The Hardy Space of Solutions to First-Order Elliptic Systems

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On a domain $D \subseteq \mathbb{C}$ in the plane, consider the first-order elliptic system

$$\frac{\partial U}{\partial y} - A\frac{\partial U}{\partial x} = 0 \qquad (1)$$

with coefficient $A \in \mathbb{C}^{l \times l}$. Ellipticity means that the matrix A has no real eigenvalues. It is well known that the solutions $U = (U_1, U_2, \ldots, U_l)$ to this system are real analytic on the domain D.

Suppose that the boundary $\Gamma = \partial D$ of the domain is a piecewise Lyapunov contour; i.e., its connected components are homeomorphic to the circle and can be represented as unions of finitely many Lyapunov arcs, which can intersect only in endpoints. Recall that an arc L is Lyapunov if it admits a smooth parameterization of class $C^{1,\mu}$ for some $0 < \mu < 1$. Take a sequence of contours $\Gamma_n \subseteq D$, where $n = 1, 2, \ldots$, converging to Γ in the sense that there exists a homeomorphism $\alpha_n: \Gamma \to \Gamma_n$ with piecewise continuous derivative α'_n such that $\alpha_n(t) - t \to 0$ as $n \to \infty$ in the sup-norm and the derivatives α'_n are uniformly bounded.

The Hardy space $H^p(D)$, where $1 < p < \infty$, of solutions to elliptic systems is defined by the condition that the norm

$$|U| = \sup_n |U|_{L^p(\Gamma_n)} \qquad (2)$$

must be finite. The domain D can be finite or infinite; in the latter case, the solutions u to system (1) are assumed to be bounded at infinity. The L^p-norm of a function $U(t)$ on Γ_n is related to the number function $|U(t)|$, where $|\cdot|$ denotes some fixed norm on $\mathbb{R}^l$. The space H^p can also be defined for $p = 1$, but we do not consider this case.

For analytic functions, this definition generalizes the classical Hardy class on the unit disk and is due to V.I. Smirnov, M.A. Lavrent'ev, and M.V. Keldysh (see [1]). This class is also denoted by $E^p(D)$.

Of special interest are systems of the form (1) for which

$$\frac{\partial \phi}{\partial y} - J\frac{\partial \phi}{\partial x} = 0, \qquad (3)$$

where the spectrum $\sigma(J)$ of the matrix $J \in \mathbb{C}^{l \times l}$ is contained in the upper half-plane. System (3) is a natural generalization of the Cauchy–Riemann equation; it was introduced by Douglis in [2]. The basic properties of analytic functions are inherent in the solutions to this system [3]; for this reason, they are called hyperanalytic functions, or functions analytic in the sense of Douglis (or, briefly, J-analytic functions).

In the general case, the solutions of system (1) can be uniquely expressed in terms of those of (3). Indeed, if an invertible matrix B reduces A to a block-diagonal form $J = \mathrm{diag}(J_1, \bar{J}_2)$ (i.e., $B^{-1}AB = J$) and the eigenvalues of J_k are contained in the upper half-plane, then we have the representation $U = B\phi$, where $\phi = (\phi_1, \bar{\phi}_2)$, of a solution to system (1) in terms of the J_k-analytic functions ϕ_k. This allows us to consider only system (3) in what follows.

If a J-analytic function ϕ is continuous on a closed simple domain $\bar{D} = D \cup \Gamma$, then the following Cauchy formula is valid [3]:

$$2\phi(z) = \frac{1}{\pi i}\int_\Gamma (t - z)_J^{-1} dt_J \phi^+(t), \quad t \in D, \qquad (4)$$

where $(x_1 + ix_2)_J = x_1 1 + x_2 J$ for $x_j \in \mathbb{R}$, the expression $(dx_1 + dx_2)_J = (dx_1)1 + (dx_2)J$ has similar meaning for the matrix differential, and the contour $\Gamma = \partial D$ has positive orientation with respect to D. Replacing the boundary value ϕ^+ by an arbitrary density φ, we obtain a generalized Cauchy-type integral, which we denote by $(I\varphi)(z)$.

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The definition of Hardy spaces can be extended to the weight case in a natural way. Let F be a finite set of points $\tau \in \Gamma$. For a family $\lambda = (\lambda_\tau, \tau \in F)$ of real numbers (of weight order), we denote the class of continuous functions on $\Gamma \backslash F$ with asymptotics $O(1)|t-\tau|^{\lambda_\tau}$ as $t \to \tau$ by $C_\lambda(\Gamma, F)$. A scalar function $\rho \in C_\lambda$ is said to be weight (of order λ) if it does not vanish and $\rho^{-1} \in C_{-\lambda}$. We indicate the dependence on λ by a subscript: $\rho = \rho_\lambda$.

In a similar way, weight functions on the domain D with respect to the corresponding class $C_\lambda = C_\lambda(\overline{D}, F)$ are defined. In this case, of main interest are J-analytic $(l \times l)$-matrix weight functions $R(z) = R_\lambda(z)$ satisfying system (3) and subject to the additional constraint $R(z)J = JR(z)$ for $z \in D$. This constraint means that the operation $\phi \to R\phi$ of multiplication by a weight function does not lead out from the class of J-analytic vector functions ϕ. Weight functions $R_\lambda(z)$ mentioned above always exist. For example, if the domain D is finite and Γ is a simple contour, then we can set

$$R_\lambda(z) = \prod_{\tau \in F} (z-\tau)_J^{\lambda_\tau}.$$

Here, in the notation of (4), the multiplier $(z-\tau)_J^\alpha = \exp[\alpha \ln(z-\tau)_J]$ is understood as a function of a matrix for the branch of the logarithm $\ln(z-\tau)$ continuous on D.

Let $L^p_\lambda(\Gamma, F)$ be the Banach space on the boundary $\Gamma = \partial D$ such that $L^p_\lambda = L^p$ for $\lambda = -\frac{1}{p}$ and the weight transformation $\varphi \to \rho_\nu \varphi$ implements an isomorphism between L^p on $L^p_{\nu-1/p}$. Note that the class $C_{\lambda+0}(\Gamma, F) = \bigcup_{\varepsilon > 0} C_{\lambda+0}$ is densely contained in $L^p_\lambda(\Gamma, F)$.

In a similar way, we define a space $L^p_\lambda(D, F)$ on the domain D by the condition $L^p_\lambda = L^p$ with $\lambda = -\frac{2}{p}$ and a family of weight Hardy spaces $H^p_\lambda(D, F)$ by the condition $H^p_\lambda = H^p$ with $\lambda = -\frac{1}{p}$ and the weight transformation $\phi \to R_\nu \phi$. Obviously, for $\lambda = 0$, the space L^p_0 on Γ and on D is an L^p-space with respect to the measures $d\mu = \rho_{-1}(t)ds$, where $t \in \Gamma$, and $d\mu = \rho_{-2}(z)dx\,dy$, where $z = x + iy \in D$, respectively.

The following theorem gives all basic properties of the Hardy spaces.

Theorem 1. (a) *Let $\phi \in H^p_\lambda(D)$ be a solution to system* (1). *Then, there exist almost everywhere angular limit values $\phi^+(t)$, which determine a function $\phi^+ \in L^p_\lambda(\Gamma)$. The space $H^p_\lambda(D)$ is Banach, the norm $|\phi| = |\phi^+|_{L^p_\lambda}$ is equivalent to* (2), *and there are continuous dense embeddings $C_{\lambda+0}(\overline{D}, F) \subseteq H^p_\lambda(D, F) \subseteq L^p_\lambda(D, F)$.*

(b) *If $\phi \in H^p_\lambda(D)$, then the restriction of ϕ to each subdomain $D_0 \subseteq D$ bounded by a piecewise Lyapunov contour belongs to $H^p_\lambda(D_0)$. Conversely, if such subdomains $D_1, D_2, \ldots, D_m$ are disjoint, $\overline{D}_1 \cup \overline{D}_2 \cup \ldots \cup \overline{D}_m = \overline{D}$, and ϕ is a solution to system* (1) *on the domain D belonging to $H^p_\lambda(D_j)$ for $j = 1, 2, \ldots, m$, then $\phi \in H_p(D)$; moreover, the norm on $H^p_\lambda(D)$ is equivalent to $|\phi| = \sum_j |\phi|_{H^p_\lambda(D_j)}$.*

(c) *The Cauchy-type integral $I\varphi$ determines a bounded linear operator $L^p_\lambda(\Gamma, F) \to H^p_\lambda(D, F)$ for $-1 < \lambda < 0$, and, for almost all $t_0 \in \Gamma$, the Sokhotskii–Plemelj formula*

$$(I\varphi)^+(t_0) = \varphi(t_0) + \frac{1}{\pi i}\int_\Gamma (t-t_0)_J^{-1} dt_J \varphi(t)$$

holds. Here, the integral is singular; it is understood in the sense of the Cauchy principal value.

The classes $C_{\lambda+0}$ and $L^p_\lambda(D)$ in assertion (b) of Theorem 1 refer to J-analytic functions. Assertions (a) and (b) imply, in particular, that the space H^p_λ does not depend on the choice of the sequence Γ_n in (2) and can be defined as the completion of the class $C_{\lambda+0}$ in the norm $|\phi| = |\phi^+|_{L^p_\lambda}$. This scheme was implemented in [4] for usual analytic functions. The above spaces for analytic functions were studied in [5] under more general assumptions about the boundary of the domain D and the weight. Theorem 1 (b) makes it possible to naturally extend the definition of $H^p_\lambda(D)$ to domains D with any piecewise Lyapunov boundary. The proofs of assertions (a) and (b) of Theorem 1 are based on assertion (c), which is proved in precisely the same way as in [6].

The analogue of N.I. Muskhelishvili's theorem about a representation of ϕ by the integral $I\varphi$ with a real vector function φ, which was proved in [3] for the Hölder classes, remains valid for the Hardy classes.

Recall that a bounded operator N: $X \to Y$ between Banach spaces X and Y is said to be Fredholm if its kernel $\ker N$ is finite-dimensional, the image $\mathrm{Im} N$ is closed, and there exists a finite-dimensional subspace $Y_0 \subseteq Y$ such that $Y = Y_0 \oplus \mathrm{Im} N$ [7]. It is convenient to call this subspace the coimage of the operator N and denote it by $\mathrm{coIm} N$, although it is not uniquely determined by N. The difference $\mathrm{ind} N$ between the dimensions of the kernel and the coimage is called the index of the operator N.

Theorem 2. (a) *If the domain D is bounded by a Lyapunov contour Γ, then the operator I: $L^p_\lambda(\Gamma, F) \to H^p_\lambda(D, F)$ $(-1 < \lambda < 0)$ is Fredholm, and its index equals $l(s-2)$, where s is the number of connected components of the contour. Moreover,* $\ker I \subseteq C^{+0}(\Gamma) = \bigcup_{\mu > 0} C^\mu(\Gamma)$, *and there exists a coimage* $\operatorname{coIm} I \subseteq C^{+0}(\bar{D})$.

(b) *Suppose that the matrix J is triangular, the domain D is bounded by a piecewise Lyapunov contour Γ without cusp points, and the set F contains all corner points of the contour.*

Then, the operator I: $L^p_\lambda(\Gamma, F) \to H^p_\lambda(D, F)$ $(-\frac{1}{2} < \lambda < 0)$ is Fredholm, and its index equals $l(2-s)$. Moreover, the elements of the kernel are constant on the connected components of Γ, and the operator has a coimage whose elements are constant on D.

More precisely, the kernel and the coimage mentioned in assertion (b) are described as follows. If the domain D is infinite, then $\ker I$ consists of all locally constant functions, and we can set $\operatorname{coIm} I = \mathbb{C}^l$. If the domain D is finite, then $\ker I$ consists of all locally constant functions vanishing on the exterior component of the contour Γ, and $\operatorname{coIm} I = \{i\xi | \xi \in \mathbb{R}^l\}$. By the exterior component we understand the connected component of Γ enclosing the domain D.

There is another approach to defining weight spaces, which is based on the consideration of homogeneity with respect to dilations. Suppose that F consists of one point, which is convenient to take for the origin. Let us cover $\Gamma \backslash \{0\}$ by smooth curves $\Gamma_1, \Gamma_2, \dots$ so that $\tilde{\Gamma}_n = 2^n \Gamma_n \subseteq \left\{ \frac{1}{2} \le |t| \le 2 \right\}$ starting with some number $n \ge m$. Setting $\tilde{\Gamma}_n = \Gamma_n$ for $n < m$, we can associate each function $\varphi \in L^p_\lambda(\Gamma, 0)$ with the sequence of functions

$$\tilde{\varphi}_n(t) = \begin{cases} \varphi(2^{-n}t), & n \ge m \\ \varphi(t), & n < m, \end{cases} \qquad \tilde{\varphi}_n(t) \in L^p(\tilde{\Gamma}_n),$$

and the space $L^p_\lambda(\Gamma, 0)$ can be described by using the equivalent norm

$$|\varphi| = \left(\sum_n |\xi_n|^p \right)^{1/p}, \quad \xi_n = 2^{-n\lambda} |\tilde{\varphi}_n|_{L^p(\tilde{\Gamma}_n)}. \tag{5}$$

A similar fact is valid for the weight Hardy space.

Theorem 3. *Suppose that the domain D is represented as the union of domains $D_1, D_2, \dots$ such that $\tilde{D}_n = 2^n D_n \subseteq \left\{ \frac{1}{2} \le |z| \le 2 \right\}$ for $n \ge m$ and the boundaries $\partial \tilde{D}_n$ converge to some contour $\tilde{\Gamma}$ in the same sense as in* (2).

Then, for $\phi \in H^p_\lambda(D, 0)$,

$$\tilde{\phi}_n(z) = \begin{cases} \phi(2^{-n}z), & n \ge m \\ \phi(z), & n < m, \end{cases} \qquad \tilde{\phi}_n(z) \in H^p(\tilde{D}_n),$$

and the space $H^p_\lambda(D, 0)$ can be described by using the equivalent norm

$$|\phi| = \left(\sum_n |\xi_n|^p \right)^{1/p}, \quad \xi_n = 2^{-n\lambda} |\tilde{\phi}_n|_{H^p(\tilde{D}_n)}. \tag{6}$$

This theorem shows that the family of spaces introduced above monotonically decreases with respect to each of the parameters p and λ_τ in the sense of an embedding of Banach spaces. Note that the contours ∂D_n in the theorem can be chosen to be smooth. Thus, the weight Hardy space can be introduced on the basis of the definition of the spaces H^p on domains with smooth boundary. Applying translations, we can extend the above definition to the case of $F = \{\tau\}$ for any point $\tau \ne 0$. In the general case, the domain D can be represented as the union of domains D_τ with $\tau \in F$, where each D_τ is bounded by a contour Γ_τ smooth outside τ. The space $H^p_\lambda(D, F)$ can be defined by the condition $\phi \in H^p_{\lambda_\tau}(D_\tau, \tau)$ for all $\tau \in F$. According to Theorem 1 (b), this definition does not depend on the choice of D_τ.

On the basis of Theorem 2, we can transfer the results obtained in [6], including those on the Riemann–Hilbert problem with piecewise continuous matrix coefficient G, from weight Hölder spaces to Hardy spaces. Consider the simplest case ($G = 1$) of the Schwarz problem

$$\operatorname{Re} \phi^+ = f \tag{7}$$

with real right-hand side $f \in L^p_\lambda(\Gamma, F)$. By the Fredholm property and the index of the problem we mean those of the $\mathbb{R}$-linear operator $H^p_\lambda \to L^p_\lambda$ of its boundary condition.

The criterion for the Fredholm property, which is given below, is stated in terms of the end symbol, i.e., the family of entire functions $x_\tau(\zeta)$ of a complex variable ζ (where $\tau \in F$) defined as follows. We associate each pair of different unit vectors $a = a_1 + ia_2$ and $b = b_1 + ib_2$ with the analytic function ω of two variables ζ and u, where $\operatorname{Im} u \ne 0$, defined by

$$\omega(a, b; u, z) = \left(-\frac{a_1 + ua_2}{b_1 + ub_2} \right)^\zeta,$$

$$\left| \arg\left(-\frac{a_1 + ua_2}{b_1 + ub_2} \right) \right| < \pi.$$

Since the spectra of the matrices J and $\bar{J}$ are contained in the upper and lower half-planes, respectively, in which ω is analytic as a function of u, it follows that the values of this function at the above matrices can be defined. Thus, we can define an entire function of the variable ζ by the formula

$$h(a, b; \zeta) = \det[e^{\pi i \zeta}\omega(a, b; J, \zeta) - e^{-\pi i \zeta}\omega(a, b; \bar{J}, \zeta)]. \tag{8}$$

It is easy to show that, on each strip $\lambda_1 < \operatorname{Re}\zeta < \lambda_2$ of finite width, the function h has finitely many zeros. Therefore, the projection of the zero set of this function onto the real axis is a discrete subset of $\mathbb{R}$, which we denote by $\Delta(a, b)$.

If $a = -b$, then $\omega = 1$ and, therefore, the function $h(a, -a; \zeta)$ coincides with $\sin^l \pi\zeta$ up to a constant multiplier. Thus,

$$\Delta(a, -a) = \mathbb{Z}. \tag{9}$$

In the scalar case ($l = 1$), in which $J = \nu \in \mathbb{C}$, definition (8) takes the form

$$h(a, b; \zeta) = 2i|q|^{\zeta}\sin\theta\zeta, \quad q = \frac{b_1 + \nu b_2}{a_1 + \nu a_2},$$

where $\theta = \arg q$, $0 < \theta < 2\pi$. Obviously, in this case, we have

$$\Delta(a, b) = \left\{\frac{\pi}{\theta k}, k \in \mathbb{Z}\right\}. \tag{10}$$

Consider problem (7). In a small neighborhood of a corner point $\tau \in \Gamma$, the set $S_\tau = D \cap \{|z - \tau| < r\}$ is a curvilinear sector; we denote its lateral sides by $\Gamma_{\tau \pm 0}$ (it is assumed that the counterclockwise rotation inside the sector is from $\Gamma_{\tau + 0}$ to $\Gamma_{\tau - 0}$). Let $q_{\tau \pm 0}$ be the unit tangent vectors to the lateral sides $\Gamma_{\tau \pm 0}$ at the point τ. Suppose that these sides are not tangent to each other, i.e., τ is not a cusp point of the contour Γ. Then, $q_{\tau - 0} \neq q_{\tau + 0}$, and we can consider the function $x_\tau(\zeta) = h(q_{\tau + 0}, q_{\tau - 0}; \zeta)$, which we call the end symbol of the problem at the point τ. For the corresponding set $\Delta(q_{\tau + 0}, q_{\tau - 0})$ we use the short notation Δ_τ. Let $\chi_\tau(t)$ be an entire function constant on the intervals from the complement to Δ and such that, for $t \in \Delta_\tau$, the jump $\chi_\tau(t - 0) - \chi_\tau(t + 0)$ equals the number of zeros of the function $x_\tau(\zeta)$ on the line $\operatorname{Re}\zeta = t$ with multiplicities taken into account. This function monotonically decreases and is defined up to an additive constant, which is determined by the condition $\chi(-0) = 0$.

Theorem 4. *Suppose that the domain D is bounded by a piecewise Lyapunov contour without cusp points and the set F contains all corner point of this contour.*

Then, problem (7) *of class $H^p_\lambda(D, F)$ is Fredholm if and only if*

$$\lambda_\tau \notin \Delta_\tau, \quad \tau \in F, \tag{11}$$

and its index κ is given by $\kappa = l(2 - s) + \sum_\tau \chi_\tau(\lambda_\tau)$, where s denotes the number of connected components in Γ.

Note that, in the scalar case ($l = 1$, $J = \nu$), the set Δ_τ is determined by (10), where the quantity $\theta = \theta_\tau$ has the geometric meaning of the angle of the sector to which the curvilinear sector S_τ is mapped under the affine transformation $x + iy \to x + \nu y$. In the special case of analytic functions ($\nu = i$), θ_τ coincides with the angle of the sector S_τ itself. In the scalar case under consideration, condition (11) reduces to $\frac{\theta_\tau\lambda_\tau}{\pi} \notin \mathbb{Z}$, and $\chi_\tau(t) = -\left[\frac{\theta_\tau t}{\pi}\right] + 1$, where the brackets denote the integer part of a number. If the contour Γ is smooth, then $q_{\tau + 0} = -q_{\tau - 0}$ at $\tau \in F$, and, according to (9), the situation is the same as in the scalar case.

So far, the boundary Γ of the domain D was assumed to be finite. Suppose that the curve Γ is unbounded, i.e., $\infty \in \Gamma$, and it is a piecewise Lyapunov contour on the Riemann sphere $\mathbb{C} \cup \infty$. The convergence of the contours $\Gamma_n \subseteq D$ to Γ is understood in the same sense as above with the additional requirement that $\infty \in \Gamma_n$ for all n (in particular, the functions $\alpha_n(t)$ must be unbounded, i.e., $\alpha_n(\infty) = \infty$). The Hardy space $H^p(D)$ under these assumptions is defined by the same condition (2). In particular, for the half-plane $\{z = x + iy, y > 0\}$, we can define Γ_n to be the lines $y = \varepsilon_n$, where $\varepsilon_n \to 0$. For analytic functions, we obtain the classical Hardy space [8].

Let us define weight spaces in the case under consideration. We include the infinite point ∞ in F and define the class $C_\lambda(\Gamma, F)$ as above with the additional requirement that $\varphi(t) = O(1)|t|^{-\lambda_\infty}$ as $t \to \infty$. For this class, we define the weight scalar functions $\rho(t) = \rho_\lambda(t)$ and the J-analytic ($l \times l$)-matrix functions $R(z) = R_\lambda(z)$ as in the finite case. Using these functions, we define the weight spaces as above, with the only difference that $L^p_\lambda(\Gamma, F) = L^p(\Gamma)$ and $H^p_\lambda(\Gamma, F) = H^p(\Gamma)$ for the weight order λ taking the value $-\frac{1}{p}$ at the finite points $\tau \in F$ and $\frac{1}{p}$ at the point $\tau = \infty$, and, similarly, $L^p_\lambda(D, F) =$

$L^p(D)$ for the weight order λ taking the value $-\frac{2}{p}$ at the finite points $\tau \in F$ and $\frac{2}{p}$ at the point $\tau = \infty$. For the half-plane, the weight Hardy space for J-analytic functions was introduced and studied in [9].

Theorems 1 and 2 remain valid in the case under consideration, with the only difference that the conditions $-1 < \lambda < 0$ and $-\frac{1}{2} < \lambda < 0$ for $\tau = \infty$ are replaced by $0 < \lambda_\infty < 1$ and $0 < \lambda_\infty < \frac{1}{2}$, respectively. An analogue of Theorem 3 is valid as well. Suppose that the set F comprises only one point $\tau = \infty$, i.e., the unbounded curve Γ is smooth. Let us cover Γ by smooth curves Γ_n, where $n \le 1$, so that $\tilde{\Gamma}_n = 2^{-n}\Gamma_n \subseteq \left\{\frac{1}{2} \le |z| \le 2\right\}$ starting with some number $n \ge m$. Let D_n be domains defined in a similar way for D. Then, the space $L^p_\lambda(\Gamma, \infty)$ can be specified by norm (5), and the space $H^p_\lambda(D, \infty)$ in Theorem 3 can be specified by (6). Thus, the families of spaces under consideration, as well as in the case of finite contours, monotonically decrease with respect to each of the parameters p and λ_τ, where $\tau \in F$.

Theorem 4 also extends to the unbounded case; we must only enumerate the arcs $\Gamma_{\tau \pm 0}$ with endpoint $\tau = \infty$ and define the corresponding unit tangent vectors $q_{\tau \pm 0}$ to these arcs at this endpoint. Recall that, for $\tau \ne \infty$, the lateral sides $\Gamma_{\tau \pm 0}$ of the curvilinear sector $S_\tau = D \cap \{|z - \tau| < \varepsilon\}$, where $\varepsilon > 0$ is sufficiently small, with vertex τ are enumerated as follows. When the boundary ∂S_τ is traversed through the point τ from $\Gamma_{\tau - 0}$ to $\Gamma_{\tau + 0}$, the sector S_τ remains on the left. The arc $\Gamma_{\tau \pm 0}$ can be specified by the parametric equation $z - \tau = rq_{\tau \pm 0}\exp[ih(r)]$, where $0 \le r \le \varepsilon$ and the real function $h(r)$ is continuous and vanishes at $r = 0$. The case of $\tau = \infty$ can be handled similarly. Namely, for sufficiently small $R > 0$, the lateral sides of the curvilinear "sector" $S_\infty = D \cap \{|z| > R\}$ are "arcs" $\Gamma_{\tau \pm 0}$ with endpoint $\tau = \infty$, which can be specified by the parametric equation $z = rq_{\tau \pm 0}\exp[ih(r)]$ on the half-axis $r \ge R$, where the real function $h(r)$ is continuous and tends to zero as $r \to \infty$. These arcs can be enumerated in the same way as for finite vertices τ.

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