Linear groups of degree four containing the subgroup $SL_2(k) \otimes SL_2(k)$

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A B S T R A C T

Let a field K be an algebraic extension of a field k. Assume that the characteristic of k is different from 2 and k contains more than 9 elements. In the article, we study subgroups of the general linear group $GL_4(K)$ that contain the subgroup $SL_2(k) \otimes SL_2(k)$.

1. Introduction

Let K be a field and n, r natural numbers such that $n \geq 2r$. Every matrix x which belongs to the general linear group $GL_n(K)$ and is conjugate in $GL_n(K)$ to the matrix

$$\operatorname{diag}\Bigg(\underbrace{\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \dots, \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}}_{r \text{ times}}, \underbrace{1, \dots, 1}_{n-2r \text{ times}}\Bigg) = \begin{pmatrix} 1 & 1 & & & & & & \\ 0 & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & 1 & & & \\ & & & 0 & 1 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}$$

satisfies the equation $(x - 1)^2 = 0$ and therefore is called a *quadratic unipotent element* of the group $GL_n(K)$ or more precisely, a *quadratic unipotent element of residue* r of this group. As is well known, the most important and interesting classical matrix groups are generated by quadratic unipotent elements which are contained in these groups. It is this reason that the problem of describing linear groups that contain quadratic unipotent elements or generated by them is critical for the study of the subgroup structure of linear groups over various fields. Also, it is quite natural that one should specialize groups considered in order to get sensible results in this area. The most challenging restriction of this kind leading to the creation of a deep and ramified theory is that the groups under consideration must contain not only individual unipotent elements but an entire group of them. Specifically, if k is a subfield of the field K, then by a *k-subgroup of quadratic unipotent elements of residue* r we understand a subgroup of $GL_n(K)$ that is conjugate in $GL_n(K)$ to the subgroup consisting of matrices

$$\mathrm{diag}\left(\underbrace{\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix},\ldots,\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}}_{r\text{ times}},\underbrace{1,\ldots,1}_{n-2r\text{ times}}\right)=\begin{pmatrix}1 & a & & & & & & \\ 0 & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & a & & & \\ & & & 0 & 1 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1\end{pmatrix}$$

with a ranging over k.

The question of describing irreducible linear groups that contain a k-subgroup of quadratic unipotent elements of residue r has been solved completely only for $r = 1$ [3,4]. Yet even the case $r = 2$ gives rise to considerable drawbacks which are not overcome to a great extent so far and which, in addition, cause the failure to document the above mentioned description as a short stated theorem. This problem makes sense only for groups of degree ≥ 4, and there is a basic difference between the case of degree 4 and degrees greater than 4. So it is reasonable to consider the former as a single question. The present paper is a constituent of the solution of this rather tough task. The paper deals with linear groups of degree 4 that contain the tensor product $SL_2(k) \otimes SL_2(k)$, where $SL_2(k)$ denotes as usual the special linear group. Recall that the special linear group $SL_n(L)$ of degree $n \geq 2$ over a field L consists of all matrices $g \in GL_n(L)$ with $\det g = 1$. As is generally known, $SL_n(L)$ is generated by quadratic unipotent elements of residue 1 and, in consequence, the group $SL_2(L) \otimes SL_2(L)$ can be generated by quadratic unipotent elements of residue 2.

Now let us suppose that the field L has a non-trivial automorphism σ of order 2 and let Φ be a non-degenerate σ-skew-Hermitian form in $n \geq 2$ variables over L. By $U_n(L, \Phi, \sigma) = U_n(L, \Phi)$ we denote the unitary group of Φ and by $SU_n(L, \Phi, \sigma) = SU_n(L, \Phi)$ the group of all matrices of $g \in U_n(L, \Phi, \sigma)$ with $\det g = 1$. We may now formulate the main result of the paper.

Theorem 1.1. *Let K be a field algebraic extension of a subfield k. Suppose that* char $k \neq 2$ *and k contains more than* 9 *elements. If $SL_2(k) \otimes SL_2(k) \leq X \leq GL_4(K)$, then X contains a normal subgroup G satisfying one of the following conditions:*

(1) *G is conjugate in $GL_4(K)$ to the group $SL_4(L)$, where L is a subfield of K such that $L \supseteq k$.*
(2) *G is conjugate in $GL_4(K)$ to the group $SU_4(L, \Phi, \sigma)$, where L is a subfield of K such that $L \supseteq k$, σ is an automorphism of L of order* 2*, Φ is a non-degenerate σ-skew-Hermitian form in four variables over L, the Witt index of Φ being* 2*.*
(3) *$G = SL_2(L_1) \otimes SL_2(L_2)$, where L_1, L_2 are subfields of K that contain k.*

It should be observed that linear groups represented as the tensor product of various families of classical matrix groups may be thought of as members of the collection obtained by fusing the classes $\mathcal{C}_4$ and $\mathcal{C}_7$ arising in Aschbacher's structure subgroup theorem [1]. The lattice of overgroups of tensored subgroups has been investigated by Li [8] who gave among other things the description of subgroups of $GL_n(K)$ with $n = lm$, l, m natural numbers both greater than 3, that contain the subgroup $SL_l(K) \otimes SL_m(K)$. Each of such groups turns out to normalize either the group $SL_l(K) \otimes SL_m(K)$ or the group $SL_n(K)$. Thus one can see the fairly dramatic change between Theorem 1.1 and the results obtained formerly. This is one of the main reasons that impelled us to carry out the present work. Also, one should note that the various features of the classification of overgroups of tensored subgroups have been highlighted in the recent survey [11].

Now we turn our attention to another remarkable aspect of the problem under consideration. Recall that if L is a field and Q is a non-degenerate quadratic form in $n \geq 2$ variables over L, then $\Omega_n(L, Q)$ denotes the commutator subgroup of the orthogonal group of Q. Arguments from [7] (ch. IV, section 8, case 7) show that the groups $SL_2(L) \otimes SL_2(L)$ and $\Omega_4(L, Q)$ can be identified provided that the Witt index of Q equals 2. From this point of view, Theorem 1.1 can be restated as follows.

Theorem 1.2. *Let a field K be an algebraic extension of a subfield k containing more than* 9 *elements. Assume that* char $k \neq 2$*, and let Q be a non-degenerate quadratic form in four variables over k. If the Witt index of Q equals* 2 *and $\Omega_4(k, Q) \leq X \leq GL_4(K)$, then X contains a normal subgroup G such as in Parts* (1)–(3) *of Theorem* 1.1*.*

Thus under the condition $n = 4$ the description of subgroups of $GL_n(K)$ containing $\Omega_n(k, Q)$ differs from the similar one given in [2] for $n > 4$. Therefore, Theorem 1 [2] is not true for $n = 4$ and this shows that Theorem 1.2 is interesting in its own right.

The background for the proof of Theorem 1.1 given in the present paper is that the group $GL_4(K)$ and each of its subgroups are treated as matrix groups of degree 2 over the full matrix ring of degree 2 over K. This approach seems to stress the proper setting of the problem but it requires, however, considerably more concepts (including those related to linear groups over non-commutative rings) needed for its exposition. All necessary prerequisites will be given in our next section.

2. Notations and terminology

Let R be an associative ring with an identity element 1 and let m, n be natural numbers. Then
$R^\times$ denotes the multiplicative group of all invertible elements of R;
${}^a b = aba^{-1}$ for any $a \in R^\times$ and $b \in R$;
$M_{m,n}(R)$ denotes the set of all $m \times n$ matrices over R;

$0_{m,n}$ denotes the zero matrix of $M_{m,n}(R)$; $0_n = 0_{n,n}$;
${}^t g$ denotes the transpose of $g \in M_{m,n}(R)$;
$M_n(R) = M_{n,n}(R)$ the ring of all matrices of degree n over R;
$GL_n(R) = M_n(R)^\times$ the general linear group;
1_n the identity matrix of degree n;
e_{ij} denotes the square matrix which has 1 in the (i,j)-entry and zeros elsewhere (the degree of e_{ij} is usually clear from the context);
for any $a \in R$ and $i \neq j$, $t_{ij}(a) = 1_n + ae_{ij}$;
if $B \subseteq R$, then $t_{ij}(B) = \{t_{ij}(b) \mid b \in B\}$ and $Ba = \{ba \mid b \in B\}$;
$\mathrm{diag}(c_1, c_2, \ldots, c_l)$ denotes a (block-) matrix which comprises the elements (matrices) $c_1, c_2, \ldots, c_l$ on its main diagonal;
$|Z|$ denotes the cardinality of the set Z.

Let Q be a subgroup of the additive group R. If r is an integer such that $2r \leq n$, then any subgroup of $GL_n(R)$ conjugate in $GL_n(R)$ to the group of matrices

$$\mathrm{diag}\left(\underbrace{\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}, \ldots, \begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}}_{r\ \text{times}}, 1_{n-2r}\right), \quad a \in R$$

is called a *quadratic unipotent Q-subgroup of residue r*. Each non-identity element of $GL_n(R)$ belonging to some quadratic unipotent subgroup of $GL_n(R)$ of residue r is called a *quadratic unipotent element of residue r*. A quadratic unipotent element of residue 1 is referred to as a *transvection* and the corresponding quadratic unipotent subgroup is called a *root Q-subgroup*. Each transvection of $GL_n(R)$ can be written in the form

$$1_n + {}^t(\lambda_1\lambda_2\ldots\lambda_n)(\mu_1\mu_2\ldots\mu_n) = \begin{pmatrix} 1+\lambda_1\mu_1 & \lambda_1\mu_2 & \cdots & \lambda_1\mu_n \\ \lambda_2\mu_1 & 1+\lambda_2\mu_2 & \cdots & \lambda_2\mu_n \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_n\mu_1 & \lambda_n\mu_2 & \cdots & 1+\lambda_n\mu_n \end{pmatrix},$$

where $\lambda_i, \mu_i \in R$ satisfy the relation $\sum_{i=1}^n \mu_i\lambda_i = 0$.

The special linear group $SL_n(R)$ is a subgroup of $GL_n(R)$ generated by all transvections of $GL_n(R)$. If R is commutative and semi-local, (in particular, if R is a field), $SL_n(R)$ coincides with the group of all matrices $g \in GL_n(R)$ such that $\det g = 1$.

Throughout the paper, a unitary R-module will be understood by a right (resp. left) R-module, that is, a module satisfying the condition $x \cdot 1 = x$ (resp. $1 \cdot x = x$) for any of its element x. If M is a right (resp. left) R-module, then M^* denotes the left (resp. right) R-module dual to M. By ${}^nR$ (resp. R^n) we denote the left (resp. right) R-module of n-rows (resp. n-columns) of elements of R under the coordinate-wise definitions of addition and multiplication by scalars. The row $(\mu_1 \ldots \mu_n) \in {}^nR$ (resp. the column ${}^t(\lambda_1 \ldots \lambda_n) \in R^n$) is called *unimodular* if the right (resp. left) ideal in R generated by $\mu_1, \ldots, \mu_n$ (resp. by $\lambda_1, \ldots, \lambda_n$) coincides with the whole of R.

There is one key concept – that of the true transvection – which is crucial for the proofs of our main results. We conclude this section with the definition of this concept.

Definition. The transvection $1_n + {}^t(\lambda_1\lambda_2\ldots\lambda_n)(\mu_1\mu_2\ldots\mu_n) \in GL_n(R)$, where $\lambda_i, \mu_i \in R$ and $\sum_{i=1}^n \mu_i\lambda_i = 0$ is called *true* if the column ${}^t(\lambda_1\ldots\lambda_n) \in R^n$ and the row $(\mu_1\ldots\mu_n) \in {}^nR$ are both unimodular.

3. Preliminary results

The notion of quadratic unipotent element admits another approach which is more geometrical in nature and therefore more convenient for applications. In order to provide a better understanding we include a summary of this approach in a more general setting than the framework of this paper requires.

Let B be an associative ring with 1, M a right B-module, $\mathrm{End}\, M$ the endomorphism ring of M. Given $V \subseteq M$ and $Y \subseteq \mathrm{End}\, M$, define V_Y to be the submodule of M generated by the set of all elements having the form $g(v) - v$ with $g \in Y$, $v \in V$ and V^Y to be the set of all $v \in V$ such that $g(v) = v$ for each $g \in Y$. It should be clear that V^Y is a submodule of M whenever V is. Now let r be a positive integer. An automorphism g of M is called an *r-transvection* if there exist r elements $s_1, \ldots, s_r$ of M and r elements $\varphi_1, \ldots, \varphi_r$ of the dual module M^* such that

$$\varphi_i(s_j) = 0, \quad i,j = 1,2,\ldots,r \tag{3.1}$$

and

$$g(x) = x + \sum_{i=1}^r s_i\varphi_i(x) \tag{3.2}$$

for every $x \in M$. It is convenient to denote the r-transvection defined in such a way as $I + \sum_{i=1}^r s_i\varphi_i$, where I designates an identity automorphism of M. It should be evident from (3.1) and (3.2) that $(g - I)^2 = 0$, that is, g is a quadratic unipotent element of End M. If g is an r-transvection defined by (3.2) and $\alpha \in B$, then the automorphism of M sending each $x \in M$ to $x + \sum_{i=1}^{r} s_i \alpha \varphi_i(x)$ is also an r-transvection denoted by g_α. Thus $g_\alpha = I + \sum_{i=1}^{r} s_i \alpha \varphi_i$. Given $Q \subseteq B$, define $g(Q)$ to be the set of all g_α with $\alpha \in Q$. If Q is a subgroup of the additive group of B, then $g(Q)$ is a subgroup of $GL(M)$ and this $g(Q)$ is called a Q-subgroup of r-transvections of M. If the submodule $\langle s_1, s_2, \ldots, s_r \rangle$ of M generated by $s_1, s_2, \ldots, s_r$ is a right free B-module with a basis $s_1, s_2, \ldots, s_r$ and the submodule $\langle \varphi_1, \varphi_2, \ldots, \varphi_r \rangle$ of M^* is a left free B-module with a basis $\varphi_1, \varphi_2, \ldots, \varphi_r$, then g is called a *true r-transvection* of M.

Assume that M is a free B-module with a basis $f_1, f_2, \ldots, f_n$, $n \geq 2$ and denote by $f_1', f_2', \ldots, f_n'$ the basis of M^* such that $f_i'(f_j) = \delta_{ij}$ the Kronecker symbol. Further let $n = lr$, where l, r are integers greater than 1 and let $g = I + \sum_{i=1}^{r} s_i \varphi_i (s_i \in M, \varphi_i \in M^*)$ be an r-transvection of M. Eqs. (3.1) and (3.2) can be written in the matrix form as follows:

$$^t(\varphi_1, \ldots, \varphi_r)(s_1, \ldots, s_r) = 0_r, \qquad g(x) = x + (s_1, \ldots, s_r)\, {}^t(\varphi_1, \ldots, \varphi_r)(x).$$

This shows that we can think of g as being a transvection, say $\tilde{g}$, of the right free module with a basis of l elements over the matrix ring $M_r(B)$. The relationship between these g and $\tilde{g}$ is especially transparent when B is a *von Neumann regular* ring, that is, when B satisfies the following condition: for every $b \in B$ the equation $bxb = x$ always has solution in B. In the sequel, we call a von Neumann regular ring simply *regular* for short. The above mentioned relationship is given by the following statement, which in spite of its technical character and slightly awkward form finds important applications in the proof of Theorem 1.1.

Lemma 3.1. *Let K be a regular ring and l, r are integers greater than 1. Denote by R the ring $M_r(K)$ and let $g = I + \sum_{i=1}^{r} s_i \varphi_i$ be an r-transvection of $GL_{lr}(K)$ with $s_1, \ldots, s_r \in K^{lr}$, $\varphi_1, \ldots, \varphi_r \in {}^{lr}K$. Set*

$$s_1 = {}^t(\lambda_{11}\lambda_{21}\ldots\lambda_{l1}), s_2 = {}^t(\lambda_{12}\lambda_{22}\ldots\lambda_{l2}), \ldots, s_r = {}^t(\lambda_{1r}\lambda_{2r}\ldots\lambda_{lr}),$$
$$\varphi_1 = (\mu_{11}\mu_{12}\ldots\mu_{1l}), \varphi_2 = (\mu_{21}\mu_{22}\ldots\mu_{2l}), \ldots, \varphi_r = (\mu_{r1}\mu_{r2}\ldots\mu_{rl}),$$

where $\lambda_{ij} \in K^r$, $\mu_{ij} \in {}^rK$ and let $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_l, \tilde{\mu}_1, \ldots, \tilde{\mu}_l$ be matrices in R defined by

$$\tilde{\lambda}_1 = (\lambda_{11}\lambda_{12}\ldots\lambda_{1r}), \tilde{\lambda}_2 = (\lambda_{21}\lambda_{22}\ldots\lambda_{2r}), \ldots, \tilde{\lambda}_l = (\lambda_{l1}\lambda_{l2}\ldots\lambda_{lr}),$$
$$\tilde{\mu}_1 = {}^t(\mu_{11}\mu_{21}\ldots\mu_{r1}), \tilde{\mu}_2 = {}^t(\mu_{12}\mu_{22}\ldots\mu_{r2}), \ldots, \tilde{\mu}_l = {}^t(\mu_{1l}\mu_{2l}\ldots\mu_{rl}).$$

If $\tilde{g}$ is the matrix $I + {}^t(\tilde{\lambda}_1 \ldots \tilde{\lambda}_l)(\tilde{\mu}_1 \ldots \tilde{\mu}_l)$ in $M_l(R)$, then $\tilde{g}$ is a transvection of $GL_l(R)$ and the following assertions are equivalent:

(1) *g is a true r-transvection of $GL_{rl}(K)$.*
(2) *$\tilde{g}$ is a true transvection of $GL_l(R)$.*
(3) *$\tilde{\lambda}_1, \ldots, \tilde{\lambda}_l$ (resp. $\tilde{\mu}_1, \ldots, \tilde{\mu}_l$) do not belong to one and the same left (resp. right) ideal in R.*

Proof. First of all we make sure that $\tilde{g}$ is really a transvection. Indeed, if $1 \leq i, j \leq l$, the (ij)-th position of the matrix $\tilde{\mu}_1\tilde{\lambda}_1 + \cdots + \tilde{\lambda}_l\tilde{\mu}_l$ is given by the expression $\mu_{i1}\lambda_{1j} + \cdots + \mu_{il}\lambda_{lj}$. But this is equal to zero because of the relation $\varphi_i s_j = 0$ which holds since g is an r-transvection of the group $GL_{rl}(K)$. To complete the proof we deduce the following chain of implications: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2). Suppose that (1) holds. We need to prove that the column ${}^t(\tilde{\lambda}_1 \ldots \tilde{\lambda}_l) \in R^l$ and the row $(\tilde{\mu}_1 \ldots \tilde{\mu}_l) \in {}^lR$ are unimodular. Assume first that ${}^t(\tilde{\lambda}_1 \ldots \tilde{\lambda}_l)$ is not unimodular. This means that there exists some maximal left ideal of R, say $\mathfrak{m}$, containing all $\tilde{\lambda}_i$. The ring R is regular as the matrix ring over the regular ring K. So $\mathfrak{m}$ is principal and it can be generated by an idempotent element $e \in R \setminus \{0, 1\}$. Thus for each $i = 1, 2, \ldots, l$ one can find $r_i \in R$ such that $\tilde{\lambda}_i = r_i e$. Set $a = 1 - e$. Then ${}^t(\tilde{\lambda}_1 \ldots \tilde{\lambda}_l)a = 0$. Write the matrix ${}^t(\tilde{\lambda}_1 \ldots \tilde{\lambda}_l)$ as $(s_1 \ldots s_r)$ and let ${}^t(a_{1j}a_{2j}\ldots a_{rj})$ be the j-th column of $a (1 \leq j \leq r)$. Then $s_1 a_{1j} + s_2 a_{2j} + \cdots + s_r a_{rj} = 0$ which implies $a_{1j} = a_{2j} = \cdots = a_{rj} = 0$ since $s_1, \ldots, s_r$ are linearly independent. Hence $a = 0$ and $e = 1$, the desired contradiction. Similarly, we prove that $(\tilde{\mu}_1 \ldots \tilde{\mu}_l)$ is a free element of ${}^lR$.

(2) $\Rightarrow$ (3). This is valid by the definition of a true transvection.

(3) $\Rightarrow$ (1). Suppose that (3) holds but the submodule $\langle s_1, \ldots, s_r \rangle$ of K^{lr} is not free. This means that $s_1\gamma_1 + \cdots + s_r\gamma_r = 0$ for some $\gamma_1, \ldots, \gamma_r \in R$ which are not equal to zero simultaneously. In particular, for any $i \in \{1, 2, \ldots, l\}$, we have $\lambda_{i1}\gamma_1 + \lambda_{i2}\gamma_2 + \cdots + \lambda_{ir}\gamma_r = 0$ which shows that each matrix $\tilde{\lambda}_i = (\lambda_{i1} \ldots \lambda_{ir})$ is contained in the proper left ideal in R consisting of all matrices $(a_1 a_2 \ldots a_r) \in R (a_i \in K^r)$ such that $a_1\gamma_1 + a_2\gamma_2 + \cdots + a_r\gamma_r = 0$. On the other hand, if the submodule $\langle \varphi_1, \ldots, \varphi_r \rangle$ of M^* is not free, then there exist $\delta_1, \ldots, \delta_r \in K$ which are not simultaneously equal to zero and such that $\tilde{\mu}_1, \ldots, \tilde{\mu}_l$ belong to the proper right ideal in R consisting of all matrices ${}^t(a_1 \ldots a_r) \in R (a_i \in {}^rK)$ with the property $\delta_1 a_1 + \cdots + \delta_r a_r = 0$. The contradiction obtained completes the proof of the lemma. □

It should be clear that if k is a field and f is a transvection of $SL_2(k)$, then the matrices $1_2 \otimes f$ and $f \otimes 1_2$ are quadratic unipotent elements of the group $H = SL_2(k) \otimes SL_2(k)$. Our next result, which finds its application in Section 5, says in fact then these elements exhaust the quadratic unipotent elements of H.

Lemma 3.2. *Let k be a field of characteristic $\neq 2$ and C a non-identity quadratic unipotent element of the group $H = SL_2(k) \otimes SL_2(k)$. Then either $C = f \otimes 1_2$ or $C = 1_2 \otimes f$, where f is a suitable unipotent element of $SL_2(k)$, that is, f is a transvection of $GL_2(k)$.*

Proof. Write $C = g \otimes h$ with $g, h \in SL_2(k)$. The equation $(C-1)^2 = 0$ shows that

$$g^2 \otimes h^2 - 2(g \otimes h) + 1_2 \otimes 1_2 = 0_4. \tag{3.3}$$

Each matrix $x \in M_2(k)$ satisfies the equation $x^2 - (\operatorname{tr} x)x + (\det x)1_2 = 0_2$, where $\operatorname{tr} x$ denotes the trace of x. Taking into account that $\det g = \det h = 1$, we have

$$g^2 = ag - 1_2, \quad h^2 = ch - 1_2, \tag{3.4}$$

where $a = \operatorname{tr} g$, $c = \operatorname{tr} h$. The substitution of (3.4) into (3.3) gives rise to

$$(ac-2)(g \otimes h) - a(g \otimes 1_2) - c(1_2 \otimes h) + 2(1_2 \otimes 1_2) = 0_4. \tag{3.5}$$

Write $h = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$ with $h_{ij} \in k$. Then (3.5) means that

$$(ac-2)gh_{11} - ag - ch_{11} \cdot 1_2 + 2 \cdot 1_2 = 0_2, \tag{3.6}$$

$$(ac-2)gh_{12} - ch_{12} \cdot 1_2 = 0_2, \tag{3.7}$$

$$(ac-2)gh_{21} - ch_{21} \cdot 1_2 = 0_2, \tag{3.8}$$

$$(ac-2)gh_{22} - ag - ch_{22} \cdot 1_2 + 2 \cdot 1_2 = 0_2. \tag{3.9}$$

Since C is not the identity, at least one of g, h must be non-scalar. For each $x, y \in M_2(k)$, let us put $\varphi(x \otimes y) = y \otimes x$. This φ can be extended by linearity to an automorphism of the k-algebra $M_4(k)$ and the restriction of the algebra automorphism φ to the group H determines an automorphism of H. Employing φ, we may assume without loss of generality that g is non-scalar. This, and Eqs. (3.7) and (3.8) are combined, to yield $h_{12} = h_{21} = 0$. Thus $h = \begin{pmatrix} h_{11} & 0 \\ 0 & h_{22} \end{pmatrix}$ with $h_{22} = h_{11}^{-1}$. Eq. (3.9) then becomes

$$-ag + 2 \cdot 1_2 = -(ac-2)gh_{11}^{-1} + ch_{11}^{-1}.$$

Using this, we obtain from (3.6) that $((ac-2)g - c1_2)(h_{11} - h_{11}^{-1}) = 0_2$. Again we employ the assumption that g is non-scalar to get from the previous equation that $h = \pm 1_2$. If $h = 1_2$, then $C = g \otimes 1_2$ and (3.5) shows that $\operatorname{tr} g = 2$, that is, g is a transvection. If $h = -1_2$, then $\operatorname{tr} g = -2$, $\operatorname{tr}(-g) = 2$. So $-g$ is a transvection and since $C = g \otimes (-1_2) = (-g) \otimes 1_2$, this transvection $-g$ may serve as f. The lemma is proved. □

4. Linear groups containing $SL_2(k) \otimes SL_2(k)$ and possessing a transvection

The present section is dedicated to the proof of Theorem 1.1 in the case when the group X appearing in this theorem contains a transvection. More precisely, the principal purpose of this section is to prove the following.

Proposition 4.1. *Let k be a field of characteristic $\neq 2$ and a field K an algebraic extension of k. Assume that $|k| > 9$. If $SL_2(k) \otimes SL_2(k) \le X \le GL_4(K)$ and X contains a transvection, then X contains a normal subgroup G which is conjugate in $GL_4(K)$ either to the group $SL_4(L)$ or to the group $SU_4(L, \Phi, \sigma)$, where L is a subfield of K containing k, σ is an automorphism of L of order two, Φ is a non-degenerate skew-Hermitian form in four variables over L, the Witt index of Φ being 2.*

The hardest part of the proof of Proposition 4.1 actually consists in establishing the following statement.

Proposition 4.2. *Let k and K be as in Proposition 4.1, E a vector space over K (E is not necessarily finite dimensional), g and h are a true 2-transvection and a transvection of $GL(E)$, respectively. If $F = \langle g(k), h \rangle$ is a subgroup of $GL(E)$ generated by a k-subgroup of 2-transvections $g(k)$ together with h, then F is either abelian or contains a root k-subgroup.*

It is convenient to preface the proof of Proposition 4.2 with two lemmas which are in effect constituents of the proof placed before it.

Lemma 4.1. *Let R be an associative ring with an identity 1. Suppose that $2 \in R^\times$ and that R contains an element v with $v^2 = 0$. Then the subgroup of $GL_2(R)$ generated by the matrix $t_{21}(v)$ together with the root k-subgroup $t_{12}(k)$ contains the root k-subgroup $t_{12}(vk)$.*

Proof. Let r be an arbitrary element in k. The lemma is immediate from the equality

$$t_{12}(-1)t_{21}(v)t_{12}(-r)t_{21}(-v)t_{12}(1)t_{21}(v)t_{12}(r)t_{21}(-v) = t_{12}(2rv). \quad \square$$

Lemma 4.2. *Let k and K be as in Proposition 4.1. If $\alpha \in K^\times$ and Z is a subgroup of $GL_4(K)$ generated by k-subgroup of true 2-transvections $\begin{pmatrix} 1_2 & k1_2 \\ 0_2 & 1_2 \end{pmatrix}$ together with the matrix $t_{42}(\alpha) \in GL_4(K)$, then Z contains a root k-subgroup.*

Proof. First observe that Z is contained in the subgroup of $GL_4(K)$ consisting of all matrices

$$n(z_1, z_2, z_3, z_4, z_5) = \begin{pmatrix} 1 & 0 & z_1 & 0 \\ 0 & z_2 & 0 & z_3 \\ 0 & 0 & 1 & 0 \\ 0 & z_4 & 0 & z_5 \end{pmatrix}$$

with $z_1 \in K$ and $\begin{pmatrix} z_2 & z_3 \\ z_4 & z_5 \end{pmatrix} \in SL_2(K)$. Applying Theorem 3 [3] to the subgroup of $GL_3(K)$ generated by the root k-subgroup $t_{13}(k)$ together with the matrix $t_{31}(\alpha) \in GL_3(K)$ shows that for any $\begin{pmatrix} y_2 & y_3 \\ y_4 & y_5 \end{pmatrix} \in SL_2(k(\alpha))$, one can find $y_1 \in K$ such that $n(y_1, y_2, y_3, y_4, y_5) \in Z$. In turn this implies that the commutator subgroup of Z coincides with the group of all matrices $n(0, x_2, x_3, x_4, x_5)$, where $\begin{pmatrix} x_2 & x_3 \\ x_4 & x_5 \end{pmatrix}$ ranges over $SL_2(k(\alpha))$. In particular, for every $r \in k$, the matrix $n(0, 1, 0, r, 1) = t_{42}(r)$ belongs to the commutator subgroup of Z, hence to Z itself. □

Proof of Proposition 4.2. Put $g = I + s\varphi + t\psi$, $h = I + p\chi$ with $s, t, p \in E$ and $\varphi, \psi, \chi \in E^*$. It is readily seen that the values of $\dim E_F$ and $\operatorname{codim} E^F$ can take only two meanings, namely, 2 and 3. Accordingly, the following four main cases can occur:

(1) $\dim E_F = 2$, $\operatorname{codim} E^F = 2 \Leftrightarrow E_h \subseteq E_g$, $E^g \subseteq E^h$.
(2) $\dim E_F = 2$, $\operatorname{codim} E^F = 3 \Leftrightarrow E_h \subseteq E_g$, $E^g \nsubseteq E^h$.
(3) $\dim E_F = 3$, $\operatorname{codim} E^F = 2 \Leftrightarrow E_h \nsubseteq E_g$, $E^g \subseteq E^h$.
(4) $\dim E_F = 3$, $\operatorname{codim} E^F = 3 \Leftrightarrow E_h \nsubseteq E_g$, $E^g \nsubseteq E^h$.

These cases will be considered separately.

Case (1). Here $p \in \langle s, t\rangle$ and so without loss of generality we may assume that the pair $\{p, t\}$ is a basis of E_F. Replacing (s, t) by $(s, t)C$, where C is a suitable matrix of $GL_2(K) \cong GL(\langle s, t\rangle)$, we may suppose $s = p$. In this case, it is easily checked that F is abelian.

Case (2). Here as above we may consider $s = p$, that is, $g = I + p\varphi + t\psi$. If $t \in E^F$, then F is abelian. Assume that $t \notin E^F$. Then the coset $t+E^F$ is a non-zero element of the quotient space E/E^F. So E/E^F has a basis of the form $\{t+E^F, w_1+E^F, w_2+E^F\}$ with $w_1, w_2 \in E$. Since $p \in E^F$, we can extend the vector p to the basis $\{p, (b_\lambda)_{\lambda\in\Lambda}\}$ of the subspace E^F. The collection $\{p, (b_\lambda)_{\lambda\in\Lambda}, t, w_1, w_2\}$ forms a basis of E and the group F can be identified with its restriction $F \mid \langle p, t, w_1, w_2\rangle$ to the subspace $\langle p, t, w_1, w_2\rangle$ generated by p, t, w_1, w_2. Having made this, we do not distinguish elements of F from its matrices relatively to the basis $\{p, t, w_1, w_2\}$. Put

$$a = \begin{pmatrix} \varphi(w_1) & \varphi(w_2) \\ \psi(w_1) & \psi(w_2) \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & \chi(t) \\ 0 & 1 \end{pmatrix}, \qquad c = \begin{pmatrix} \chi(w_1) & \chi(w_2) \\ 0 & 0 \end{pmatrix}.$$

Then

$$g_r = \begin{pmatrix} 1_2 & ra \\ 0_2 & 1_2 \end{pmatrix} (r \in k), \qquad h = \begin{pmatrix} b & c \\ 0_2 & 1_2 \end{pmatrix}, \qquad g_r h g_r^{-1} h^{-1} = \begin{pmatrix} 1_2 & (1-b)ar \\ 0_2 & 1_2 \end{pmatrix}.$$

Since $\chi(t) \neq 0$ and at least one of $\psi(w_1)$, $\psi(w_2)$ is non-zero, the matrix

$$(1-b)a = \begin{pmatrix} \chi(t)\psi(w_1) & \chi(t)\psi(w_2) \\ 0 & 0 \end{pmatrix}$$

is non-zero, hence the collection $\{g_r h g_r^{-1} h^{-1} \mid r \in k\}$ forms a root k-subgroup of F.

Case (3). Here $E_F = \langle p, s, t\rangle$. If $p \in E^F$, then straightforward verifying shows that F is abelian. Assume that $p \notin E^F$. Then E has some basis of the form $\{s, t, (b_\lambda)_{\lambda\in\Lambda}, p, w\}$, where $\{s, t, (b_\lambda)_{\lambda\in\Lambda}\}$ is a basis of E^F and $\{p + E^F, w + E^F\}$ is a basis of E/E^F. Let us identify F with $F \mid \langle s, t, p, w\rangle$. With this made, we do not distinguish linear transformations of the subspace $\langle s, t, p, w\rangle$ from their matrices respectively to the basis $\{s, t, p, w\}$. Put

$$a = \begin{pmatrix} \varphi(p) & \varphi(w) \\ \psi(p) & \psi(w) \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & \chi(w) \\ 0 & 1 \end{pmatrix}.$$

Then

$$g_r = \begin{pmatrix} 1_2 & ra \\ 0_2 & 1_2 \end{pmatrix} (r \in k), \qquad h = \begin{pmatrix} 1_2 & 0_2 \\ 0_2 & b \end{pmatrix}.$$

Since g is a true 2-transvection, the matrix

$$a(b - 1_2) = \begin{pmatrix} 0 & \varphi(p)\chi(w) \\ 0 & \psi(p)\chi(w) \end{pmatrix}$$

is non-zero, so the collection of matrices

$$h^{-1}g_rhg_r^{-1} = \begin{pmatrix} 1_2 & a(b-1_2)r \\ 0_2 & 1_2 \end{pmatrix}$$

with r ranging over k yields a root k-subgroup of F.

Case (4). Denote $S = E_F \cap E^F$. Since $\dim S \le 3$, one of the following four subcases occur:

(a) $\dim S = 0$.
(b) $\dim S = 1$.
(c) $\dim S = 2$.
(d) $\dim S = 3$.

Notice first that subcase (a) can be excluded at once for under its assumptions we have $E_g = \langle s\varphi(p) + t\psi(p)\rangle$ which is false because g is a true 2-transvection. Moreover, if $\dim S = 3$, then F is abelian. So there remains to consider subcases (b) and (c).

Consider (b). Assume that $S \not\subseteq E_g$, and so $\langle p\rangle = S$. It is easily seen that E has a basis of the form $\{p, (b_\lambda)_{\lambda\in\Lambda}, s, t, w\}$, where $\{p, (b_\lambda)_{\lambda\in\Lambda}\}$ is a basis of the subspace E^F and the collection $\{s+E^F, t+E^F, w+E^F\}$ is a basis of the quotient space E/E^F. Hence $E_g = \langle s\varphi(w) + t\psi(w)\rangle$ contradicting again the hypothesis that g is a true 2-transvection. So $S \subseteq E_g$ and we have to consider the following situation: $\dim S = 1$, $S \subseteq E_g = \langle s, t\rangle$.

Let us take a vector $s\alpha + t\beta \in E_g\,(\alpha, \beta \in K)$ which spans S and extend $s\alpha + t\beta$ to a basis $\{s\alpha+t\beta, t_1\}$ of $E_g\,(t_1 \in E)$. The group $GL(\langle s, t\rangle) \cong GL_2(K)$ contains an element sending s to $s\alpha + t\beta$, so we may assume that $\langle s\rangle = S$. Therefore, $p \notin E^F$ and $t \notin E^F$. Consequently, the quotient space E/E^F possesses a basis $\{p+E^F, t+E^F, w+E^F\}$ with $w \in E$. Put $V = \langle p, t, w\rangle$. Since $\dim S = 1$, $V \not\subseteq E^h$ and so $V + E^h = E$. Hence $V/(V\cap E^h) \cong (V+E^h)/E^h = E/E^h$ which implies that $\dim(V\cap E^h) = 2$. Denote by W_1 a subspace of V complemented to $V \cap E^h$, that is, assume that V admits a direct sum decomposition $(V\cap E^h) \oplus W_1$. Further, let us choose $w_1 \in V\cap E^h$ so that $\{p, w_1\}$ is a basis of $V \cap E^h$. Since $\dim S = 1$, $t \notin V\cap E^h$ and consequently, $\{t, p, w_1\}$ is a basis of V. Accordingly, replacing w by w_1, we may assume that $w \in E^h$. The space E has a basis $\{s, (b_\lambda)_{\lambda\in\Lambda}, t, p, w\}$, where $\{s, (b_\lambda)_{\lambda\in\Lambda}\}$ is a basis of E^F. Identify F with $F \mid \langle s, t, p, w\rangle$ and denote

$$a = \begin{pmatrix} \varphi(p) & \varphi(w) \\ \psi(p) & \psi(w) \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & \chi(t) \\ 0 & 0 \end{pmatrix}, \qquad u = \begin{pmatrix} 1_2 & 0_2 \\ 0_2 & a \end{pmatrix}.$$

Then with respect to the basis $\{s, t, p, w\}$, the transformations $g_r\,(r \in k)$ and h have the matrices $\begin{pmatrix} 1_2 & ar \\ 0_2 & 1_2 \end{pmatrix}$ and $\begin{pmatrix} 1_2 & 0_2 \\ b & 1_2 \end{pmatrix}$, respectively. So the group ${}^uF$ is generated by the subgroup $\begin{pmatrix} 1_2 & k1_2 \\ 0_2 & 1_2 \end{pmatrix}$ together with the matrix $\begin{pmatrix} 1_2 & 0_2 \\ c & 1_2 \end{pmatrix}$, where

$$c = ab = \begin{pmatrix} 0 & \varphi(p) \\ 0 & \psi(p) \end{pmatrix} \chi(t).$$

Letting $\alpha = \psi(p)\chi(t)$, we see that $c^2 = c\alpha$. If $\alpha = 0$, then $c^2 = 0$ and F contains a root k-subgroup according to Lemma 4.1 with $M_2(K)$ as R. Suppose now that $\alpha \ne 0$. Then there exists $d \in GL_2(K)$ such that ${}^dc = \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}$. Taking account of this equation, we have

$$\begin{pmatrix} d & 0_2 \\ 0_2 & d \end{pmatrix} \begin{pmatrix} 1_2 & 0_2 \\ c & 1_2 \end{pmatrix} \begin{pmatrix} d & 0_2 \\ 0_2 & d \end{pmatrix}^{-1} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{1_2} & 0_2 \\ \hline 0 & 0 & \\ 0 & \alpha & 1_2 \end{array}\right),$$

$$\begin{pmatrix} d & 0_2 \\ 0_2 & d \end{pmatrix} \begin{pmatrix} 1_2 & r1_2 \\ 0_2 & 1_2 \end{pmatrix} \begin{pmatrix} d & 0_2 \\ 0_2 & d \end{pmatrix}^{-1} = \begin{pmatrix} 1_2 & r1_2 \\ 0_2 & 1_2 \end{pmatrix}.$$

Hence F contains a root k-subgroup by Lemma 4.2.

Subcase (c). Recall that here we are concerned with the situation defined by the following conditions: $\dim E_F = 3$, $\operatorname{codim} E^F = 3$, $\dim S = 2$. An arbitrary element of S has the form $s\alpha + t\beta + p\gamma$, where $\alpha, \beta, \gamma \in K$ and $\varphi(p)\gamma = \psi(p)\gamma = \chi(s\alpha + t\beta) = 0$. So $p \in S$ if $\gamma \ne 0$. Thus if $S \not\subseteq \langle s, t\rangle$, then $p \in S$. Hence we can conclude that either $S \subseteq \langle s, t\rangle$ (and so $S = \langle s, t\rangle$) or $S = \langle p, s_1\rangle$, where $s_1 \in \langle s, t\rangle$, and we may assume $S = \langle p, s\rangle$. Accordingly, we need to consider the following two possibilities:

(I) $S = \langle s, t\rangle$; (II) $S = \langle p, s\rangle$.

First suppose that we deal with (I). Let us add a set $(b_\lambda)_{\lambda\in\Lambda} \subseteq E$ to the pair $\{s, t\}$ so that the collection $\{s, t, (b_\lambda)_{\lambda\in\Lambda}\}$ forms a basis of E^F. Since $p \notin E^F$, the quotient space E/E^F has a basis of the form $\{p+E^F, w_1+E^F, w_2+E^F\}$, where w_1, w_2 are some elements of E. Set $V = \langle p, w_1, w_2\rangle$. Evidently, $\dim V = 3$ and the collection $\{s, t, (b_\lambda)_{\lambda\in\Lambda}, p, w_1, w_2\}$ constitutes a basis of E. Since $V \not\subseteq E^h$, $V + E^h = E$. So $V/(V \cap E^h) \cong E/E^h$ whence it follows that $\dim(V \cap E^h) = 2$. Let V_1 be the complement of $V \cap E^h$ in V. Choose $w_1' \in V \cap E^h$ so that the elements p, w_1' form a basis of $V \cap E^h$ and let w_2' generate V_1. Then $\{s, t, (b_\lambda)_{\lambda\in\Lambda}, p, w_1', w_2'\}$ is a basis of E and the group F may be identified with $F \mid \langle s, t, p, w_1', w_2'\rangle$. Denote

$$a = \begin{pmatrix} \varphi(p) & \varphi(w_1') & \varphi(w_2') \\ \psi(p) & \psi(w_1') & \psi(w_2') \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 & \chi(w_2') \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

With respect to the basis $\{s, t, p, w_1', w_2'\}$, the transformations g_r and h have the matrices $\begin{pmatrix} 1_2 & ra \\ 0_{3,2} & 1_3 \end{pmatrix}$, $\begin{pmatrix} 1_2 & 0_{2,3} \\ 0_{3,2} & b \end{pmatrix}$ respectively. If the matrix

$$a(b - 1_3) = \begin{pmatrix} 0 & 0 & \varphi(p)\chi(w_2') \\ 0 & 0 & \psi(p)\chi(w_2') \end{pmatrix}$$

were zero, p would belong to S contradicting the condition $\dim S = 2$ which defines subcase (c). Hence $a(b - 1_3) \neq 0_{2,3}$ and the transformations

$$h^{-1}g_r h g_r^{-1} = \begin{pmatrix} 1_2 & ra(b - 1_3) \\ 0_{3,2} & 1_3 \end{pmatrix}$$

with r ranging over k form a root k-subgroup of F.

Consider (II). Extend the pair $\{p, s\}$ to the basis $\{p, s, (b_\lambda)_{\lambda\in\Lambda}, \}$ of E^F. There exists a basis of the quotient space E/E^F having the form $\{t + E^F, w_1 + E^F, w_2 + E^F\}$ $(w_1, w_2 \in E)$. Set $V = \langle t, w_1, w_2\rangle$. Since $t \notin \langle p, s\rangle$, $\dim(V \cap E^h) = 2$. Let V_1 be the complement of $V \cap E^h$ in V. Fix some basis of $V \cap E^h$, say w_1', w_2'. Since $t \notin E^h$ and $t \in V$, the collection $\{t, w_1', w_2'\}$ forms a basis of V. Consequently, $\{p, s, (b_\lambda)_{\lambda\in\Lambda}, t, w_1', w_2'\}$ is a basis of the whole E, and F may be identified with its restriction to the subspace $\langle s, t, p, w_1', w_2'\rangle$. Denote

$$a = \begin{pmatrix} 0 & \varphi(w_1') & \psi(w_1') \\ 0 & \varphi(w_2') & \psi(w_2') \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & \chi(t) \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

With respect to the basis $\{s, t, p, w_1', w_2'\}$, the transvections $g_r (r \in k)$ and h have the matrices $\begin{pmatrix} 1_2 & ar \\ 0_{3,2} & 1_3 \end{pmatrix}$, $\begin{pmatrix} 1_2 & 0_{2,3} \\ b & 1_3 \end{pmatrix}$, respectively. Since $ab = 0_2$, we have $aba = 0_2$, and so

$$g_r h g_r^{-1} h^{-1} = \begin{pmatrix} 1_2 & 0_{2,3} \\ 0_{3,2} & 1_3 - bar \end{pmatrix}.$$

But

$$ba = \begin{pmatrix} 0 & \chi(t)\psi(w_1') & \chi(t)\psi(w_2') \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and at least one of the elements $\chi(t)\psi(w_1')$, $\chi(t)\psi(w_2')$ is different from zero. Hence $\{g_r h g_r^{-1} h^{-1} \mid r \in k\}$ is a root k-subgroup of F. The proposition is proved completely. □

Our next step is to establish a preliminary assertion which permits us to exclude some groups provided that the conditions of Proposition 4.1 hold.

To begin with, let us recall that if A is a non-degenerate alternating matrix of degree 4 over a field L, then the symplectic group $Sp_4(L, A)$ is the group of all $g \in GL_4(L)$ such that ${}^t gAg = A$. The group consisting of all $g \in GL_4(L)$ such that ${}^t gAg = \lambda_g A$ with $\lambda_g \in L$ depending on g is called the group of symplectic similitudes and is denoted by $GSp_4(L, A)$. If $A = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix}$ we write simply $Sp_4(L)$ and $GSp_4(L)$ instead of $Sp_4(L, A)$ and $GSp_4(L, A)$ respectively.

Lemma 4.3. *Let L be a field of characteristic $\neq 2$, and σ an automorphism of L such that $\sigma^2 = \mathrm{id}$. Let E be a four-dimensional vector space over a field L and A a σ-sesquilinear form on E which is linear in its first argument. Fix a basis $\{e_1, e_2, e_3, e_4\}$ of E. We do not distinguish endomorphisms of E from their matrices with respect to the basis chosen. Let*

$$h_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \otimes 1_2, \qquad h_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \otimes 1_2, \qquad h_3 = 1_2 \otimes \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad h_4 = 1_2 \otimes \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{4.1}$$

Suppose that

$$A(x, y) = A(h_j(x), h_j(y)) \tag{4.2}$$

for all $j \in \{1, 2, 3, 4\}$ and all $x, y \in E$. Then

(a) *If* $\sigma = \mathrm{id}$, *that is, if A is bilinear, then* $A = 0$.
(b) *If* $\sigma \neq \mathrm{id}$, *that is, if A is* σ*-skew-Hermitian, then there exists* $\theta \in L$ *with* $\theta^\sigma = -\theta$ *such that A has the matrix*

$$\begin{pmatrix} 0 & 0 & 0 & -\theta \\ 0 & 0 & \theta & 0 \\ 0 & \theta & 0 & 0 \\ -\theta & 0 & 0 & 0 \end{pmatrix}$$

with respect to the basis $\{e_1, e_2, e_3, e_4\}$. *In particular, if A is non-degenerate, its Witt index equals 2.*

Proof. First we take $j = 1$ and make the pair (x, y) to run over the set consisting of the pairs (e_1, e_2), (e_1, e_4), (e_3, e_4), (e_2, e_4). Then we get from (4.2) that

$$A(e_1, e_1) = 0, \tag{4.3}$$
$$A(e_1, e_3) = 0, \tag{4.4}$$
$$A(e_3, e_3) = 0, \tag{4.5}$$
$$A(e_1, e_4) + A(e_2, e_3) = 0. \tag{4.6}$$

If $j = 2$ and the pair (x, y) runs over the set $\{(e_1, e_2), (e_1, e_3), (e_3, e_4)\}$, Eq. (4.2) gives

$$A(e_2, e_2) = 0, \tag{4.7}$$
$$A(e_2, e_4) = 0, \tag{4.8}$$
$$A(e_4, e_4) = 0. \tag{4.9}$$

Next put $j = 3$ and let $(x, y) \in \{(e_1, e_4), (e_3, e_4)\}$. By (4.2), we obtain

$$A(e_1, e_2) = 0, \tag{4.10}$$
$$A(e_1, e_4) + A(e_3, e_2) = 0. \tag{4.11}$$

Finally, if $j = 4$, $x = e_1$, $y = e_4$, then

$$A(e_3, e_4) = 0. \tag{4.12}$$

Suppose now $\sigma = \mathrm{id}$. Since $A(e_3, e_2) = -A(e_2, e_3)$ and $\mathrm{char}\, k \neq 2$, we find from (4.6) and (4.11) that $A(e_1, e_4) = A(e_2, e_3) = 0$. Combining this and (4.3)–(4.5), (4.7)–(4.10), (4.12) yields $A = 0$ as required.

Next suppose that $\sigma \neq \mathrm{id}$ and let $\theta \in L$ be the value of $A(e_2, e_3)$. Then $A(e_1, e_4) = -\theta$ by (4.6) and $\theta^\sigma = -\theta$ by comparing (4.6) with (4.11). Taking into account relations (4.3)–(4.5), (4.7)–(4.10) and (4.12), we conclude that the matrix of A with respect to the basis $\{e_1, e_2, e_3, e_4\}$ has the required form. The lemma is proved completely. □

We are now in a position to prove Proposition 4.1.

Proof of Proposition 4.1. For the sake of brevity, denote the group $SL_2(k) \otimes SL_2(k)$ by H. Let τ be a transvection which is contained in X. The group H is absolutely irreducible since SL_2 is. So if τ commutes with any true 2-transvection of H, then the matrix τ is scalar. But $(\tau - 1)^2 = 0$, and so τ must be trivial. The contradiction obtained shows that H contains a true 2-transvection, say h, which is not permutable with τ. Since $h(k) \leq H$, the application of Proposition 4.2 to the group $\langle \tau, h(k) \rangle$ implies that X contains a root k-subgroup. Since H is a primitive subgroup of $GL_4(K)$, we can use Theorem 1.1 [3] to conclude that X contains a normal subgroup G which is conjugate in $GL_4(K)$ to one of the groups $SL_4(L)$, $SU_4(L, \Phi, \sigma)$, $Sp_4(L)$, where L is a subfield of K containing k, σ is an automorphism of L with $\sigma^2 = \mathrm{id}$, Φ is a non-degenerate σ-skew-Hermitian form in four variables over L, the Witt index of Φ being non-zero.

Let us make sure that the symplectic group $Sp_4(L)$ cannot occur in this situation. Indeed, assume that $G = {}^{y}Sp_4(L)$ for some $y \in GL_4(K)$. Since $G \trianglelefteq X$, ${}^{x}G = G$ for any $x \in X$. It follows that x^y normalizes the group $Sp_4(L)$. Recall that the group $Sp_4(L)$ is absolutely irreducible, so according to 1.19 [12], $x^y = \lambda g$ with $\lambda \in K^\times$, $g \in GL_4(L)$. But then g normalizes $Sp_4(L)$, and since $g \in GL_4(L)$, we have, by the well known result (see, for instance, [10], §5) that g belongs to the group $GSp_4(L)$. Therefore, $\lambda g \in GSp_4(K)$. Thus for any $x \in X$, $x^y \in GSp_4(K)$. If x is a quadratic unipotent element of H, then so is x^y and by Proposition 2 [5] (see also [9]), x^y must belong to the symplectic group itself. Thus if $\Psi = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix}$ is the alternating matrix defining $Sp_4(K)$, then ${}^{t}(x^y)\Psi x^y = \Psi$ for any quadratic unipotent $x \in H$. Set ${}^{t}y^{-1}\Psi y^{-1} = A$. Since ${}^{t}A = -A$, invoking Lemma 4.3 yields $A = 0_4$, so $\Psi = 0_4$.

Suppose next that $G = {}^{y}SU_4(L, \Phi, \sigma)$ $(y \in GL_4(K))$ and show that the Witt index of Φ equals 2. Let x be an arbitrary element of X. Then since the group $SU_4(L, \Phi, \sigma)$ is absolutely irreducible, we obtain as above that $x^y = \lambda_x g_x$ for some $\lambda_x \in K^\times$ and some $g_x \in GL_4(L)$. Furthermore, this g_x normalizes $SU_4(L, \Phi, \sigma)$, hence it is contained in the similitude group $GU_4(L, \Phi, \sigma)$ of the form Φ. Now let us choose x to be a quadratic unipotent element of X and assume that $\lambda = \lambda_x \notin L$. Write $g = g_x = \sum g_{ij} e_{ij}$ $(g_{ij} \in L)$. If $1 \leq i \neq j \leq 4$, the equation $\lambda^2 - 2\lambda g + 1_4 = 0_4$ and the condition $\lambda \neq 0$ show that $\lambda(g_{i1}g_{1j} + g_{i2}g_{2j} + g_{i3}g_{3j} + g_{i4}g_{4j}) - 2g_{ij} = 0$, hence $g_{ij} = 0$. Thus the matrix g is diagonal, $g = \mathrm{diag}(g_{11}, g_{22}, g_{33}, g_{44})$, and $(\lambda g_{ii} - 1)^2 = 0$ for all $i = 1, 2, 3, 4$. It follows that $\lambda g_{ii} = 1$ and so $x = 1$ which contradicts the choice of x. The contradiction obtained shows that if x is a quadratic unipotent element of X, then $\lambda_x \in L$ and so $x^y \in GL_4(L)$. Moreover, since $g_x \in GU_4(L, \Phi, \sigma)$ and since x^y is a quadratic unipotent element, it must belong to the unitary group $U_4(L, \Phi, \sigma)$ (and even to the group $SU_4(L, \Phi, \sigma)$ for the determinant of x^y is equal to 1). In particular, this shows that if $i \in \{1, 2, 3, 4\}$ and h_i is the matrix defined by (4.1), then $SU_4(L, \Phi, \sigma)$ contains matrices l_i such that $h_i y = y l_i$ for all $i = 1, 2, 3, 4$. Write $y = \sum y_{ij} e_{ij}$ ($y_{ij} \in K$) and let us treat these four matrix equations as a system of linear homogeneous equations in 16 variables y_{ij}. This system is consistent and all of its coefficients belong to the field L. Therefore, it has a solution, with all components lying in L. This means that from the very beginning we may think of y as a matrix over L. In this case, the matrix y^σ is defined, hence too is the matrix $A = {}^t(y^{-1})^\sigma \Phi y^{-1}$. Since A is a σ-skew-Hermitian matrix over L(${}^tA^\sigma = -A$), we can use Lemma 4.3 to deduce that the Witt index of A is 2. Hence that of Φ is also 2 and the proposition is proved completely. □

5. Linear groups without transvections containing $SL_2(k) \otimes SL_2(k)$

This section concerns the case of Theorem 1.1 when the group X of this theorem does not contain any transvections. Here our primary effort will be devoted to the proof of the following.

Proposition 5.1. *Let K be a field of characteristic $\neq 2$. Assume that K is an algebraic extension of a field k such that $|k| > 9$. If $SL_2(k) \otimes SL_2(k) \leq X \leq GL_4(K)$ and X does not possess any transvections, then X contains a normal subgroup $SL_2(L_1) \otimes SL_2(L_2)$, where L_1, L_2 are subfields of K containing k.*

To prove Proposition 5.1 we treat the group $GL_4(K)$ as the general linear group $GL_2(R)$ over the ring $R = M_2(K)$. In turn $GL_2(R)$ is regarded as the automorphism group $GL(M)$ of the right free R-module M with a basis of two elements f_1, f_2. Having all these conventions been made, Lemma 3.1 says that the set of all true 2-transvections of $GL_4(K)$ is exactly the set of all true 1-transvections of the group $GL_2(R)$. Recall that 1-transvections are called simply transvections. So, the fact that X viewed as a subgroup of $GL_4(K)$ contains no transvections means exactly that each transvection of X as a subgroup of $GL_2(R)$ is a true one. Now let us agree on certain notations which will be used throughout this section.

Denote a unique symplectic type involution acting on R by J, that is, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^J = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Next define N to be the set of all free elements $f_1\alpha_1 + f_2\alpha_2 \in M$ ($\alpha_i \in R$) subject to

$$\alpha_1\alpha_1^J = \alpha_2\alpha_2^J = \alpha_1\alpha_2^J + \alpha_2\alpha_1^J = 0_2.$$

In the sequel, we shall make the repeated use the following simple property of the set N.

Lemma 5.1. *If $f_1\alpha_1 + f_2\alpha_2 \in N$ ($\alpha_i \in R$), then $\alpha_1^J\alpha_2 = \alpha_2^J\alpha_1 = 0_2$.*

Proof. The ring R is regular, so α_1 and α_2 do not lie in one and the same maximal left ideal of R. Hence replacing α_1, α_2 by ${}^\xi\alpha_1, {}^\xi\alpha_2$, where ξ is a suitable element of $R^\times$, we may assume that $\alpha_1 = \begin{pmatrix} a_1 & 0 \\ a_2 & 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 & a_3 \\ 0 & a_4 \end{pmatrix}$ with $a_i \in K$. The condition $\alpha_1\alpha_2^J + \alpha_2\alpha_1^J = 0_2$ shows then that $a_1a_4 - a_2a_3 = 0$. So

$$\alpha_1^J\alpha_2 = \begin{pmatrix} 0 & 0 \\ 0 & a_1a_4 - a_2a_3 \end{pmatrix} = 0_2,$$

hence $\alpha_2^J\alpha_1 = (\alpha_1^J\alpha_2)^J = 0_2$ as we intended to prove. □

Lemma 5.1 can be restated as follows.

Lemma 5.2. *If $(\alpha_1, \alpha_2) \in {}^2R$ is unimodular and $\alpha_1\alpha_1^J = \alpha_2\alpha_2^J = \alpha_1\alpha_2^J + \alpha_2\alpha_1^J = 0_2$, then $\alpha_1^J\alpha_2 = \alpha_2^J\alpha_1 = 0_2$.* □

Our next purpose is to derive the form which has a true transvection $I + s\psi$ ($s \in M$, $\psi \in M^*$) of the group X when s does not belong to N. For the sake of clarity, we carry out this in a sequence of lemmas.

Lemma 5.3. *Let $g = I + s\psi$ be a true transvection of $GL_2(R) \cong GL(M)$ ($s \in M$, $\psi \in M^*$). Set $s = f_1\alpha_1 + f_2\alpha_2$, $\psi = \beta_1 f_1' + \beta_2 f_2'$, where $\alpha_i, \beta_i \in R$. If $\alpha_1, \alpha_2 \in R^\times$, then $\beta_1, \beta_2 \in R^\times$.*

Proof. Let us assume contrary to our claim that at least one of β_1, β_2 is not invertible. The case when only one of β_1, β_2 does not belong to $R^\times$ is impossible for if this were true, the equation $\beta_1\alpha_1 = -\beta_2\alpha_2$ would show that an invertible element of R is equal to a non-invertible one. Therefore, both β_1, β_2 are not invertible. Then since $\alpha_2 \in R^\times$, we have $\beta_2 = -\beta_1\alpha_1\alpha_2^{-1} \in \beta_1 R$ which tells us that both β_1, β_2 are contained in one and the same maximal right ideal of R. However, this means that ψ is not a free element of the dual module M^* contradicting the assumption that g is a true transvection of $GL_2(R)$. The lemma is proved. □

Further, for any $X \leq GL_2(R)$ let us introduce into consideration the set

$$L(X) = \{\gamma \in R \mid t_{12}(\gamma) \in X\},$$

which is easily seen to be an additive subgroup of R. Moreover, this $L(X)$ has the following important property.

Lemma 5.4. *If $X \leq GL_2(R)$ satisfies the conditions of Proposition 5.1, then $L(X) \subseteq K1_2$, that is, $L(X)$ consists of scalar matrices only.*

Proof. Let $b \in L(X)$. Since for any $a \in SL_2(k)$, the matrix $\begin{pmatrix} a & 0_2 \\ 0_2 & a \end{pmatrix} = a \otimes 1_2$ is contained in X, we have

$$\begin{pmatrix} a & 0_2 \\ 0_2 & a \end{pmatrix}\begin{pmatrix} 1_2 & b \\ 0_2 & 1_2 \end{pmatrix}\begin{pmatrix} a & 0_2 \\ 0_2 & a \end{pmatrix}^{-1} = \begin{pmatrix} 1_2 & aba^{-1} \\ 0_2 & 1_2 \end{pmatrix} \in X,$$

which implies that

$$aba^{-1} \in L(X) \quad \text{for any } a \in SL_2(k). \tag{5.1}$$

Write $b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ with $b_i \in K$. Let r be an arbitrary element of k. In (5.1), we take $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$ as a to get

$$\begin{pmatrix} b_1 + rb_3 & -b_1r - r^2b_3 + b_2 + b_4r \\ b_3 & -b_3r + b_4 \end{pmatrix} \in L(X). \tag{5.2}$$

Now let us replace r by $-r$ in (5.2) and then add the relation obtained with (5.2). Taking into account the condition $\operatorname{char} k \neq 2$, we get

$$b' = \begin{pmatrix} b_1 & -r^2b_3 + b_2 \\ b_3 & b_4 \end{pmatrix} \in L(X),$$

whence it follows that $b - b' = \begin{pmatrix} 0 & r^2b_3 \\ 0 & 0 \end{pmatrix} \in L(X)$. This means that

$$\left(\begin{array}{c|cc} 1_2 & 0 & b_3 \\ & 0 & 0 \\ \hline 0_2 & \multicolumn{2}{c}{1_2} \end{array}\right) \in X$$

which forces b_3 to be zero since X contains no transvections. Similarly we can take $\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$ as a in (5.1) to obtain $b_2 = 0$. In (5.2), we make the substitution $b_2 = b_3 = 0$. This yields $b'' = \begin{pmatrix} b_1 & (b_4 - b_1)r \\ 0 & b_4 \end{pmatrix} \in L(X)$, whence it follows that $b'' - b = \begin{pmatrix} 0 & r(b_4 - b_1) \\ 0 & 0 \end{pmatrix} \in L(X)$. Employing once again the assumption that X has no transvections, we conclude that $b_1 = b_4$ and hence $b = b_1 1_2$. The lemma is proved. □

Lemma 5.5. *Let s be a free element of M such that $s \notin N$. The group $SL_2(k)$ treated as the subgroup $SL_2(k1_2)$ of the group $GL_2(R)$ contains an element h such that if $h(s) = f_1\xi_1 + f_2\xi_2$ with $\xi_i \in R$, then both ξ_1, ξ_2 are invertible elements of R.*

Proof. Express s as a sum $f_1\alpha_1 + f_2\alpha_2$ with $\alpha_i \in R$. If both α_1, α_2 are contained in $R^\times$, then there is nothing to prove. Suppose therefore that at least one of α_1, α_2 is not invertible. If α_1, α_2 are both not invertible, then $\alpha_1\alpha_2^J + \alpha_2\alpha_1^J \neq 0_2$, so $\alpha_1 + \alpha_2 \in R^\times$. Hence taking h_0 to be the transvection $I + f_2f_1'$ and replacing s by $h_0(s)$, we may assume that only one of α_1, α_2 is non-invertible. Next define h_1 to be the element of $SL_2(k1_2)$ such that $h_1(f_1) = -f_2$, $h_1(f_2) = f_1$. Replacing, if necessary, s by $h_1(s)$, we may assume that $\alpha_1 \notin R^\times$, $\alpha_2 \in R^\times$. By virtue of the inequality $|k| > 2$, one can find $r \in k^\times$ such that $(\alpha_1\alpha_2^J + \alpha_2\alpha_1^J) + r\alpha_2\alpha_2^J \neq 0_2$, and so we may take $I + f_1rf_2'$ as h. □

With all these preparations made, we give the desired form of the true transvection $I + s\psi \in X$ with $s \in M \setminus N$ ($\psi \in M^*$).

Lemma 5.6. *If $g = I + s\psi \in X (s \in M, \psi \in M^*)$ is a true transvection of $GL_2(R)$ and $s \notin N$, then g as a 2-transvection of $GL_4(K)$ is a member of the subgroup $1_2 \otimes SL_2(K)$, or more precisely, has the form $1_2 \otimes g'$, where g' is a transvection of $SL_2(K)$.*

Proof. Set $s = f_1\alpha_1 + f_2\alpha_2$, $\psi = \beta_1f_1' + \beta_2f_2'$ with $\alpha_i, \beta_i \in R$. By Lemma 5.5, we may assume that $\alpha_1, \alpha_2 \in R^\times$. Then by Lemma 5.3, $\beta_1, \beta_2 \in R^\times$. Further, by Lemma 5.4, $L(X) \subseteq K1_2$. Proceeding as in the proof of Lemma 5.2 [6], we find that

$$\alpha_2\beta_1 \in K1_2, \quad \alpha_1\beta_2 \in K1_2, \quad \alpha_1\beta_1 - \alpha_2\beta_2 \in K1_2. \tag{5.3}$$

Since $g = I + (s\alpha_2^{-1})(\alpha_2\psi)$, we may assume $\alpha_2 = 1_2$. Relations (5.3) then show that $\alpha_i, \beta_i \in K1_2$, so $g \in SL_2(K1_2)$, or in other words, $g \in 1_2 \otimes SL_2(K)$. Thus $g = 1_2 \otimes g'$, where g' is a transvection of $SL_2(K)$ by Lemma 3.2. The lemma is proved completely. □

Lemmas 5.7 and 5.9 below give the form which has a true transvection $g = I + s\psi \in X$ with $s \in N$. They show that g as an element of $GL_4(K)$ is contained in the group $SL_2(K) \otimes 1_2$.

Lemma 5.7. (a) *Let* $g = I + s\psi \in X (s \in M, \psi \in M^*)$ *be a true transvection of* $GL_2(R)$ *and let* $s = f_1\alpha_1 + f_2\alpha_2$, $\psi = \beta_1 f_1' + \beta_2 f_2' (\alpha_i, \beta_i \in R)$. *Assume that* $s \in N$. *Then* $\beta_1 = c_1\alpha_1^J + c_2\alpha_2^J$, $\beta_2 = c_3\alpha_1^J + c_4\alpha_2^J$ *for some* $c_1, c_2, c_3, c_4 \in K$.

(b) *If* $s = f_1\alpha_1 + f_2\alpha_2 (\alpha_1, \alpha_2 \in R)$ *belongs to N and* $\beta_1 = c_1\alpha_1^J + c_2\alpha_2^J$, $\beta_2 = c_3\alpha_1^J + c_4\alpha_2^J$ *with* $c_i \in K$, *then* $\beta_1\alpha_1 + \beta_2\alpha_2 = 0_2$.

Proof. After an inner automorphism by a suitably chosen an invertible element of R, we may assume that $\alpha_1 = \begin{pmatrix} a_1 & 0 \\ a_2 & 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 & a_3 \\ 0 & a_4 \end{pmatrix}$, where $a_1, a_2, a_3, a_4 \in K$ subject to

$$a_1a_4 - a_2a_3 = 0. \tag{5.4}$$

Write

$$\beta_1 = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}, \qquad \beta_2 = \begin{pmatrix} b_5 & b_6 \\ b_7 & b_8 \end{pmatrix}, \quad b_i \in K.$$

The equation $\beta_1\alpha_1 + \beta_2\alpha_2 = 0_2$ is tantamount to the system

$$\begin{cases} b_1a_1 + b_2a_2 = 0, \\ b_3a_1 + b_4a_2 = 0, \\ b_5a_3 + b_6a_4 = 0, \\ b_7a_3 + b_8a_4 = 0. \end{cases} \tag{5.5}$$

Since α_1 and α_2 are not contained in one and the same maximal left ideal of R, the pairs (a_1, a_2) and (a_3, a_4) are different from the pair (0, 0). This and (5.4) show that

$$a_3 = ra_1, \qquad a_4 = ra_2 \tag{5.6}$$

for some $r \in K^\times$. The first and the second equations of (5.5) together with (5.4) imply then that the rank of the matrix

$$\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \\ a_4 & -a_3 \end{pmatrix}$$

equals 1. So $(b_1, b_2) = c_2(a_4, -a_3)$ for some $c_2 \in K$, and $(b_3, b_4) = c_1'(a_4, -a_3) = -c_1'r(-a_2, a_1)$ for some $c_1' \in K$, that is, $(b_3, b_4) = c_1(-a_2, a_1)$, where $c_1 = -c_1'r \in K$. Hence

$$\beta_1 = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} c_2a_4 & -c_2a_3 \\ -c_1a_2 & c_1a_1 \end{pmatrix} = c_1\alpha_1^J + c_2\alpha_2^J.$$

The substitution of (5.6) into the third and the forth equations of (5.5) yields $b_5a_1 + b_6a_2 = b_7a_1 + b_8a_2 = 0$. Combined with (5.4), these equalities mean that the rank of the matrix

$$\begin{pmatrix} b_5 & b_6 \\ b_7 & b_8 \\ a_4 & -a_3 \end{pmatrix}$$

is 1, and so one can deduce as above that $b_5 = c_4a_4$, $b_6 = -c_4a_3$, $b_7 = -c_3a_2$, $b_8 = c_3a_1$ for some $c_3, c_4 \in K$. Hence $\beta_2 = c_3\alpha_1^J + c_4\alpha_2^J$ proving part (a) of the lemma.

To prove part (b) we observe that by Lemma 5.1, $\alpha_1^J\alpha_2 = \alpha_2^J\alpha_1 = 0_2$. Consequently, $\beta_1\alpha_1 + \beta_2\alpha_2 = c_1\alpha_1\alpha_1^J + c_2\alpha_2^J\alpha_1 + c_3\alpha_1^J\alpha_2 + c_4\alpha_2\alpha_2^J$ and this sum is equal to zero because this is valid for each of its terms. □

Note that the preceding lemma admits the following restatement.

Lemma 5.8. *Let the column* ${}^t(\alpha_1, \alpha_2)$ *and the row* (β_1, β_2) *be unimodular elements of* R^2 *and* ${}^2R$, *respectively, and* $\beta_1\alpha_1 + \beta_2\alpha_2 = 0_2$. *Then* $\alpha_1\alpha_1^J = \alpha_2\alpha_2^J = \alpha_1\alpha_2^J + \alpha_2\alpha_1^J = 0_2$ *if and only if* $\beta_1 = c_1\alpha_1^J + c_2\alpha_2^J$, $\beta_2 = c_3\alpha_1^J + c_4\alpha_2^J$ *for some* $c_i \in K$.

Lemma 5.9. *If* $g = I + s\psi$ $(s \in M, \psi \in M^*)$ *is a true transvection of X such that* $s \in N$, *then g as an element of* $GL_4(K)$ *has the form* $g' \otimes 1_2$, *where* g' *is a transvection of* $SL_2(K)$.

Proof. Put $s = f_1\alpha_1 + f_2\alpha_2$, $\psi = \beta_1 f_1' + \beta_2 f_2' (\alpha_i, \beta_i \in R)$. By Lemma 5.7, $\beta_1 = c_1\alpha_1^J + c_2\alpha_2^J$, $\beta_2 = c_3\alpha_1^J + c_4\alpha_2^J$ for some $c_i \in K$, and so

$$g = \begin{pmatrix} 1 + c_2\alpha_1\alpha_2^J & c_4\alpha_1\alpha_2^J \\ -c_1\alpha_1\alpha_2^J & 1 - c_3\alpha_1\alpha_2^J \end{pmatrix}.$$

Set $y = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (more precisely, $y = \begin{pmatrix} 1_2 & 0_2 \\ 1_2 & 1_2 \end{pmatrix}$, $h = \begin{pmatrix} 1_2 & 1_2 \\ 0_2 & 1_2 \end{pmatrix}$ if y and h are thought of as being elements of $GL_4(K)$). It is straightforward to verify that

$$^{yg}h = I + (f_1(1_2 + c_2\alpha_1\alpha_2^J) + f_2(1_2 + (c_2 - c_1)\alpha_1\alpha_2^J))((-1_2 + (c_1 - c_3)\alpha_1\alpha_2^J)f_1' + (1_2 + c_3\alpha_1\alpha_2^J)f_2').$$

Since $\alpha_1\alpha_2^J$ is nilpotent, all the elements $1_2 + c_2\alpha_1\alpha_2^J$, $1_2 + (c_2 - c_1)\alpha_1\alpha_2^J$, $-1_2 + (c_1 - c_3)\alpha_1\alpha_2^J$, $1_2 + c_3\alpha_1\alpha_2^J$ are invertible in R. According to the proof of Lemma 5.6 (see relations (5.3)),

$$(1_2 + c_2\alpha_1\alpha_2^J)(1_2 + c_3\alpha_1\alpha_2^J) \in K1_2, \tag{5.7}$$

$$[1_2 + (c_2 - c_1)\alpha_1\alpha_2^J][-1_2 + (c_1 - c_3)\alpha_1\alpha_2^J] \in K1_2. \tag{5.8}$$

Relation (5.7) shows that $(c_2 + c_3)\alpha_1\alpha_2^J \in K1_2$. Thus $(c_2 + c_3)\alpha_1\alpha_2^J$ is a non-invertible scalar matrix of $M_2(K)$, that is, $(c_2 + c_3)\alpha_1\alpha_2^J = 0_2$. Since $\alpha_1\alpha_2^J \neq 0_2$ for $g \neq 1_2$, we conclude that $c_2 + c_3 = 0$, and so from relation (5.8) we obtain $c_1 = 0$. Consequently,

$$g = \begin{pmatrix} 1_2 + c_2\alpha_1\alpha_2^J & c_4\alpha_1\alpha_2^J \\ 0_2 & 1_2 + c_2\alpha_1\alpha_2^J \end{pmatrix}.$$

It follows that

$$^{hg}y = I + (f_1(1_2 + (c_2 + c_4)\alpha_1\alpha_2^J) + f_2(1_2 + c_2\alpha_1\alpha_2^J))((1_2 - c_2\alpha_1\alpha_2^J)f_1' + (-1_2 + (c_2 - c_4)\alpha_1\alpha_2^J)f_2'),$$

and so we have

$$[1_2 + (c_2 + c_4)\alpha_1\alpha_2^J][-1_2 + (c_2 - c_4)\alpha_1\alpha_2^J] \in K1_2,$$

whence it follows that $c_4 = 0$. Thus we eventually obtain

$$g = \begin{pmatrix} 1_2 + c_2\alpha_1\alpha_2^J & 0_2 \\ 0_2 & 1_2 + c_2\alpha_1\alpha_2^J \end{pmatrix} = (1_2 + c_2\alpha_1\alpha_2^J) \otimes 1_2 \in SL_2(K) \otimes 1_2,$$

where $1_2 + c_2\alpha_1\alpha_2^J$ must be a transvection of $SL_2(K)$ by Lemma 3.2 (notice that this can be deduced directly from the observation that $\alpha_1\alpha_2^J$ is nilpotent). The lemma is proved. □

Proof of Proposition 5.1. Let G be a subgroup of X generated by all true 2-transvections that are contained in X. Clearly $G \trianglelefteq X$. Here we again regard X as a subgroup of $GL_2(R)$ with $R = M_2(K)$. Then G is generated by those true transvections of $GL_2(R)$ that are contained in X. For each true transvection $g = I + s\psi \in X (s \in M, \psi \in M^+)$, we have one of the following: either $s \in N$ or $s \notin N$. If $s \in N$, then by Lemma 5.9, g as an element of $GL_4(K)$ can be written in the form $h \otimes 1_2$ with h a transvection of $GL_2(K)$. If $s \notin N$, then by Lemma 5.6, g viewed as an element of $GL_4(K)$ can be represented as $1_2 \otimes h'$ with h' a transvection of $GL_2(K)$. Thus if T denotes the set of all true 2-transvections of $GL_4(K)$ that are contained in X, T_1 denotes the set of all elements of T having the form $h \otimes 1_2$ with $h \in SL_2(K)$, and T_2 denotes the set of all elements of T having the form $1_2 \otimes h'$, where h' is a transvection of $GL_2(K)$, then $T = T_1 \cup T_2$. By virtue of [3], $\langle T_1 \rangle = SL_2(L_1) \otimes 1_2$, $\langle T_2 \rangle = 1_2 \otimes SL_2(L_2)$, where L_1, L_2 are subfields of K containing k. So

$$G = \langle T \rangle = \langle SL_2(L_1) \otimes 1_2, 1_2 \otimes SL_2(L_2) \rangle = SL_2(L_1) \otimes SL_2(L_2).$$

The proposition is proved. □

6. Proofs of main results

Proof of Theorem 1.1. With the help of the results presented in the two preceding sections it is now an easy matter to prove Theorem 1.1. Indeed, if the group X contains a transvection, then by Proposition 4.1, X contains a normal subgroup which is conjugate in $GL_4(K)$ either to $SL_4(L)$ or to $SU_4(L, \Phi, \sigma)$, where L is a subfield of K containing k, σ is an automorphism of order 2 of L, Φ is a non-degenerate σ-skew-Hermitian form in four variables over L such that the Witt index of Φ equals 2. If X contains no transvections, then by Proposition 5.1, X contains a normal subgroup $SL_2(L_1) \otimes SL_2(L_2)$, where L_1, L_2 are subfields of K containing k. The theorem is proved completely. □

We close the paper with an interesting fact that Theorem 1.1 gives actually the description of linear groups of degree four over the field K containing the group $\Omega_4(k, Q)$, where Q is a non-degenerate quadratic form in four variables over the field k such that the Witt index of Q equals 2. This can be explained due to the fact that the groups $SL_2(k) \otimes SL_2(k)$ and $\Omega_4(k, Q)$ are isomorphic. The explicit form of this isomorphism can be easily deduced from [7] (ch. IV, section 8, case 7). Moreover, endow the k-vector space $E = k^4$ with a quadratic form Q and denote by Φ the symmetric bilinear form associated with Q. In other words,

$$\Phi(x, y) = \frac{1}{2}(Q(x + y) - Q(x) - Q(y)) \; \forall x, \quad y \in E.$$

Fix a basis $\{e_1, e_2, e_3, e_4\}$ of E such that the matrix of Φ with respect to this one is

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

If we do not distinguish endomorphisms of E from their matrices with respect to the basis $\{e_1, e_2, e_3, e_4\}$, then according to the above mentioned result from [7], $\Omega_4(k, Q) = SL_2(k) \otimes SL_2(k)$.

Combined, this and Theorem 1.1 lead immediately to the following result.

Theorem 6.1. *Let k be a field of characteristic $\neq 2$, K a field algebraic extension of k, Q a non-degenerate quadratic form in four variables over k. Assume that the Witt index of Q equals 2 and $|k| > 9$. If $\Omega_4(k, Q) \leq X \leq GL_4(K)$, then X contains a normal subgroup G such that one of the following holds:*

(1) *G is conjugate in $GL_4(K)$ to $SL_4(L)$, where L is a subfield of K containing k.*
(2) *G is conjugate in $GL_4(K)$ to $SU_4(L, \Phi, \sigma)$, where L is a subfield of K containing k, σ is a non-trivial automorphism of L with $\sigma^2 = \mathrm{id}$, Φ is a non-degenerate σ-skew-Hermitian form in four variables over L, the Witt index of Φ being equal to 2.*
(3) *$G = SL_2(L_1) \otimes SL_2(L_2)$, where L_1 and L_2 are subfields of K containing k.*

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