], as well as [2], [3] and their extensive bibliography). In the second chapter of the monograph [1], the degree of ill-posedness of general boundary-value problems for a first-order differential operator equation and for an abstract nonsingular second-order equation was investigated (case $k = 0$ in Eq. (1.1)).

We will present statements of various boundary-value problems for Eq. (1.1) depending on the parameter $k \in \mathbb{R}$ and establish the corresponding criteria for the uniqueness of their solutions. It will be shown that the uniqueness of solutions depends only on the location on the complex plane $\mathbb{C}$ of the eigenvalues of the operator A and is related to the distribution of zeros of certain analytic functions. Since some very general conditions are imposed on the operator A, we naturally do not consider the solvability of boundary-value problems in this paper. Let us only point out that results on the solvability of boundary-value problems in the half-space for the Euler–Poisson–Darboux partial differential equation were given in [4], and boundary-value problems on the semiaxis for abstract singular equations were studied in [5] and [6].

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An important role in establishing the uniqueness criterion will be played by the eigenvalue problem for the ordinary differential equation

$$v''(t) + \frac{k}{t}v'(t) = \lambda v(t), \qquad 0 < t < 1.$$

The structure of the general solution of this differential equation for all values of the parameter $k \in \mathbb{R}$ was specified in [7], and we will recall the results needed for our purposes in what follows.

2. THE CASE $k < 1$. THE DIRICHLET CONDITION FOR $t = 0$

Consider the problem of determining the function

$$u(t) \in C([0,1], E) \cap C^2((0,1], E) \cap C((0,1), D(A)),$$

satisfying the Euler–Poisson–Darboux equation (1.1), condition (1.2), and the Dirichlet boundary condition

$$u(0) = u_0, \qquad u_0 \in E. \tag{2.1}$$

The investigation of the uniqueness of the solution of problem (1.1), (1.2), (2.1) reduces to the question of the absence of nontrivial solutions $u(t)$ to Eq. (1.1) satisfying the zero conditions

$$u(0) = 0, \tag{2.2}$$

$$\alpha u(1) + \beta u'(1) = 0, \tag{2.3}$$

because this problem always has the zero solution $u(t) \equiv 0$.

We will search for nontrivial solutions $u(t)$ of the homogeneous problem (1.1), (2.2), (2.3) by the method of separating variables in the form $u(t) = v(t)h$, where $v(t) \in C[0,1] \cap C^2(0,1]$ is a nonzero scalar complex-valued function and $h \in D(A)$, $h \neq 0$.

Substituting $u(t) = v(t)h$ into problem (1.1), (2.2), (2.3), we obtain the equation

$$v''(t)h + \frac{k}{t}v'(t)h = v(t)Ah \tag{2.4}$$

and the conditions

$$v(0) = 0, \tag{2.5}$$

$$\alpha v(1) + \beta v'(1) = 0. \tag{2.6}$$

It follows from Eq. (2.4) that

$$Ah = \frac{v''(t) + k/t\, v'(t)}{v(t)}\,h; \tag{2.7}$$

this equality must hold on the set $\{t \in (0,1) : v(t) \neq 0\}$.

Obviously, equality (2.7) can only be valid if

$$Ah = \lambda h \tag{2.8}$$

with some constant $\lambda \in \mathbb{C}$.

Thus, by virtue of (2.8), the element $h \in D(A)$, $h \neq 0$, must be an eigenvector of the operator A with eigenvalue $\lambda \in \mathbb{C}$, and Eq. (2.4) will become

$$v''(t) + \frac{k}{t}v'(t) = \lambda v(t). \tag{2.9}$$

It was established in [7] that the general solution of the ordinary differential Eq. (2.9) is

$$v(t) = c_1 Y_k(t;\lambda) + c_2 t^{1-k} Y_{2-k}(t;\lambda), \qquad c_1, c_2 \in \mathbb{R}, \tag{2.10}$$

where $Y_k(t;A)$, $Y_k(0;A) = I$, is the solving operator of the Cauchy problem for Eq. (1.1) constructed in [7] or the Bessel operator function [8]. Concrete representations for $Y_k(t;\lambda)$ and $Y_{2-k}(t;\lambda)$ will be indicated in the proof.

Therefore, a solution of Eq. (2.9) satisfying the initial condition (2.5) has the form

$$v(t) = t^{1-k} Y_{2-k}(t;\lambda), \tag{2.11}$$

where, for $k < 1$,

$$Y_{2-k}(t;\lambda) = \Gamma\left(\frac{3}{2} - \frac{k}{2}\right)\left(\frac{t\sqrt{\lambda}}{2}\right)^{k/2-1/2} I_{1/2-k/2}(t\sqrt{\lambda}\,), \tag{2.12}$$

$\Gamma(\,\cdot\,)$ is the Euler gamma function and $I_\nu(\,\cdot\,)$ is the modified Bessel function. The scalar function $Y_{2-k}(t;\lambda)$ is also called the *normalized Bessel function* and denoted by $j_{1/2-k/2}(t\sqrt{\lambda}\,)$.

To find suitable eigenvalues $\lambda \in \mathbb{C}$, it remains to use the boundary condition (2.6); substituting into the function (2.11), we obtain the transcendental equation

$$(\alpha + \beta(1-k))Y_{2-k}(1;\lambda) + \beta Y'_{2-k}(1;\lambda) = 0. \tag{2.13}$$

Denoting $\sqrt{\lambda} = i\mu$ and taking into account representation (2.12), we write Eq. (2.13) in terms of the Bessel function $J_\nu(\,\cdot\,)$ of the first kind as

$$\frac{(\alpha + \beta(1-k)/2)J_{1/2-k/2}(\mu) + \beta\mu J'_{1/2-k/2}(\mu)}{\mu^{1/2-k/2}} = 0. \tag{2.14}$$

It is known (see [9, Sec. 18.3]) that Eq. (2.14) has an infinite set of positive roots μ_m, $m \in \mathbb{N}$, numbered in increasing order. Substituting $\lambda_m = -\mu_m^2$ into (2.11), we obtain the functions

$$v_m(t) = t^{1-k}(Y_{2-k}(t;\lambda_m), \qquad m \in \mathbb{N}, \tag{2.15}$$

which are nontrivial solutions of problem (2.9), (2.5), (2.6); at the same time, equality (2.8) becomes the following equation for finding $h_m \neq 0$:

$$Ah_m = \lambda_m h_m, \qquad m \in \mathbb{N}.$$

Let us further assume that, for some $m \in \mathbb{N}$, the number λ_m is an eigenvalue of the operator A with eigenvector $h_m \neq 0$. Then we find a nontrivial solution to the homogeneous problem (1.1), (2.2), (2.3) of the following form:

$$u_m(t) = t^{1-k} Y_{2-k}(t;\lambda_m) h_m. \tag{2.16}$$

Let us now formulate a uniqueness criterion for the solution of the boundary-value problem (1.1), (1.2), (2.1).

Theorem 1. *Let $k < 1$, and let A be a linear closed operator in E. Suppose that the boundary-value problem* (1.1)*,* (1.2)*,* (2.1) *has a solution $u(t)$. For this solution to be unique, it is necessary and sufficient that no zero λ_m, $m \in \mathbb{N}$, of the function*

$$\Upsilon_k^{\alpha,\beta}(\lambda) = (\alpha + \beta(1-k))Y_{2-k}(1;\lambda) + \beta Y'_{2-k}(1;\lambda) \tag{2.17}$$

be an eigenvalue of the operator A, i.e., $\lambda_m \notin \sigma_p(A)$.

Proof. As noted earlier, the study of the uniqueness of the solution of problem (1.1), (1.2), (2.1) reduces to the question of the absence of nontrivial solutions $u(t)$ to Eq. (1.1) satisfying the zero conditions (2.2), (2.3).

Necessity. Suppose the contrary; let some zero λ_m, $m \in \mathbb{N}$, from the countable set $\Lambda_k^{\alpha,\beta}$ of zeros of the function $\Upsilon_k^{\alpha,\beta}(\lambda)$ defined by Eq. (2.17), be an eigenvalue of the operator A with eigenvector $h_m \neq 0$, i.e., $\Lambda_k^{\alpha,\beta} \cap \sigma_p(A) \neq \phi$. Then the function $u_m(t)$ defined by (2.16) serves as a nontrivial solution to the homogeneous boundary-value problem (1.1), (2.2), (2.3), which contradicts the uniqueness of the solution of this problem, and hence necessity is proved.

We now prove *sufficiency*. Suppose that $\Lambda_k^{\alpha,\beta} \cap \sigma_p(A) = \phi$, and let $u(t)$ be a solution of the homogeneous boundary-value problem (1.1), (2.2), (2.3). We will show that, in this case, $u(t) \equiv 0$.

We introduce the function $U(\lambda)$ of a variable $\lambda \in \mathbb{C}$ with values in the Banach space E:

$$U(\lambda) = \int_0^1 tY_{2-k}(t;\lambda)u(t)\,dt, \tag{2.18}$$

where $Y_{2-k}(t;\lambda)$ is the scalar function defined by Eq. (2.12), is a solution of Eq. (2.9) with parameter $2-k$ instead of k, and satisfies the conditions $Y_{2-k}(0;\lambda) = I$ and $Y'_{2-k}(0;\lambda) = 0$.

Since the operator A is closed, taking into account equality (1.1), we calculate $AU_\delta(\lambda)$, where

$$U_\delta(\lambda) = \int_\delta^1 tY_{2-k}(t;\lambda)u(t)\,dt, \qquad \delta > 0.$$

Twice integrating by parts, we obtain

$$\begin{aligned} AU_\delta(\lambda) &= \int_\delta^1 tY_{2-k}(t;\lambda)Au(t)\,dt = \int_\delta^1 tY_{2-k}(t;\lambda)\Big(u''(t) + \frac{k}{t}u'(t)\Big)\,dt \\ &= tY_{2-k}(t;\lambda)u'(t)|_\delta^1 + \int_\delta^1 ((k-1)Y_{2-k}(t;\lambda) - tY'_{2-k}(t;\lambda))u'(t)\,dt \\ &= tY_{2-k}(t;\lambda)u'(t)|_\delta^1 + ((k-1)Y_{2-k}(t;\lambda) - tY'_{2-k}(t;\lambda))u(t)|_\delta^1 \\ &\quad + \int_\delta^1 t\Big(Y''_{2-k}(t;\lambda) + \frac{2-k}{t}Y'_{2-k}(t;\lambda)\Big)u(t)\,dt. \end{aligned}$$

Letting $\delta \to 0$, we obtain

$$AU(\lambda) = Y_{2-k}(1;\lambda)u'(1) + \big((k-1)Y_{2-k}(1;\lambda) - Y'_{2-k}(1;\lambda)\big)u(1) + \lambda U(\lambda). \tag{2.19}$$

Let, for example, $\beta \neq 0$ in the boundary condition (2.3). (The case $\alpha \neq 0$ is considered in a similar way.) Then equality (2.19) will take the form

$$(\lambda I - A)U(\lambda) = \frac{1}{\beta}\big((\alpha + \beta(1-k))Y_{2-k}(1;\lambda) + \beta Y'_{2-k}(1;\lambda)\big)u(1). \tag{2.20}$$

Thus, for all numbers $\lambda_m \in \Lambda_k^{\alpha,\beta}$ from the countable set of zeros of the function $\Upsilon_k^{\alpha,\beta}(\lambda)$ defined by equality (2.17), Eq. (2.20) implies the relation

$$AU(\lambda_m) = \lambda_m U(\lambda_m).$$

By assumption, none of these numbers λ_m is an eigenvalue of the operator A. But, in that case, all the values of $U(\lambda_m)$ must be zero:

$$U(\lambda_m) = 0, \qquad m \in \mathbb{N}. \tag{2.21}$$

Let μ_m, $m \in \mathbb{N}$, be the positive roots of Eq. (2.14) numbered in increasing order, and let $(i\mu_m)^2 = \lambda_m$. Then equalities (2.21) take the form

$$U_m = \int_0^1 t^{k/2+1/2}J_{1/2-k/2}(t\mu_m)u(t)\,dt = 0, \qquad m \in \mathbb{N}. \tag{2.22}$$

Applying a linear continuous functional $f \in E^*$ to the vector coefficients U_m defined by equality (2.22), we obtain the scalar function $\varphi(t) = f(t^{k/2-1/2}u(t))$ satisfying the conditions

$$f(U_m) = \int_0^1 tJ_{1/2-k/2}(t\mu_m)\varphi(t)\,dt = 0, \qquad m \in \mathbb{N}. \tag{2.23}$$

The further arguments depend on the coefficients α, β, and k.

(a) If the coefficients α, β, and k in the boundary condition (2.3) satisfy

$$\frac{\alpha}{\beta} + 1 - k > 0, \tag{2.24}$$

then, up to a multiplier, the scalar coefficients $f(U_m)$ are the coefficients of the Dini series (see [9, Sec. 18.3])

$$\sum_{m=1}^{\infty} b_m J_{1/2-k/2}(t\mu_m) \tag{2.25}$$

of the expansion of the function $\varphi(t)$ in the functions $J_{1/2-k/2}(t\mu_m)$.

The function $\varphi(t)$ is completely defined by the coefficients of its Dini series regardless of whether this series converges; therefore, $\varphi(t) \equiv 0$, $0 \le t \le 1$. Since the choice of the functional $f \in E^*$ was arbitrary, we have $u(t) \equiv 0$, $0 \le t \le 1$.

(b) If

$$\frac{\alpha}{\beta} + 1 - k = 0, \tag{2.26}$$

then, to the Dini expansion (2.25), we must additionally add (see [9, Sec. 18.3]) the summand $b_0 t^{1/2-k/2}$, where $b_0 = 2f(U_0)$ for U_0 determined by (2.22) with $\mu_0 = 0$.

In this case, given equality (2.26) and the boundary condition (2.3), we obtain

$$\varphi(t) = b_0 t^{1/2-k/2}, \qquad \alpha\varphi(1) + \beta\varphi'(1) = b_0 \frac{\beta(k-1)}{2} = 0.$$

Since $\beta \ne 0$, $k < 1$, we have $b_0 = 0$ and $\varphi(t) \equiv 0$, $u(t) \equiv 0$, $0 \le t \le 1$.

(c) Let

$$\frac{\alpha}{\beta} + 1 - k < 0. \tag{2.27}$$

In this case, to the Dini expansion (2.25), we must additionally add (see [9, Sec. 18.3]) the summand $b_0 I_{1/2-k/2}(t\mu_0)$, where the $\pm i\mu_0$ ($\mu_0 > 0$) are two purely imaginary roots of Eq. (2.14), and then, given the boundary condition (2.3), we can write

$$\varphi(t) = b_0 I_{1/2-k/2}(t\mu_0),$$

$$\alpha\varphi(1) + \beta\varphi'(1) = b_0(\alpha I_{1/2-k/2}(\mu_0) + \beta\mu_0 I'_{1/2-k/2}(\mu_0)) = b_0 \frac{\beta(k-1)}{2} I_{1/2-k/2}(\mu_0) = 0.$$

The modified Bessel function $I_{1/2-k/2}(\mu_0)$ does not vanish at real values of μ_0, so again $b_0 = 0$ and $\varphi(t) \equiv 0$, $u(t) \equiv 0$, $0 \le t \le 1$.

Thus, it is established that the solution $u(t)$ of the homogeneous problem (1.1), (2.2), (2.3) and, hence, of problem (1.1)–(2.1) can only be zero.

(d) If, finally, $\beta = 0$, then the proof is similar to the arguments in item (a) in which, instead of the Dini series (2.25), where the μ_m are the roots of Eq. (2.14), we use the Fourier–Bessel series constructed from the zeros of the function $J_{1/2-k/2}(\mu)$. □

For $k < 1$, it is convenient to determine the zeros of the function $\Upsilon_k^{\alpha,\beta}(\lambda)$ from the equality $\lambda_m = -\mu_m^2$, where μ_m are the roots of Eq. (2.14). For example, if $k = 0$, then Eq. (2.14) takes the form

$$\alpha \frac{\sin\mu}{\mu} + \beta\cos\mu = 0.$$

In some cases, the roots of this equation can be explicitly calculated; in particular, for $\alpha = 0$, we have $\mu_m = \pi/2 + \pi m$, $m \in \mathbb{N}$, and hence $\Lambda_0^{0,\beta} = \{-(\pi/2 + \pi m)^2,\ m \in \mathbb{N}\}$, and if $\beta = 0$, then $\mu_m = \pi m$, $m \in \mathbb{N}$, and $\Lambda_0^{\alpha,0} = \{-(\pi m)^2,\ m \in \mathbb{N}\}$.

Thus, in order to solve the question of the uniqueness of the solution of the boundary-value problem under consideration, one must determine the eigenvalues of the operator A and find out whether they belong to the set $\Lambda_k^{\alpha,\beta}$ of zeros of the function $\Upsilon_k^{\alpha,\beta}(\lambda)$. Numerous examples of finding eigenvalues for differential operators A acting on spatial variables can be found, for example, in [10, Chap. 2]; in each concrete case, they must be compared with the zeros of the function $\Upsilon_k^{\alpha,\beta}(\lambda)$.

We will consider an *example* of a singular operator A acting on the spatial variable x. For the differential Bessel operator $A = B_{q,x}$, where

$$B_{q,x} = \frac{d^2}{dx^2} + \frac{q}{x}\frac{d}{dx}, \qquad q > 0,$$

given on the set of functions

$$D(A) = H^2(0,1) \cap H_0^1(0,1) \subset E = L_2(0,1),$$

the problem of the uniqueness of the solution of the boundary-value problems under consideration for the hyperbolic equation reduces to the study of the location of the zeros of the function $I_{q/2-1/2}(\sqrt{z}\,)$, which are the eigenvalues of the operator $B_{q,x}$ and to that of the zeros of the function $\Upsilon_k^{\alpha,\beta}(\lambda)$ defined by equality (2.17).

In particular, for $\beta = 0$, $k < 1$, it is necessary to investigate the location of the zeros of the functions $I_{q/2-1/2}(\sqrt{z}\,)$ and $I_{1/2-k/2}(\sqrt{\lambda}\,)$. Depending on the parameters k and q, these Bessel functions may or may not have common zeros located on $(-\infty, 0)$, and hence the uniqueness of the solution of boundary-value problems may or may not take place. For more information about the location of the zeros of Bessel functions, see, for example, item 2 in [11]. Also note that the ranges of variation of the variables $0 < t < T$ and $0 < x < l$ also play an important role in the study of uniqueness, because the position of the zeros of each one of the Bessel functions depends on them. Similar facts for the solution of the Dirichlet problem for hyperbolic partial differential equations were established earlier in [12].

In the cases $A = -B_{q,x}$ and $A = iB_{q,x}$, where i is the imaginary unit, the eigenvalues of the operator A lie either on $(0, +\infty)$ or on the imaginary axis and do not fall in $(-\infty, 0)$, and, therefore, the corresponding boundary-value problems have a unique solution.

3. THE CASE $k \geq 0$. THE NEUMANN WEIGHT CONDITION FOR $t = 0$

It is seen from the representation (2.10) for the general solution of Eq. (2.9) that Eq. (1.1) can also have unbounded solutions at $t = 0$ if, instead of the Dirichlet condition, we set the Neumann weight condition. The method for proving uniqueness proposed in Sec. 2 makes use of Dini and Fourier–Bessel series expansions, so, setting the boundary condition at $t = 0$ must be such that the general solution of the ordinary differential Eq. (2.9) has only one of the constants c_1 and c_2 equal to zero. The first such option was implemented in Sec. 2 ($c_1 = 0$, $c_2 = 1$). Another option is a Neumann weight boundary condition of the form

$$\lim_{t \to 0+} t^k u'(t) = u_2 \in E. \tag{3.1}$$

In this case, the solution of Eq. (2.9) satisfying the homogeneous initial condition (3.1) has the form ($c_1 = 1$, $c_2 = 0$)

$$v(t) = Y_k(t;\lambda),$$

where, for $k \geq 0$,

$$Y_k(t;\lambda) = \Gamma\left(\frac{k}{2} + \frac{1}{2}\right)\left(\frac{t\sqrt{\lambda}}{2}\right)^{1/2-k/2} I_{k/2-1/2}(t\sqrt{\lambda}\,). \tag{3.2}$$

The further scheme for establishing a uniqueness criterion is similar to that of Sec. 2, but, instead of relations (2.13), (2.14), (2.16), and (2.18), the following equalities must be used, respectively:

$$\begin{gathered} \alpha Y_k(1;\lambda) + \beta Y_k'(1;\lambda) = 0, \\ \frac{(\alpha + \beta(1-k)/2) J_{k/2-1/2}(\mu) + \beta\mu J'_{k/2-1/2}(\mu)}{\mu^{k/2-1/2}} = 0, \\ u_m(t) = Y_k(t;\lambda_m) h_m, \qquad U(\lambda) = \int_0^1 t^k Y_k(t;\lambda) u(t)\, dt. \end{gathered} \tag{3.3}$$

As a result, we arrive at the following uniqueness criterion.

Theorem 2. *Let $k \ge 0$, and let A be a linear closed operator in E. Suppose that the boundary-value problem* (1.1), (1.2), (3.1) *has a solution $u(t)$. For this solution to be unique, it is necessary and sufficient that no zero λ_m, $m \in \mathbb{N}$, of the function*

$$\Upsilon_k^{\alpha,\beta}(\lambda) = \alpha Y_k(1;\lambda) + \beta Y_k'(1;\lambda) \tag{3.4}$$

be an eigenvalue of the operator A.

The zeros of the function $\Upsilon_k^{\alpha,\beta}(\lambda)$ defined by equality (3.4) are conveniently determined from the equality $\lambda_m = -\mu_m^2$, where the μ_m are the roots of Eq. (3.3).

If $k = 0$, then Eq. (3.3) takes the form

$$\alpha \cos\mu - \beta\mu \sin\mu = 0.$$

In particular, for $\beta = 0$, we have $\mu_m = \pi/2 + \pi m$ and $\lambda_m = -\mu_m^2$ for $m \in \mathbb{N}$, and if $\alpha = 0$, then $\mu_0 = \lambda_0 = 0$, $\mu_m = \pi m$, and $\lambda_m = -(\pi m)^2$ for $m \in \mathbb{N}$.

Note that, in determining the unique solvability of boundary-value problems, the presence of the zero $\lambda_0 = 0$ of the function $\Upsilon_k^{\alpha,\beta}(\lambda)$ necessarily leads to the existence of the inverse operator A^{-1}.

If $k = 2$, then Eq. (3.3) takes the form

$$\frac{(\alpha - \beta)\sin\mu + \beta\mu\cos\mu}{\mu} = 0.$$

In particular, for $\beta = 0$, we have $\mu_m = \pi m$ and $\lambda_m = -(\pi m)^2$, $m \in \mathbb{N}$, and if $\alpha = \beta \neq 0$, then $\mu_m = \pi/2 + \pi m$ and $\lambda_m = -\mu_m^2$, $m \in \mathbb{N}$.

4. THE CASE $0 \le k < 1$

For the specified values of the parameter k, both Theorem 1 with the Dirichlet condition for $t = 0$ and Theorem 2 with the Neumann weight condition for $t = 0$ are valid.

5. THE CASE $-1 < k < 0$. THE NEUMANN WEIGHT CONDITION FOR $t = 0$. THE NEUMANN OR DIRICHLET CONDITION FOR $t = 1$

Along with the Dirichlet condition for $t = 0$, which was the subject of study in Sec. 2, we will also consider the Neumann weight condition. Unlike in Sec. 3, for the negative values of the index $-1 < k < 0$, we must use (see [7]) the following representation of the function $v(t) = Y_k(t;\lambda)$ via the Bessel functions with positive index instead of (3.2):

$$\begin{aligned} v(t) = Y_k(t;\lambda) &= Y_{k+2}(t;\lambda) + \frac{t}{k+1} Y_{k+2}'(t;\lambda) \\ &= \frac{\Gamma(k/2+3/2)}{(t\sqrt{\lambda})^{k/2+1/2}} \biggl(\frac{1}{2} I_{k/2+1/2}(t\sqrt{\lambda}\,) + \frac{t\sqrt{\lambda}}{k+1} I_{k/2+1/2}'(t\sqrt{\lambda}\,)\biggr). \end{aligned} \tag{5.1}$$

To find suitable eigenvalues $\lambda \in \mathbb{C}$, we use the boundary condition (2.6); substituting the function (5.1) into this condition, we obtain the transcendental equation

$$\biggl(\alpha + \frac{\beta\lambda}{k+1}\biggr) Y_{k+2}(1;\lambda) + \frac{\alpha}{k+1} Y_{k+2}'(1;\lambda) = 0,$$

or

$$(\sqrt{\lambda}\,)^{-k/2-1/2}\biggl(\biggl(\frac{\alpha}{2} + \frac{\beta\lambda}{k+1}\biggr) I_{k/2+1/2}(\sqrt{\lambda}\,) + \frac{\alpha\sqrt{\lambda}}{k+1} I_{k/2+1/2}'(\sqrt{\lambda}\,)\biggr) = 0. \tag{5.2}$$

Due to the presence of the multiplier λ in front of the function $I_{k/2+1/2}(\sqrt{\lambda}\,)$, the roots of Eq. (5.2) are, in general, not related to the Fourier–Bessel and Dini series expansions, except for the cases $\alpha = 0$ and $\beta = 0$. It is these cases that we will be considered further.

5.1. The Neumann Condition for $t = 1$

If $\alpha = 0$, then the boundary condition (1.2) becomes a Neumann condition for $t = 1$:

$$u'(1) = u_1, \qquad u_1 \in E, \tag{5.3}$$

and Eq. (5.2) takes the form

$$\lambda^{3/4-k/4} I_{k/2+1/2}(\sqrt{\lambda}\,) = 0,$$

or, for $\lambda = -\mu^2$, the form

$$\mu^{3/2-k/2} J_{k/2+1/2}(\mu) = 0. \tag{5.4}$$

We denote the roots of Eq. (5.4) by $\mu_0 = 0$ and $\mu_m > 0$, $m \in \mathbb{N}$, and set $\lambda_m = -\mu_m^2$.

The further scheme used to establish the uniqueness criterion is similar to the proof of Theorem 1, (d) or Theorem 1, (a). As a result, we obtain the following uniqueness criterion.

Theorem 3. *Let $-1 < k < 0$, and let A be a linear closed operator in E. Suppose that the boundary-value problem* (1.1), (3.1), (5.3) *has a solution $u(t)$. For this solution to be unique, it is necessary and sufficient that no zero λ_m, $m = 0, 1, 2\dots$, of the function*

$$\Upsilon_k^{0,\beta}(\lambda) = \lambda Y_{k+2}(1; \lambda)$$

be an eigenvalue of the operator A.

5.2. The Dirichlet Condition for $t = 1$

If $\beta = 0$, then the boundary condition (1.2) becomes a Dirichlet condition for $t = 1$:

$$u(1) = u_1, \qquad u_1 \in E, \tag{5.5}$$

and Eq. (5.2) takes the form

$$\frac{(k+1) I_{k/2+1/2}(\sqrt{\lambda}\,) + 2\sqrt{\lambda}\, I'_{k/2+1/2}(\sqrt{\lambda}\,)}{\lambda^{k/4+1/4}} = 0,$$

or, for $\lambda = -\mu^2$, the form

$$\frac{(k+1) J_{k/2+1/2}(\mu) + 2\mu J'_{k/2+1/2}(\mu)}{\mu^{k/2+1/2}} = 0. \tag{5.6}$$

We denote the positive roots of Eq. (5.6) by μ_m, $m \in \mathbb{N}$, and set $\lambda_m = -\mu_m^2$. Just as in the proof of Theorem 1, (a), we obtain the following criterion.

Theorem 4. *Let $-1 < k < 0$, and let A be a linear closed operator in E. Suppose that the boundary-value problem* (1.1), (3.1), (5.5) *has a solution $u(t)$. For this solution to be unique, it is necessary and sufficient that no zero λ_m, $m = 0, 1, 2\dots$, of the function*

$$\Upsilon_k^{\alpha,0}(\lambda) = Y_{k+2}(1; \lambda) + \frac{1}{k+1} Y'_{k+2}(1; \lambda)$$

be an eigenvalue of the operator A.

6. SINGULAR SOBOLEV-TYPE EQUATION

The results of the previous sections concerning the uniqueness criterion can be generalized to the case of a singular Sobolev-type equation

$$B\left(u''(t) + \frac{k}{t} u'(t)\right) = Au(t), \qquad 0 < t < 1, \tag{6.1}$$

where, as well as A, B is a linear closed operator on E whose domain $D(B) \subset E$ is not necessarily dense in E.

The scheme of proof of these results is similar. The distinctive feature is the replacement of equality (2.8) $Ah = \lambda h$, which determines the eigenvalues of the operator A, by the operator equation $Ah = \lambda Bh$ if it has nontrivial solutions, as well as the replacement of the spectrum $\sigma_p(A)$ by the spectrum $\sigma_p(B, A)$ of the the operator A with respect to B. In addition, the definition of the solution must naturally additionally include the requirement that the operators belong to the space $C^2((0, 1), D(B))$.

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