Riemann–Hilbert Problems for the System of String Vibration Equations in a Domain with a Characteristic Part of Boundary

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Abstract—For the simplest hyperbolic system of string vibration equations in a special admissible domain whose boundary contains a characteristic segment of the system, we consider two problems of Riemann–Hilbert type and find existence and uniqueness conditions for the generalized solution of these problems in weighted classes of continuous functions.

1. STATEMENTS OF PROBLEMS

Consider the simplest hyperbolic system of string vibration equations,

$$u_x = -v_y, \qquad u_y = -v_x. \tag{1}$$

Let D be a domain convex with respect to the characteristics $x \pm y = c$, $c = \text{const}$, of system (1); i.e., the sets $D \cap \{x \pm y = c\}$ are connected for any number c. (The empty set is considered to be connected.) Suppose that the boundary Γ of the domain D consists of smooth arcs $\Gamma_i = [\tau, \tau_i]$, $i = 1, 2$, with the common point τ and a segment $l = [\tau_1, \tau_2]$ of one of the characteristics, which is denoted by $l = l_1$ for the characteristic $x + y = c$ and $l = l_2$ for $x - y = c$. We assume that the arcs Γ_i, $i = 1, 2$, are not tangent to characteristics of system (1). The numbering of the arcs Γ_i is chosen so as to ensure that the rotation from the arc Γ_2 to the arc Γ_1 around the point τ in the domain D is counterclockwise.

In the domain D, we consider two problems of the Riemann–Hilbert type given by the following boundary conditions.

Problem $\mathbf{R}^+$: $(au + bv)_{\Gamma\setminus l} = f$.

Problem $\mathbf{R}^-$: $(au + bv)_{\Gamma\setminus l} = f$. $(u \pm v)|_l = h$.

Here the upper sign in the assumptions of Problem $\mathbf{R}^-$ corresponds to the case $l = l_1$ and the lower sign, to $l = l_2$. The coefficients a and b are continuous functions on each of the arcs Γ_i, and we assume that $a^2 + b^2 \neq 0$ everywhere on Γ_i, $i = 1, 2$. The right-hand side h in the assumptions of Problem $\mathbf{R}^-$ is also assumed to be a continuous function on l.

Riemann–Hilbert problems were studied by numerous authors. A system equivalent to the equation of string vibrations with a Riemann–Hilbert boundary condition was studied in [1], where an existence and uniqueness condition for that problem was obtained. Vakhaniya [2] developed Sobolev's idea and reduced the problem in question to the Dirichlet problem. A similar system with an irrational ratio of sides and a Riemann–Hilbert condition was studied in [3]. Soldatov [4] analyzed the case of a characteristic part of the domain boundary for the Lavrent'ev–Bitsadze system. By using the approach in [4], in the present paper, we consider the hyperbolic case of this system and find conditions for the unique solvability of the two posed Riemann–Hilbert problems in a specially defined weighted function classes.

For functions $u, v \in C^1(D)$, system (1) is equivalent to the fact that

$$u \pm v = \text{const} \quad \text{on} \quad D \cap \{x \mp y = \text{const}\}. \tag{2}$$

A pair of functions $u, v \in C(\bar{D}\backslash\{\tau\})$ satisfying this property is called a generalized solution of system (1).

We construct a generalized solution of Problems $\mathbf{R}^+$ and $\mathbf{R}^-$ in the weighted classes $C_\lambda\left(\bar{D}, \tau\right)$ of continuous functions on $\bar{D}\backslash\{\tau\}$. We introduce these classes as follows. Let λ be an arbitrary real number. Then, by definition, functions $u(z) = u(x, y)$ of the complex variable $z = x + iy$ belong to the class $C_\lambda\left(\bar{D}, \tau\right)$ if they can be represented in the form $u(z) = |z - \tau|^\lambda u_0(z)$, where $u_0(z)$ belongs to $C\left(\bar{D}\backslash\{\tau\}\right)$ and is bounded in the domain D. The space $C_\lambda(\Gamma, \tau)$ has a similar meaning for functions defined on the boundary $\partial D = \Gamma$.

2. PROBLEM $\mathbf{R}^+$

Suppose that

$$(a - b)(t) \neq 0 \quad \text{for any} \quad t \in \Gamma_1, \quad (a + b)(t) \neq 0 \quad \text{for each} \quad t \in \Gamma_2, \tag{3}$$

including the limit values $(a - b)(\tau, \Gamma_1)$ and $(a + b)(\tau, \Gamma_2)$. Here and throughout the following, $c(\tau, \Gamma_i)$ is treated as the value of a function c that is continuous on each of the arcs Γ_i, $i = 1, 2$, at the point τ from the side of the arc Γ_i.

In accordance with condition (3), one can introduce the special weight order λ_+ by the relation

$$\lambda_+ = -\frac{\ln|q_+|}{\ln|\tan\theta_2/\tan\theta_1|}, \qquad q_+ = \left(\frac{a+b}{a-b}\right)(\tau, \Gamma_1)\left(\frac{a-b}{a+b}\right)(\tau, \Gamma_2), \tag{4}$$

where θ_k is the angle between the tangent to the curve Γ_k at the point τ and the characteristic $x + y = 0$. If $q_+ = 0$, then we set $\lambda_+ = -\infty$.

Theorem 1. *Let condition* (3) *be satisfied, and let* $\lambda > \lambda_+$. *Then Problem* $\mathbf{R}^+$ *is uniquely solvable in the class* $C_\lambda\left(\bar{D}, \tau\right)$.

Proof. Let the boundary of the domain D consist of a segment $l = l_1$ of the characteristic $x + y = c$ with endpoints τ_1 and τ_2 and smooth arcs $\Gamma_i = [\tau, \tau_i]$, $i = 1, 2$. The case $l = l_2$ will be considered below. Let $\alpha_1 : \Gamma_2 \to \Gamma_1$ be a shift along the characteristic $x + y = c$ such that points $t \in \Gamma_2$ and $\alpha_1(t) \in \Gamma_1$ lie on the characteristic $x + y = c$ for some value c. By virtue of the above-described choice of the numbering of the arcs Γ_i, there exists an interior point τ_+ on the arc Γ_2 such that the shift α_2 acting along the characteristic $x - y = c$ brings the arc Γ_1 to $\Gamma_2' = [\tau, \tau_+]$, where $\tau_+ = \alpha_2(\tau_1)$. Obviously, the point τ is fixed under these shifts.

One can readily show that the shifts α_i, $i = 1, 2$, are continuously differentiable and the derivative α_i' with respect to the arc length parameter is nonzero everywhere. This follows from the fact that the arcs Γ_i are nontangent to the characteristics of system (1). The composition $\alpha = \alpha_2 \circ \alpha_1$ of these shifts will be called the contraction shift, since $\Gamma_2' = \alpha(\Gamma_2) \subseteq \Gamma_2$. By virtue of the above-mentioned properties of the shifts α_i, the shift operator $T_\alpha\varphi = \varphi \circ \alpha$ maps the class $C_\lambda(\Gamma_2)$ into the class $C_\lambda(\Gamma_2')$.

We reduce the boundary condition of Problem $\mathbf{R}^+$ to the form

$$(a + b)(u + v) + (a - b)(u - v) = 2f$$

and write it out for each of the arcs Γ_i. Then, by taking into account the fact that, by virtue of (2), $(u + v)(\alpha_1(t)) = (u + v)(\alpha(t))$ and $(u - v)(\alpha_1(t)) = (u - v)(t)$ and by setting $c_2 = (a + b)(t)$, $d_2 = (a - b)(t)$, $c_1 = (a + b)(\alpha_1(t))$, and $d_1 = (a - b)(\alpha_1(t))$, we reduce Problem $\mathbf{R}^+$ for the function $\varphi(t) = (u + v)(t)$, $t \in \Gamma_2$, by an equivalent transformation to the functional equation

$$\varphi - q(\varphi \circ \alpha) = g, \tag{5}$$

where

$$g = 2\frac{fd_1 - (f \circ \alpha_1)\, d_2}{c_2 d_1}, \qquad q = \frac{c_1 d_2}{c_2 d_1} \in C(\Gamma_2).$$

By evaluating the coefficient q at the point τ, we find that it coincides with the value q_+ defined in (4); i.e., $q(\tau) = q_+$.

Equation (5) corresponds to the operator $1-A$, where $A = qT_\alpha$. By [5], this operator is invertible in the class $C_\lambda(\Gamma_2, \tau)$ provided that

$$|q_+||\alpha'(\tau)|^\lambda < 1.$$

Therefore, if this condition is satisfied, then Problem $\mathbf{R}^+$ is uniquely solvable. It remains to show that this condition is equivalent to the inequality $\lambda > \lambda_+$.

This assertion is obvious for $q_+ = 0$. From geometric viewpoint, it is clear that

$$|\alpha'(\tau)| = \frac{\tan\theta_2}{\tan\theta_1}$$

if $|q_+| \neq 0$. When computing the value of λ_+, we take the last quantity in the absolute value, since we choose any of the angles formed by the tangents to the curves Γ_1 and Γ_2 and the characteristics $x + y = 0$.

If the boundary of the domain D contains a segment l_2 of a characteristic from the family $x - y = c$, considerations can be performed in a similar way. As α one should consider the composition $\alpha_1 \circ \alpha_2$ of the above-defined shifts. Then α maps the arc Γ_1 into $\Gamma_1' = [\tau_+, \tau_1]$, where $\tau_+ = \alpha_1(\tau_2)$. For the function $\varphi(t) = (u-v)(t)$, $t \in \Gamma_1$, Problem $\mathbf{R}^+$ can also be reduced to Eq. (5), but, in this case, we have

$$g = 2\frac{fc_2 - (f \circ \alpha_2)\, c_1}{c_2 d_1}, \qquad q = \frac{c_1 d_2}{c_2 d_1} \in C(\Gamma_1),$$

where we have set $c_2 = (a+b)(\alpha_2(t))$, $d_2 = (a-b)(\alpha_2(t))$, $c_1 = (a+b)(t)$, and $d_1 = (a-b)(t)$. Then one should repeat the above-performed considerations. The proof of the theorem is complete.

3. PROBLEM $\mathbf{R}^-$

Suppose that

$$(a+b)(t) \neq 0 \quad \text{for any} \quad t \in \Gamma_1, \qquad (a-b)(t) \neq 0 \quad \text{for each} \quad t \in \Gamma_2, \tag{6}$$

including the limit values $(a+b)(\tau, \Gamma_1)$ and $(a-b)(\tau, \Gamma_2)$.

Just as in the case of Problem $\mathbf{R}^+$, we introduce the critical weight order λ_- by the formula

$$\lambda_- = -\frac{\ln|q_-|}{\ln|\tan\theta_1/\tan\theta_2|}, \qquad q_- = \left(\frac{a+b}{a-b}\right)(\tau, \Gamma_2)\left(\frac{a-b}{a+b}\right)(\tau, \Gamma_1).$$

If $q_- = 0$, then one should set $\lambda_- = +\infty$.

Theorem 2. *Let condition (6) be satisfied, let $\lambda < \lambda_-$, and let the right-hand sides of the boundary value problem $\mathbf{R}^-$ be related by the matching condition*

$$A_1 f(\tau_1) - A_2 f(\tau_2) - B_1 h(\tau_1) + B_2 h(\tau_2) = 0, \tag{7}$$

where

$$A_1 = 2(a-b)(\tau_2), \qquad A_2 = 2(a-b)(\tau_1),$$
$$B_1 = (a+b)(\tau_1)(a-b)(\tau_2), \qquad B_2 = (a-b)(\tau_1)(a+b)(\tau_2)$$

for $l = l_1$ and

$$A_1 = 2(a+b)(\tau_2), \qquad A_2 = 2(a+b)(\tau_1),$$
$$B_1 = (a-b)(\tau_1)(a+b)(\tau_2), \qquad B_2 = (a+b)(\tau_1)(a-b)(\tau_2)$$

for $l = l_2$. Then Problem $\mathbf{R}^-$ is uniquely solvable in the class $C_\lambda(\bar{D}, \tau)$.

Proof. Let the boundary of the domain D contain a segment $l = l_1 = [\tau_1, \tau_2]$ of the characteristic $x + y = c$. The case of a segment of a characteristic of another family will be considered below. Consider the composition $\alpha = \alpha_2 \circ \alpha_1$ of shifts defined as in Theorem 1. We have the shift $\alpha_1 : \Gamma_1' \to \Gamma_2'$, where $\Gamma_1' = [\tau, \tau_-]$, $\tau_- = \alpha^{-1}(\tau_1)$, $\Gamma_2' = [\tau, \tau_+]$, $\tau_+ = \alpha_2^{-1}(\tau_1)$, acting along the characteristics $x + y = c$ and the shift $\alpha_2 : \Gamma_2' \to \Gamma_1$ along the characteristics $x - y = c$. The shift operator $T_\alpha \varphi = \varphi \circ \alpha$ maps the class $C_\lambda(\Gamma_1')$ into the class $C_\lambda(\Gamma_1)$. If we consider this shift on the entire Γ_1, then $\alpha(\Gamma_1) \supseteq \Gamma_1$. Such a shift will be called an *extension shift*.

By writing out the boundary condition of Problem $\mathbf{R}^-$ on each of the arcs Γ_i just as in Theorem 1 and by setting $c_2 = (a+b)(\alpha_1(t))$, $d_2 = (a-b)(\alpha_1(t))$, $c_1 = (a+b)(t)$, and $d_1 = (a-b)(t)$, we find that, for the function $\varphi(t) = (u+v)(t)$, $t \in \Gamma_1$, this problem can be reduced in an equivalent way to the system of functional equations

$$\varphi - q(\varphi \circ \alpha) = g \quad \text{on} \quad \Gamma_1', \qquad \varphi - q(h \circ \alpha) = g \quad \text{on} \quad \Gamma_1 \backslash \Gamma_1', \tag{8}$$

where

$$g = 2\frac{f_1 d_2 - (f_2 \circ \alpha_1)\, d_1}{c_1 d_2}, \qquad q = \frac{c_2 d_1}{c_1 d_2} \in C(\Gamma_1).$$

By evaluating q at the point τ, we find that it coincides with the above-defined quantity q_-.

System (8) can be represented in the form of a single equation similar to (5):

$$\varphi - q(\varphi \circ \alpha) = G, \tag{9}$$

where

$$G = \begin{cases} g & \text{on} \ \ \Gamma_1' \\ g + q(h \circ \alpha) & \text{on} \ \ \Gamma_1 \backslash \Gamma_1', \end{cases}$$

and the function G has a simple discontinuity at the point τ_-.

Let $u_1, v_1 \in C_\lambda(\bar{D}, \tau)$ be functions such that $\varphi_1(t) = (u_1 + v_1)(t)$ at points lying on l_1 is some function $h_1(t)$; moreover, $h_1(\tau_1) = h(\tau_1)$. Such functions can always be found. Consider the transformation $\tilde{u} = u - u_1$, $\tilde{v} = v - v_1$. Then $\tilde{\varphi} = (\tilde{u} + \tilde{v}) \in C_\lambda(\Gamma_1, \tau)$. Let the relation $\tilde{\varphi} = \tilde{h}$ be valid at points lying on l_1, where $\tilde{h}$ is some function of points of the characteristic segment l_1; then $\tilde{\varphi}(\tau_1) = 0$.

Under the substitution $\varphi = \tilde{\varphi} + \varphi_1$, the functional equation (9) acquires the form

$$\begin{aligned} &\tilde{\varphi} - q(\tilde{\varphi} \circ \alpha) = \tilde{G}, \\ &\tilde{G} = \begin{cases} g - \varphi_1 + q(\varphi_1 \circ \alpha) & \text{on} \ \ \Gamma_1' \\ g - \varphi_1 + q(\varphi_1 \circ \alpha) + q(\tilde{h} \circ \alpha) & \text{on} \ \ \Gamma_1 \backslash \Gamma_1'. \end{cases} \end{aligned} \tag{10}$$

By writing out (10) at the point τ_1, we obtain the matching condition (7) in the form

$$0 = g(\tau_1) - h(\tau_1) + q(\tau_1)\, h(\tau_2).$$

Therefore, the function $\tilde{G}$ is zero at the point τ_1. By evaluating the function $\tilde{G}$ from the left and right at the point τ_-, we find that it is continuous at this point by virtue of the choice of the functions u_1 and v_1.

Obviously, the functional equation (10) is solvable if $1 - qT_\alpha$, where

$$(T_\alpha \tilde{\varphi})(t) = \begin{cases} (\tilde{\varphi} \circ \alpha)(t) & \text{for} \quad t \in \Gamma_1' \\ 0 & \text{for} \quad t \in \Gamma_1 \backslash \Gamma_1', \end{cases}$$

is an invertible operator. As to be shown in Lemma 1, a sufficient condition for the invertibility of this operator in the class $C_\lambda(\Gamma_1, \tau)$ and the solvability of Problem $\mathbf{R}^-$ is given by the inequality

$$|q_-| |\alpha'(\tau)|^\lambda < 1. \tag{11}$$

Following the proof of Theorem 1, we evaluate the special weight order and find that if $|q_-| \neq 0$, then the solvability condition (11) is equivalent to the inequality $\lambda < \lambda_-$. If $q_- = 0$, then $\lambda_- = +\infty$. The value of $|\alpha'(\tau)|$ can be evaluated just as in Theorem 1.

If the boundary of the domain D contains a segment l_2 of a characteristic from the family $x - y = c$, then considerations can be performed in a similar way. For the shift α one should take the composition $\alpha_1 \circ \alpha_2$ of the above-defined shifts. Then α maps the arc $\Gamma_2' = [\tau, \tau_-]$ into Γ_2, where $\tau_- = \alpha^{-1}(\tau_2)$. For the function $\varphi(t) = (u - v)(t)$, $t \in \Gamma_2$, Problem $\mathbf{R}^-$ can be reduced to the system of functional equations (8) on Γ_2' and $\Gamma_2 \backslash \Gamma_2'$, respectively, but in this case, we have

$$g = 2\,\frac{fc_1 - (f \circ \alpha_2)c_2}{c_1 d_2}, \qquad q = \frac{c_2 d_1}{c_1 d_2} \in C\left(\Gamma_2\right),$$

where $c_2 = (a+b)(t)$, $d_2 = (a-b)(t)$, $c_1 = (a+b)\left(\alpha_2(t)\right)$, and $d_1 = (a-b)\left(\alpha_2(t)\right)$. Then one should repeat the above-performed considerations. The proof of the theorem is complete.

Lemma 1. *Let Γ be a given smooth arc with endpoints τ and τ_1, and let $\Gamma_0 = [\tau, \tau_0] \subseteq \Gamma$, where τ_0 is an interior point. Let $a \in C(\Gamma)$, let $\alpha \in C^1(\Gamma)$, and let α implement a homeomorphism of the arc Γ_0 on Γ. Suppose that $|\alpha'(\tau)| \neq 0$, $\alpha(t) \neq t$ for $t \in \Gamma_0$, $t \neq \tau$, and $a(t) = 0$ on $\Gamma \backslash \Gamma_0$. If*

$$|a(\tau)||\alpha'(\tau)|^\lambda < 1, \tag{12}$$

then the operator $1 - A$, where

$$(A\varphi)(t) = \begin{cases} a(t)(\varphi \circ \alpha)(t) & \text{for} \quad t \in \Gamma_0 \\ 0 & \text{for} \quad t \in \Gamma \backslash \Gamma_0, \end{cases}$$

is invertible in the class $C_\lambda(\Gamma, \tau)$.

Proof. The shift $\alpha(t)$ described in the lemma is an extension shift, since $\alpha(\Gamma) \supseteq \Gamma$. Consider the weight transformation $\varrho_\lambda(t) = |t - \tau|^\lambda$, which realizes an isomorphism of the space $C_0(\Gamma)$ onto $C_\lambda(\Gamma)$ by the rule $\varphi_0 \to \varphi = \varrho_\lambda \varphi_0$. Then for the operator $1 - A$, we have

$$((1 - A)\varphi)(t) = \varrho_\lambda(t)\left[\varphi_0(t) - a(t)\frac{|\alpha(t) - \tau|^\lambda}{|t - \tau|^\lambda}\varphi_0(\alpha(t))\right], \qquad t \in \Gamma_0.$$

Therefore, it suffices to consider the operator $1 - A_0$, where $A_0 = a_0(\varphi \circ \alpha)$ with the coefficient

$$a_0(t) = a(t)\frac{|\alpha(t) - \tau|^\lambda}{|t - \tau|^\lambda}$$

in the space $C_0(\Gamma)$. Condition (12) for the invertibility of the operator $1 - A_0$ can be represented in the form $|a_0(\tau)| < 1$.

Consider the product

$$\left(A_0^k \varphi\right)(t) = a_0(t)\, a_0(\alpha(t))\, a_0\left(\alpha_2(t)\right) \cdots\, a_0\left(\alpha_{(k-1)}(t)\right)\, \varphi\left(\alpha_k(t)\right), \qquad t \in \Gamma_0.$$

By using the transformation $\gamma : [0, 1] \to \Gamma$, one can reduce conditions on the curve Γ to the interval $[0, 1]$ and analyze the invertibility of the operator $1 - A_0$ in the class $C_0([0, 1], 0)$. In what follows, α is treated as the composition $\gamma^{-1} \circ \alpha \circ \gamma$ and a as the composition $\gamma^{-1} \circ a \circ \gamma$.

Consider the composition of shifts $\alpha_{(n)} = \underbrace{\alpha \circ \alpha \circ \cdots \circ \alpha}_{n}$. In the case of an extension shift, $\alpha(s) > s$ for any $s \in [0, 1]$. Then $\alpha_{(n+1)} > \alpha_{(n)}$ for each $n \in \mathbb{N}$, and $\alpha_{(n)} \to 0$ as $n \to -\infty$.

Consider the inverse shift: $\alpha^{-1}(s) = t$ whenever $\alpha(t) = s$, $s \in [0, 1]$. Since

$$a(t)\varphi(\alpha(t)) = \varphi(s) a\left(\alpha^{-1}(s)\right)$$

for any $t \in [0, \tau_0]$, it follows that, by setting $\alpha_{(k-1)}(t) = s$, one can rewrite the product $\left(A_0^k\varphi\right)(t)$ in the form

$$\left(A_0^k\varphi\right)\left(\alpha_{(k-1)}^{-1}(s)\right) = a_0\left(\alpha_{(k-1)}^{-1}(s)\right) a_0\left(\alpha_{(k-2)}^{-1}(s)\right) \cdots a_0(s)\,\varphi(\alpha(s))$$

and obtain the case described in [5], because $\alpha_{(n)}^{-1} \to 0$ as $n \to +\infty$. The subsequent proof is similar to that in [5]. The proof of the lemma is complete.

If $a^2 - b^2 \neq 0$ everywhere on Γ_i, $i = 1, 2$, then both of the above-introduced numbers q_+ and q_- are well defined; moreover, $q_- = 1/q_+$. The type of the problem to be solved for a given domain D depends on the value of the product

$$|q_+||\tan\theta_2/\tan\theta_1|^{\lambda}\,. \tag{13}$$

If this value is less than unity, then Theorem 1 is valid, and Problem $\mathbf{R}^+$ is to be solved. If the quantity (13) exceeds unity and condition (9) is satisfied, then Problem $\mathbf{R}^-$ is to be solved in the domain D.

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