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SOME RESULTS ON PRIME LABELINGS OF GRAPHS

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A graph is said to be prime if it has a relatively prime labeling on its vertices which satisfies certain properties. The purpose of this paper is to give some new families of graphs that have a prime labeling and give some necessary and sufficient conditions for some families of prime graphs.

Key words: Graph, prime labeling.

Introduction

It is well known that graph theory has applications in many other fields of study, including physics, chemistry, biology, communication, psychology, sociology, economics, engineering, operations research, and especially computer science. For all standard notation and terminology in graph theory we follow [10]. Graph labelings where the vertices are assigned real values subject to certain conditions like as Graceful [9,12], Harmonious [9,12], Cordial [3-8], Prime [9] and others, have often been motivated by practical problems such as coding theory, communication networks and astronomy, but they are also of logico- mathematical interest in their own right. An enormous body of literature has grown around the subject especially in the last thirty years or so, and is presented in a survey by Gallian [9].

A graph G is an ordered pair G(V,E), where V(G) stands for a finite set of elements called vertices, while E(G) - a finite set of unordered pairs of vertices called edges. The cardinality of the set of vertices V(G) is denoted by the symbol $|V|$ and called the order of graph G. Likewise, the cardinality of the set of edges E(G) is denoted by the symbol $|E|$ and called the size of graph G. The vertices u, v $\in V(G)$ are called adjacent (or neighbors) if $\{u, v\}$ in E(G) and nonadjacent if $\{u, v\}$ not in E(G). The degree deg(v) of vertex v in graph G is the number of edges incident to vertex v in graph G, i.e., $|e \in E(G): v \in e|$. The maximum degree of a vertex in graph G is denoted by $\Delta(G)$, while the minimum degree is denoted by $\delta(G)$. A set of vertices V of a graph G is said to be independent if any two vertices u and v in V are not adjacent in G.

A graph G with vertex set V(G) is said to have a prime labeling if its vertices can be labeled with distinct integers 1, 2, 3, ... , $|V|$ such that for each $xy \in E(G)$ the labels assigned to x and y are relatively prime. A graph that admits a prime labeling is called a prime graph. The notion of a prime labeling originated with Entringer and was introduced in a paper by Tout, Dabboucy, and Howalla [14]. Around the classes of trees known to have a prime labelings are: paths, stars, caterpillars , complete binary trees, spiders and all tree of order up to 50. Also, other graphs with prime labelings include all cycles and a complete graph K_n does not have a prime labeling for $n \geq 4$ [9].

Youssef [15] has shown that $K_n \odot \overline{K_1}$ is prime if and only if $n \leq 7$. In section 3, we extend this result to show that $K_n \odot \overline{K_2}$ is prime if and only if $n \leq 16$. Moreover, we show that the graph $\overline{K_m} \odot K_n$ is prime if and only if m = 1 and n = 2, or n = 1 and for all $m \geq 1$.

Deretsky, Lee and Mitchem [2] have shown that the disjoint union $C_{2k} \cup C_n$ of two cycles C_{2k} and C_n is prime. In section 4, we extend this result to show that the union $C_n \cup C_m$ of two cycles C_n and C_m is not prime if and only if both n and m are odd. Moreover, we show that the joint $C_n + C_m$ of two cycles C_n and C_m is not prime for all $n \geq 3$ and $m \geq 3$, the union $P_n \cup P_m$ of two paths P_n and P_m is prime for all $n \geq 1$ and $m \geq 1$, the joint $P_n + P_m$ of two paths P_n and P_m is prime if and only if n =1 and $m \geq 1$(or vice versa), or n =2 and m odd (or vice versa), the union $C_n \cup P_m$ of cycles C_n and paths P_m is prime for all $n \geq 3$ and $m \geq 1$, and the joint $C_n + P_m$ of cycles C_n and paths P_m is not prime for all $n \geq 3$ and $m \geq 1$.

Seoud, Diab, and Elsakhawi [12], have shown the following complete bipartite graphs are prime: $K_{2,m}$ and $K_{3,m}$ unless m = 3 or m = 7. In section 5, we extend those results to some complete tripartite, namely, $K_{1,1,n}$ is prime for all $n \geq 1$ and $K_{1,2,n}$ is prime for all $n \geq 1$ except n = 3 or n = 7. Finally, we show that the following graphs are prime: $P_n \cup K_{1,m}$ for all $n \geq 1$ and $m \geq 1$, $C_n \cup K_{1,m}$ for all $n \geq 3$ and $m \geq 1$, $\overline{K_n} \cup K_{1,m}$ for all $n \geq 1$ and $m \geq 1$, $\overline{K_1} + K_{1,m}$ for all $m \geq 1$, $P_n \odot \overline{K_1}$ for all $n \geq 1$, $\overline{K_1} \odot P_n$ for all $n \geq 1$, and $\overline{K_n} \cup \overline{K_m}$ for all $n \geq 1$ and $m \geq 1$.

2. Notation and Preliminaries

We introduce some basic properties of graphs and we will study the effect of these properties on the prime graphs. The maximum cardinality of an independent set of vertices of a graph G is called the vertex independence number and is denoted by $\beta(G)$.

A coloring of a graph G is an assignment of colors (which are actually considered as elements of some set) to the vertices of G, one color to each vertex, so that adjacent vertices are assigned different colors. A graph G for which there exists a vertex - coloring which requires k colors is called k- colorable, while such a coloring is called a k- coloring. The smallest number k for which there exists a k-coloring of graph G is called the chromatic number of graph G and is denoted by $\chi(G)$. Such a graph G is called k-chromatic, while any coloring of G which requires k = $\chi(G)$ colors is called chromatic or optimal. Harary [10] has shown that for any graph G, we have $\chi(G) \leq |V| - 1 - \beta(G)$.

The clique number $\omega(G)$ of a graph G is the maximum order among the complete subgraph of G. Clearly, $\omega(G) = \beta(\overline{G})$ for every graph G, where the complement $\overline{G}$ of a graph G is that graph with vertex set V(G) such that two vertices are adjacent in $\overline{G}$ if and only if these vertices are not adjacent in G. If $K_n \subseteq G$ for some n, where K_n is a complete graph of order n, then $\chi(G) \geq n$. It follows that $\chi(G) \geq \omega(G)$ (for more details, one can refer to [1,2,9,10]).

We follow the basic notation and terminology of the theory of numbers as in [11]. In particular, we let p_r be the rth prime number, where $p_1 = 2$; $\pi(n)$ is the number of primes less than or equal to n ; the Euler's Φ - function $\phi(n)$ is defined as the number of positive integers less than or equal to n that are relatively prime to n.

Let x be any real number, then $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the greatest integer less than or equal x, the smallest integer greater than or equal x respectively.

Seoud and Youssef [13,15] proved the following result, which gives some necessary conditions for a prime graph.

Theorem 2.1. *If G is a prime graph of order n, then*
(1) $|E| \leq \rho(n)$, *where* $\rho(n)$ *is the maximum number of edges in G,*
(2) $\beta(G) \geq \lfloor \frac{n}{2} \rfloor$

(3) $\omega(G) \leq \pi(n) + 1$.

Proof. For the proof, the reader may be referred to the articles [13,15].

Finally, for specific labelings of $K_n \odot \overline{K_m}$, we let V($K_n \odot \overline{K_m}$) = { u_i : $1 \leq i \leq n\} \cup \{ v_{ji} : 1 \leq j \leq m \}$, and we write the vertices of $K_n \odot \overline{K_m}$ in m- tuples ($u_i, v_{1i}, v_{2i}, \cdots, v_{mi}$), where $1 \leq i \leq n$ and we let ($x_i, y_{1i}, y_{2i}, \cdots, y_{mi}$) denote the joint labeling.

3. Coronas of Complete Graphs and Null Graphs

It is well known that the corona $G_1 \odot G_2$ of G_1 and G_2 is the graph obtained by taking one copy of G_1 (which has n_1 vertices) and n_1 copies of G_2, and then joining i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2. So the order of $G_1 \odot G_2$ is $n_1 + n_1 n_2$, where n_2 is the order of G_2 and its size is $m_1 + n_1 n_2 + n_1 m_2$, where m_i is the size of G_i for all i =1,2.

Youssef [15] has shown that $K_n \odot \overline{K_1}$ is prime if and only if $n \leq 7$. In this section, we extend this result to show that $K_n \odot \overline{K_2}$ is prime if and only if $n \leq 16$. Moreover, we show that the graph $\overline{K_m} \odot K_n$ is prime if and only if m = 1 and n = 2, or n = 1 and for all $m \geq 1$. Now, we present the proof of above result due to Youssef using our notation.

Lemma 3.1. *The graph $K_n \odot \overline{K_1}$ is prime if and only if $n \leq 7$.*

Proof. Let $n \geq 8$, then $\pi(2n) \leq n - 2$ and since $\omega(K_n \odot \overline{K_1}) = n$, we get $\omega(K_n \odot \overline{K_1}) > \pi(2n) + 1$ and hence by theorem 2.1, $K_n \odot \overline{K_1}$ is not prime. Conversely, if $n \leq 7$, then $K_1 \odot \overline{K_1} \equiv P_2$, $K_2 \odot \overline{K_1} \equiv P_4$ are trivially prime and $K_3 \odot \overline{K_1}$ is prime with the following prime labeling (1,2), (3,4), (5,6), i.e., we label the vertices of K_3 as 1,3,5 and we label the corresponding pendent vertices as 2,4,6. For $4 \leq n \leq 7$, we let $V(K_n \odot \overline{K_1}) = (u_1, u_2, u_3, \cdots, u_n) \cup (v_1, v_2, v_3, \cdots, v_n)$, where $E(K_n \odot \overline{K_1})$ = {u_iu_j:1 i<j n } { $u_i v_i$: $1 \leq i \leq n$ }, then $K_n \odot \overline{K_1}$, where $4 \leq n \leq 7$ are prime with the following prime labeling functions: f : $V(K_4 \odot \overline{K_1}) \to \{1,2,3,..,8\}$ such that f(u_i) = 2i-1, f(v_i) = 2i, where $1 \leq i \leq 4$ (i.e., (1,2), (3,4), (5,6), (7,8)), f : $V(K_5 \odot \overline{K_1}) \to \{1,2,3,..,10\}$ such that f(u_1) = 1, f(u_2) = 2, f(u_3) = 3, f(u_4) = 5, f(u_5) =7, f(v_1) = 10, f(v_2) = 9, f(v_3) = 4, f(v_4) = 6, f(v_5) = 8 (i.e., (1,10), (2,9), (3,4), (5,6), (7,8)), f : $V(K_6 \odot \overline{K_1}) \to \{1,2,3,..,12\}$ such that f(u_1) = 1, f(u_2) = 2, f(u_3) = 3, f(u_4) = 5, f(u_5) =7, f(u_6) =11, f(v_1) = 10, f(v_2) = 9, f(v_3) = 4, f(v_4) = 6, f(v_5) = 8, f(v_6) = 12 (i.e., (1,10), (2,9), (3,4), (5,6), (7,8),(11,12)) and f : $V(K_7 \odot \overline{K_1}) \to \{1,2,3,..,14\}$ such that f(u_1) = 1, f(u_2) = 2, f(u_3) = 3, f(u_4) = 5, f(u_5) =7, f(u_6) =11, f(u_7) =13, f(v_1) = 10, f(v_2) = 9, f(v_3) = 4, f(v_4) = 6, f(v_5) = 8, f(v_6) = 12, f(v_7) = 14 (i.e., (1,10), (2,9), (3,4), (5,6), (7,8), (11,12),(13,14)), the lemma follows.

Lemma 3.2. *The graph $K_n \odot \overline{K_2}$ is prime if and only if $n \leq 16$.*

Proof. Let $n \geq 17$, then $\pi(3n) \leq n - 2$ and since $\omega(K_n \odot \overline{K_2}) = n$, we get $\omega(K_n \odot \overline{K_2}) > \pi(3n) + 1$ and hence by theorem 2.1, $K_n \odot \overline{K_2}$ is not prime. Conversely, if $n \leq 16$, the first n-tuples suffice as a prime labeling for $K_n \odot \overline{K_2}$: (1,2,3), (5,4,6), (7,8,9), (11,10,12), (13,14,15), (17,16,18),(19,20,21), (23,22,24), (26,25,27), (29,28,30), (31,32,33), (35,34,36), (37,38,39), (41,40,42), (43,44,45), (47,46,48), the lemma follows.

Lemma 3.3. *The graph* $\overline{K_m} \odot K_1$ *is prime for all* $m \geq 1$.

Proof. Let $V(K_m \odot \overline{K_1}) = (u_1, u_2, u_3, \cdots, u_m) \cup (v_1, v_2, v_3, \cdots, v_m)$, then the graph $\overline{K_m} \odot K_1$ is prime with the following prime labeling function

f : $V(\overline{K_m} \odot \overline{K_1}) \rightarrow \{1,2,3,..,2m\}$ such that f(u_i) = 2i-1, f(v_i) = 2i, where $1 \leq i \leq m$ (i.e.,(1,2),(3,4),(5,6),⋯,(2m-1, 2m)), the lemma follows.

Example 3.1. The graph $\overline{K_1} \odot K_2$ is prime.

Solution. This follows directly since $\overline{K_1} \odot K_2 \equiv K_3$ and K_3 is clearly prime.

Lemma 3.4. *If either* $m \geq 1$ *and* $n > 2$, *or* $m > 1$ *and* $n = 2$, *then the graph* $\overline{K_m} \odot K_n$ *is not prime.*

Proof. It is easy to verify that $\beta(\overline{K_m} \odot K_n)$ = m and m $< \frac{m(n+1)}{2}$ for all $m \geq 1$ and $n > 2$, or for all $m > 1$ and n = 2, and hence by theorem 2.1, $\overline{K_m} \odot K_n$ is not prime, the lemma follows.

Theorem 3.1. *The graph* $\overline{K_m} \odot K_n$ *is prime if and only if m = 1 and n = 2, or n = 1 and for all* $m \geq 1$.

Proof. The proof follows directly from lemma 3.3, example 3.1 and lemma 3.4, the theorem follows.

4. Joins and Unions of Cycles and Paths

As stated in [9], every path P_n is prime and every cycle C_n is prime. In this section we extend those results to pairs of paths, pairs of cycles, and graphs consisting of one cycle and one path. Deretsky, Lee and Mitchem [2] have shown that the disjoint union $C_{2k} \cup C_n$ of two cycles C_{2k} and C_n is prime. The following theorem generalizes this result as follows:

Theorem 4.1. *The union* $C_n \cup C_m$ *of two cycles* C_n *and* C_m *is not prime if and only if both n and m are odd.*

Proof. If both n and m are odd, then it is clear that β(C_n C_m) = $\beta(C_n) + \beta(C_m) = \frac{n}{2} + \frac{m}{2} < \frac{n+m}{2}$ and hence by theorem 2.1, the graph $C_n \cup C_m$ is not prime. Conversely, we have two cases:

Case 1. n and m are even.

We label the vertices of C_n as 1,2,3, ⋯ n, and we label the vertices of C_m as n+1, n+2,…, n + m. So, the reader can easily verify that the graph $C_n \cup C_m$ is prime.

Case 2. n even and m odd (or vice versa).

We label the vertices of C_n as 2,3, $\cdots$,n+1, and we label the vertices of C_m as 1, n+2, $\cdots$, n + m, which clearly give a graph $C_n \cup C_m$ is prime, the theorem follows.

Youssef [15], has shown that the joint $C_n + C_m$ of two cycles C_n and C_m is not prime for all $n \geq 3$ and $m \geq 3$ as follows.

Lemma 4.1. *The joint $C_n + C_m$ of two cycles C_n and C_m is not prime for all $n \geq 3$ and $m \geq 3$.*

Proof. Since β(C_n + C_m) = max {β(C_n), β(C_m) }= max $\{\frac{n}{2}, \frac{m}{2}\} < \frac{n+m}{2}$ for all $n \geq 3$ and $m \geq 3$, hence by theorem 2.1, the graph $C_n + C_m$ is not prime, the lemma follows.

Lemma 4.2. *The union $P_n \cup P_m$ of two paths P_n and P_m is prime for all $n \geq 1$ and $m \geq 1$.*

Proof. Without loss of generality, we assume that $n \leq m$, then let V(P_n) = $(u_1, u_2, u_3, \cdots, u_n)$ and V(P_m) =$(v_1, v_2, v_3, \cdots, v_m)$. Therefore the graph $P_n \cup P_m$ is prime with the following prime labeling function f : V($P_n \cup P_m$)$\rightarrow \{1,2,3,\cdots, n+m\}$ such that f((u_i)=i, where $1 \leq i \leq n$, and f(v_j) = j, where $n+1 \leq j \leq n+m$, the lemma follows.

Lemma 4.3. *The joint $P_1 + P_n$ of two paths P_1 and P_n is prime for all $n \geq 1$.*

Proof. Let V(P_1)= $\{u\}$ and V(P_n)= $(u_1, u_2, u_3, \cdots, u_n)$. Therfore the graph $P_1 + P_n$ is prime with the following labeling function f :V($P_1 + P_n$) $\rightarrow \{1,2,3,\cdots, n+1\}$ such that f(u) = 1 and f(u_j)= j, where $2 \leq j \leq n+1$, the lemma follows.

Lemma 4.4. *The joint $P_2 + P_n$ of two paths P_2 and P_n is prime for all $n \geq 1$ if and only if n odd.*

Proof. If n is even, then it is clear that β(P_2+ P_n) = max $\{\frac{n}{2}, \frac{m}{2}\} < \frac{n+m}{2}$ for all even $n \geq 2$, hence the necessity follows directly from theorem 2.1. Conversely, suppose that n is odd, and let V($P_2 + P_n$)= $\{u, v\} \cup \{u_1, u_2, u_3, \cdots, u_n\}$, then we have two cases.

Case 1. n+2 is prime.

The graph $P_2 + P_n$ is prime with the following prime labeling function f : V($P_2 + P_n$)$\rightarrow \{1,2,3,\cdots, n+2\}$ such that f(u) = 1, f(v) = n+2, and f(u_i) = i+1, where $1 \leq i \leq n$.

Case 2. n+2 is not prime.

The graph $P_2 + P_n$ is prime with the following prime labeling function f : V($P_2 + P_n$)$\rightarrow \{1,2,3,\cdots, n+2\}$ such that f(u) = 1, f(v) = p, where p is the greatest prime less than n+2, f(u_1) = p+1, f(u_2) = p+2, f(u_3) = p+3, $\cdots$, f(u_{p-3}) = n+2, f(u_{p-2}) = 2, $\cdots$, f(u_n) = p-1, i.e., we label the vertices of P_2 as 1, p and we label the vertices of P_n as p+1, p+2, $\cdots$, n+2, 2, 3, $\cdots$, p-1. Therefore the sufficiency follows, the lemma follows.

Lemma 4.5. *The joint $P_n + P_m$ of two paths P_n and P_m is not prime for all $n \geq 1$ and $m \geq 1$ except for n = 1 and for all $m \geq 1$(or vice versa), or n = 2 and m odd (or vice versa).*

Proof. The proof follows directly from the fact that β(P_n+ P_m) = max $\{\frac{n}{2}, \frac{m}{2}\} < \frac{n+m}{2}$ for all $n \geq 3$ and $m \geq 1$ (or vice versa), or n = 2 and m even (or vice versa), and theorem 2.1, the lemma follows.

Theorem 4.2. *The joint $P_n + P_m$ of two paths P_n and P_m is prime if and only if n =1 and $m \geq 1$(or vice versa), or n =2 and m odd (or vice versa)*

Proof. The proof follows directly from lemma 4.3, lemma 4.4 and lemma 4.5, the theorem follows.

Lemma 4.6. *The union $C_n \cup P_m$ of cycls C_n and paths P_m is prime for all $n \geq 3$ and $m \geq 1$.*

Proof. Without loss of generality, we assume that $n \leq m$, then let V(C_n) = $(u_1, u_2, u_3, \cdots, u_n)$ and V(P_m) =$(v_1, v_2, v_3, \cdots, v_m)$. Therefore the graph $C_n \cup P_m$ is prime with the following prime labeling function f : V $(C_n \cup P_m) \rightarrow \{1,2,3,\cdots, n+m\}$ such that f(u_i)=i, where $1 \leq i \leq n$, and f(v_j) = j, where $n+1 \leq j \leq n+m$, the lemma follows.

Lemma 4.7. *The joint $C_n + P_m$ of cycles C_n and paths P_m is not prime for all $n \geq 3$ and $m \geq 1$.*

Proof. If n = 3 and $m \leq 2$, then from the fact that K_n is not prime for all $n \geq 4$, $C_3 + P_1 \equiv K_4$ and $C_3 + P_2 \equiv K_5$, we get $C_n + P_m$ is not prime. For $n \geq 3$ and $m \geq 3$, it is clear that β(C_n+ P_m) = max $\{(C_n),(P_m)\}$= max $\{\frac{n}{2}, \frac{m}{2}\} < \frac{n+m}{2}$, hence by theorem 2.1, the graph $C_n + P_m$ is not prime, the lemma follows.

5. Complete Tripartite Graphs with Other Graphs

Seoud, Diab, and Elsakhawi [12], have shown the following complete bipartite graphs are prime: $K_{2,m}$ and $K_{3,m}$ unless m = 3 or m = 7. In this section, we extend those results to show that $K_{1,1,n}$ is prime for all $n \geq 1$ and $K_{1,2,n}$ is prime for all $n \geq 1$ except n = 3 or n = 7. Moreover, we show that the following graphs are prime: $P_n \cup K_{1,m}$ for all $n \geq 1$ and $m \geq 1$, $C_n \cup K_{1,m}$, for all $n \geq 3$ and $m \geq 1$, $\overline{K_n} \cup K_{1,m}$ for all $n \geq 1$ and $m \geq 1$, $\overline{K_1} + K_{1,m}$ for all $m \geq 1$, $P_n \odot \overline{K_1}$ for all $n \geq 1$, $\overline{K_1} \odot P_n$ for all $n \geq 1$, and $\overline{K_n} \cup \overline{K_m}$ for all $n \geq 1$ and $m \geq 1$.

Lemma 5.1. *The complete tripartite graph $K_{1,1,n}$ is prime for all $n \geq 1$.*

Proof. Let the set of vertices of $K_{1,1,n}$ be L = $\{u\}$, M =$\{v\}$ and N = $(u_1, u_2, u_3, \cdots, u_n)$, then we label the vertices of $K_{1,1,n}$ as u =1, v = p, where p is the greatest prime less than or equal to n+2, and we label the vertices of N by the remaining labels, which give a prime labeling and hence the considered graph is prime, the lemma follows.

Lemma 5.2. *The complete tripartite graph $K_{1,2,n}$ is prime for all $n \geq 1$ except for n = 3, or n = 7.*

Proof. Let the set of vertices of $K_{1,2,n}$ be L = $\{u\}$, M = $\{u, w\}$ and N = $(u_1, u_2, u_3, \cdots, u_n)$, then we label the vertices of $K_{1,2,n}$ as as u = 1, v = p_1, w = p_2, where $p_1 < p_2$ are two greatest prime less than or equal to n+2, and we label the vertices of N by the remaining labels, which give a prime labeling and hence the considered graph is prime, the lemma follows.

Lemma 5.3. *The union $P_n \cup K_{1,m}$ of paths P_n and stars $K_{1,m}$ is prime for all $n \geq 1$ and $m \geq 1$.*

Proof. Without loss of generality, we assume that $1 \leq m \leq n$. If m = 1, then it is easy to verify that $P_n \cup K_{1,1} \equiv P_n \cup P_2$, and hence by lemma 4.2, we get $P_n \cup K_{1,1}$ is prime. Now, let $m > 1$ and the vertex sets of $K_{1,m}$ be L = $\{u\}$ and M = $\{u_i: 1 \leq i \leq m\}$, and let the vertex set of P_n be N =$\{v_j: 1 \leq i \leq n\}$, then we label the vertices of the sets L and M as u = 1 and u_i = i + 1 for all $1 \leq i \leq m$, and we label the vertices of the set N by remaining labels, i.e, v_j = j + m for all $2 \leq j \leq n$. Hence we can easily verify that the considered graph is prime, the lemma follows.

Lemma 5.4. *The union $C_n \cup K_{1,m}$ of cycles C_n and stars $K_{1,m}$ is prime for all $n \geq 3$ and $m \geq 1$.*

Proof. Without loss of generality, we assume that $m \leq n$. If m = 1, then it is easy to verify that $C_n \cup K_{1,1} \equiv C_n \cup P_2$, and hence by lemma 4.6, we get $C_n \cup K_{1,m}$ is prime. Now, let $m > 1$ and the vertex sets of $K_{1,m}$ be L = $\{u\}$ and M =$\{u_i: 1 \leq i \leq m\}$, and let the vertex set of C_n be N =$\{v_j: 1 \leq i \leq n\}$, then we label the vertices of the sets L and N as u = p, where p is the greatest prime number in the set $\{1,2,3,\cdots, n+m+1\}$ and v_j = j for all $1 \leq j \leq m$, and we label the vertex set M by remaining labels. Hence we can easily verify that the considered graph is prime, the lemma follows.

Lemma 5.5. *The following graphs are prime:*

(a) $\overline{K_n} \cup K_{1,m}$ *for all* $n \geq 1$ *and* $m \geq 1$,

(b) $\overline{K_1} + K_{1,m}$ *for all* $m \geq 1$,

(c) $P_n \odot \overline{K_1}$ *for all* $n \geq 1$,

(d) $\overline{K_1} \odot P_n$ *for all* $n \geq 1$,

(e) $\overline{K_n} \cup \overline{K_m}$ *for all* $n \geq 1$ *and* $m \geq 1$.

Proof. (a) Without loss of generality, we assume that $m \leq n$. Now, let the vertex sets of $K_{1,m}$ be L = $\{u\}$ and M =$\{u_i: 1 \leq i \leq m\}$, and let the vertex set of $\overline{K_n}$ be N =$\{v_j: 1 \leq i \leq n\}$, then we label the vertices of the sets L and M as u =1 and u_i = i for $2 \leq i \leq m$, and we label the vertex set N by remaining labels. Hence we can easily verify that the considered graph is prime.

(b) Let the vertex sets of $K_{1,m}$ be L = $\{u\}$ and M =$\{u_i: 1 \leq i \leq m\}$, then we label the vertex of $\overline{K_1}$ as 1, we label the vertex set L as u = p, where p is the greatest prime number in the set $\{1,2,3,\cdots, m+2\}$, and we label the vertex set M by remaining labels. Hence we can easily verify that the considered graph is prime.

(c) We label the vertices of P_n as 1, 3, 5, ⋯, 2n-1, and we label the pendent vertices as 2, 4, 6, ⋯, 2n. Hence we can easily verify that the considered graph is prime.

(d) The proof follows directly from the fact that $\overline{K_1} \odot P_n \equiv P_1 + P_n$ and theorem 4.2.

(e) Without loss of generality, we assume that $m \leq n$. Now, we label the vertices of $\overline{K_m}$ as 1, 2, 3,⋯, m, and we label the vertices of $\overline{K_n}$ as m+1, m+2, m+3, ⋯, n + m. Hence we can easily verify that the considered graph is prime, the lemma follows.

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НЕКОТОРЫЕ РЕЗУЛЬТАТЫ О ПЕРВИЧНОЙ МАРКИРОВКЕ ГРАФОВ

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Граф называется первичным, если он имеет относительтельную первичную маркировку своих граней, удовлетворяющих определенным свойствам. Целью настоящей статьи является описание некоторых новых семейств графов, обладающих первичной маркировкой, в терминах необходимых и достаточных условий.

Ключевые слова: Граф, маркировка граней.