Transmutation Operators as a Solvability Concept of Abstract Singular Equations

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Abstract One of the methods of studying differential equations is the transmutation operators method. Detailed study of the theory of transmutation operators with applications may be found in the literature. Application of transmutation operators establishes many important results for different classes of differential equations including singular equations with Bessel operator. In this paper transmutation operators are used in more general case when in Euler–Poisson–Darboux equation as the space-variable Laplace operator is replaced by some abstract operator acting in Banach space. Also some other abstract singular equations are studied by this method.

1 Introduction

One of the method of studying to differential equations is transmutation operators method. Detailed study of the theory of transmutation operators with applications may be found in [1, 2]. Application of transmutation operators establishes many important results for different classes of differential equations including singular differential equations with Bessel operator

$$B_k = \frac{d^2}{dt^2} + \frac{k}{t}\frac{d}{dt}, \quad k \in \mathbb{R}.$$

For example, singular PDE named Euler–Poisson–Darboux equation (EPD) has the form

$$\frac{\partial^2 u(t,x)}{\partial t^2} + \frac{k}{t}\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x), \quad k > 0, \ \ x \in \mathbb{R}^n,$$

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where Δ is the space-variable Laplace operator. In the paper [3] singular EPD was leading to a simpler wave equation (with $k = 0$) using the appropriate transmutation operator. In this case, the formulas for the solution are written using spherical means acting by spatial variables.

In this paper transmutation operators are used in more general case when in EPD equation the space-variable Laplace operator is replaced by some abstract operator acting in Banach space. Also some other abstract singular equations will be studied by this method.

In the future we will assume that A is a closed operator in a Banach space E with a dense in E domain $D(A)$.

2 Euler–Poisson–Darboux Equation: Bessel Operator Function

Consider the Euler–Poisson–Darboux equation expressed as follows:

$$u''(t) + \frac{k}{t}u'(t) = Au(t), \quad t > 0 \tag{1}$$

for $k > 0$ in Banach space E.

As we will see further the correct initial condition for the EPD equation (1) are

$$u(0) = u_0, \quad u'(0) = 0. \tag{2}$$

Wherein, if $k \geq 1$ then the initial condition $u'(0) = 0$ is not needed that is usual situation for some equations with a singularity in the coefficients at $t = 0$.

Correct choice of initial conditions depending on the parameter $k \in \mathbb{R}$ and solution to the problem (1)–(2) when A is space-variable Laplace operator is given in Chapter 1 in [3]. Next results on the theory of singular equations in partial derivatives can be found, for example, in the papers [4–10] and their bibliography.

The problem (1)–(2) for $k = 0$ studied in details in [11–13]. In these papers the fact that the problem (1)–(2) is uniformly correct only if the operator A is the generator of the cosine operator function (COF) $C(t)$ was established. For terminology, see [11–14]. In the same papers, necessary and sufficient conditions that the operator A is a generator COF are given. These conditions are formulated in terms of the estimation of the norm of the resolvent $R(\lambda) = (\lambda I - A)^{-1}$ and its derivatives of the operator A

As for abstract EPD equation (1), then is was studied in [15], in Chapter 1 in [16, 17] under various assumptions about the operator A.

The Cauchy problem (1)–(2) was studied in [18], in which the necessary and sufficient solvability conditions are formulated in terms of the estimation of the norm of the resolvent $R(\lambda)$ and its weighted derivatives. In the present paper, unlike in [18], we give the necessary and sufficient condition for operator A is formulated in terms of the fractional degree of the resolvent and its non-weight derivatives as in the case $k = 0$.

Denote by $C^n(I, E_0)$ a space of n times strongly continuously differentiable for $t \in I$ functions with values in $E_0 \subset E$. Let $L(E)$ is the space of linear bounded operators.

Definition 1 A solution of Eq. (1) is a function $u(t)$ that is twice strongly continuously differentiable for $t \geq 0$ which takes values belonging to $D(A)$ for $t > 0$. That means $u(t) \in C^2(\bar{R}_+, E) \cap C(R_+, D(A))$, and satisfies Eq. (1).

Definition 2 Problem (1)–(2) is called uniformly well posed if there exists a commuting on $D(A)$ with the A operator function $Y_k(\cdot) : [0, \infty) \to L(E)$ and numbers $M \geq 1$, $\omega \geq 0$, such that for all $u_0 \in D(A)$ function $Y_k(t)u_0$ is its unique solution and

$$\|Y_k(t)\| \leq M \exp(\omega t), \tag{3}$$

$$\left\|Y_k'(t)u_0\right\| \leq Mt \ \exp(\omega t) \, \|Au_0\| \, . \tag{4}$$

Function $Y_k(t)$ is the Bessel operator function (OFB) of the problem (1)–(2) and the set of operators for which the problem (1)–(2) is uniformly correct, denoted by G_k. Moreover, G_0 is the set of generators of the operator cosine function, and $Y_0(t) = C(t)$.

In Definition 2 and throughout the following, we use the notation $Y_k'(t)u_0 = (Y_k(t)u_0)'$.

Theorem 1 ([19]) *Let problem (1)–(2) be uniformly well posed for values of parameter $m \geq 0$ ($A \in G_m$). Then this problem is also uniformly well posed fork $> m \geq 0$ ($A \in G_k \supset G_m$). The corresponding Bessel operator function $Y_k(t)$ has the form*

$$Y_k(t) = \Pi_{k,m} Y_m(t) = \mu_{k,m} \int\limits_0^1 s^m (1 - s^2)^{(k-m)/2-1} Y_m(ts)\, ds, \tag{5}$$

$$\mu_{k,m} = \frac{2\Gamma(k/2 + 1/2)}{\Gamma(m/2 + 1/2)\Gamma(k/2 - m/2)},$$

where $\Gamma(\cdot)$ is the Euler gamma-function.

The equality (5) written on the initial element u_0 is called the translation formula by the parameter k for the solution of the Cauchy problem for Eq. (1).

The integral on the right side of Eq. (5) called the Poisson integral, and $\Pi_{k,m}$ is transmutation operator intertwining differential operators B_m and B_k (for terminology see [1]). Operator $\Pi_{k,m}$ is the particular case of Erdelyi–Kober operator (see. [20]) preserving the initial conditions (2).

Note that in this paper we get along with the concept of an integral of a continuous function, but if necessary, we can use the Bochner integral of a function with a value in a Banach space.

If operator $A \in G_0$ is a COF generator $C(t)$ then (see. [21]) uniformly by $t \in [0, t_0]$, $t_0 > 0$ for $u_0 \in E$. When $k \to 0$ operator $P_{k,0}$ strongly converges to a identity operator I and OFB $Y_k(t)$ strongly converges to a COF $C(t)$:

$$\lim_{k \to 0} Y_k(t)u_0 = C(t)u_0.$$

Let $\rho(A)$ is resolvent operator set of A, $K_\nu(\cdot)$ is Macdonald function or modified Bessel function of the third kind of order ν.

Theorem 2 ([19]) *If problem (1)–(2) is uniformly well posed and Re $\lambda > \omega$, then $\lambda^2 \in \rho(A)$ and the representation for resolvent of operator A*

$$\lambda^{(1-k)/2} R(\lambda^2)x = \frac{2^{(1-k)/2}}{\Gamma(k/2 + 1/2)} \int\limits_0^\infty K_{(k-1)/2}(\lambda t) t^{(k+1)/2} Y_k(t)x \, dt.$$

holds for each $x \in E$.

Theorem 3 ([19]) *Let the problem (1)–(2) is uniformly well posed and $Y_k(t)$ is OFB of this problem. Then the operator A is the generator of a C_0—semigroup $T(t)$, and this semigroup admits the representation*

$$T(t)x = \frac{1}{2^k \, \Gamma(k/2 + 1/2) \, t^{k/2+1/2}} \int\limits_0^\infty s^k \exp\left(-\frac{s^2}{4t}\right) Y_k(s)x \, ds, \quad x \in E. \tag{6}$$

The semigroup $T(t)$ defined by the equality (6) can be extended to an operator function that is analytic in some sector Ξ_φ and get the representation (see [22], p. 269)

$$T(z) = \frac{1}{2\pi i} \int\limits_{\Gamma_1 \bigcup \Gamma_2} e^{\lambda z} R(\lambda) \, d\lambda,$$

where $\Gamma_1 \bigcup \Gamma_2$ is a contour consisting of rays $\lambda = \sigma + \rho \exp(-i\varphi)$, $0 \le \rho < \infty$ and $\lambda = \sigma + \rho \exp(i\varphi)$, $0 \le \rho < \infty$, $\sigma \ge \omega_0$, $\dfrac{\pi}{2} < \varphi < \dfrac{\pi}{2} + \arcsin \dfrac{1}{M_0(k)}$. Therefore, to find a criterion for the uniform well-posedness of problem (1)–(2) one can restrict considerations to the class of operators that are generators of analytic C_0–semigroups $T(t)$. We denote this class of operators by G. In [23] can be found that if $A \in G$ then for Re $\lambda > \omega$ and for $\alpha > 0$ there exists a fractional degree of the resolvent $R(\lambda)$ which has the form

$$R^\alpha(\lambda)x = \frac{1}{\Gamma(\alpha)} \int\limits_0^\infty t^{\alpha-1} \exp(-\lambda t) T(t)x \, dt, \quad x \in E.$$

A necessary condition for the uniform well-posedness of problem (1)–(2) is obtained in the following assertion.

Theorem 4 ([19]) *If problem (1)–(2) is uniformly well posed and Re* $\lambda > \omega$, *then* λ^2 *belongs to the resolvent set* $\rho(A)$ *of the operator A, and the fractional power of the resolvent admits the representation*

$$R^{1+k/2}(\lambda^2) = \frac{1}{\Gamma(k+1)\,\lambda} \int\limits_0^\infty t^k \exp(-\lambda t) Y_k(t)\, dt$$

in addition,

$$\left\| \frac{d^n}{d\lambda^n} \left(\lambda R^{1+k/2} \left(\lambda^2 \right) \right) \right\| \le \frac{M\,\Gamma(k+n+1)}{(Re\,\lambda - \omega)^{k+n+1}}, \quad n = 0,\ 1,\ 2, \dots \tag{7}$$

In fact the estimates (7) are sufficient for the uniform well-posedness of problem (1)–(2).

Theorem 5 (Criterion of the Uniform Well-Posedness [19]) *Let* $A \in G$ *is a generator of an analytic* C_0*–semigroup. For the problem (1)–(2) to be uniformly well posed it is necessary and sufficient that for some constants* $M \ge 1,\ \omega \ge 0$ *the number* λ^2 *with Re* $\lambda > \omega$ *belonged to the resolvent set of the operator A and for the fractional degree of the resolvent of the A operator estimates (7) were correct.*

Example 1 Let $m > 0$ and $E = L^2_{x^m}(0, \infty)$ is a Hilbert space of complex-valued functions $v(x)$, $x \in (0, \infty)$, squared integrable with the weight x^m and with the norm

$$\|v(x)\|^2 = \int_0^\infty x^m\ |v(x)|^2\, dx.$$

Consider presented into[24] set of the form

$$S = \left\{ v(x) \in C^\infty(-\infty, \infty),\ v(-x) = v(x),\ \left| v^{(n)}(x) \right| \le \frac{M_n}{(1+x^2)^N} \right\},$$

where $n \ge 0,\ N \ge 0$ are arbitrary integers, M_n are constants independent of x, and operator A is Bessel operator

$$A = \frac{d^2}{dx^2} + \frac{m}{x}\frac{d}{dx}$$

on functions from the set S considering on $[0, \infty)$. Obviously, $\overline{D(A)} = L^2_{x^m}(0, \infty)$ and operator A is a symmetric upper semibounded operator, i.e. $(Av, v) \le 0$. By the Friedrichs theorem, its closure $\overline{A}$ is a selfadjoint operator.

Following [24, 25], we define the Fourier–Bessel transform on functions in S by the formulas

$$\hat{v}(s) = \int_0^\infty x^{2p+1}\, j_p(sx)\, v(x)\, dx,$$

$$v(x) = \gamma_p \int_0^\infty s^{2p+1}\, j_p(sx)\, \hat{v}(s)\, ds,$$

$$m = 2p+1,\ \gamma_p = \frac{1}{2^{2p}\,\Gamma^2(p+1)},\ j_p(x) = \frac{2^p\,\Gamma(p+1)}{x^p}\, J_p(x),$$

where $J_p(x)$ is the Bessel function. The set S is invariant under the one-to-one Fourier–Bessel transform.

For $\operatorname{Re}\lambda > 0$, the operator $\overline{A}$ has the resolvent $R(\lambda)$ defined by the formula

$$R(\lambda)v(x) = \gamma_p \int_0^\infty \frac{s^{2p+1}}{s^2+\lambda}\, j_p(sx)\, \hat{v}(s)\, ds, \qquad v(x) \in L^2_{x^{2p+1}}(0,\infty),$$

and, by virtue of the Parseval relation, the following estimate holds:

$$\|R(\lambda)v(x)\|^2 = \gamma_p^2 \int_0^\infty s^{2p+1}\frac{|\hat{v}(s)|^2}{|s^2+\lambda|^2}\, ds \le \frac{\gamma_p^2}{|\lambda|^2}\|\hat{v}(s))\|^2 = \frac{\gamma_p^2}{|\lambda|^2}\|v(x))\|^2,\ \ \operatorname{Re}\lambda > 0.$$

Consequently, the operator $\overline{A} \in G$, i.e., it is the generator of an analytic semigroup, which admits the representation

$$T_{2p+1}(t)v(x) = \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} e^{\lambda t} R(\lambda)v(x)\, d\lambda = \frac{\gamma_p}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} e^{\lambda t}\left(\int_0^\infty \frac{s^{2p+1}}{s^2+\lambda} j_p(sx)\hat{v}(s)\, ds\right) d\lambda =$$

$$= \gamma_p \int_0^\infty s^{2p+1} j_p(sx)\hat{v}(s)\left(\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\frac{e^{\lambda t}}{s^2+\lambda}\, d\lambda\right) ds = \gamma_p \int_0^\infty \exp(-s^2 t)s^{2p+1} j_p(sx)\hat{v}(s)\, ds =$$

$$= \gamma_p^2 \int_0^\infty \exp(-s^2 t)s^{2p+1} j_p(sx)\left(\int_0^\infty \tau^{2p+1} j_p(s\tau)v(\tau)\, d\tau\right) ds =$$

$$= \frac{1}{x^p}\int_0^\infty \tau^{p+1}v(\tau)\left(\int_0^\infty s\ \exp(-s^2 t)J_p(sx)J_p(s\tau)\, ds\right) d\tau =$$

$$= \frac{1}{2t\, x^p}\int_0^\infty \tau^{p+1}\exp\left(-\frac{x^2+\tau^2}{4t}\right) I_p\left(\frac{x\tau}{2t}\right) v(\tau)\, d\tau, \tag{8}$$

here we have used the integral 2.12.39.3 [26], where $I_p(\cdot)$ is the modified Bessel function.

Let us show that the resolvent of the operator $\overline{A}$ satisfies the estimates (7). By using relation (8) we obtain

$$R^{1+k/2}(\lambda^2)v(x) = \frac{1}{\Gamma(k/2+1)}\int_0^\infty t^{k/2}\exp(-\lambda^2 t)T_{2p+1}(t)v(x)\,dt =$$

$$= \frac{\gamma_p}{\Gamma(k/2+1)}\int_0^\infty t^{k/2}\exp(-\lambda^2 t)\int_0^\infty \exp(-s^2 t)s^{2p+1}j_p(sx)\hat{v}(s)\,dsdt =$$

$$= \gamma_p\int_0^\infty \frac{s^{2p+1}}{(s^2+\lambda^2)^{1+k/2}}j_p(sx)\hat{v}(s)\,ds, \quad v(x)\in L^2_{x^{2p+1}}(0,\infty).$$

Next, by virtue of the Parseval relation, the representation

$$\left\|\frac{d^n}{d\lambda^n}\left(\lambda R^{1+k/2}\left(\lambda^2\right)\right)v(x)\right\|^2 = \gamma_p^2\int_0^\infty s^{2p+1}\left|\frac{d^n}{d\lambda^n}\left(\frac{\lambda}{(s^2+\lambda^2)^{1+k/2}}\right)\right|^2|\hat{v}(s)|^2\,ds \tag{9}$$

holds for $\operatorname{Re}\lambda > 0$.

By differentiating the relation (see [26] 2.12.8.4)

$$\frac{\lambda}{(s^2+\lambda^2)^{1+k/2}} = \frac{\sqrt{\pi}}{2(2s)^{(k-1)/2}\Gamma(k/2+1)}\int_0^\infty t^{(k+1)/2}e^{-\lambda t}J_{(k-1)/2}(ts)\,dt,$$

with respect to λ, we obtain

$$\frac{d^n}{d\lambda^n}\left(\frac{\lambda}{(s^2+\lambda^2)^{1+k/2}}\right) = \frac{(-1)^n\sqrt{\pi}}{2(2s)^{(k-1)/2}\Gamma(k/2+1)}\int_0^\infty t^{(k+1)/2+n}e^{-\lambda t}J_{(k-1)/2}(ts)\,dt. \tag{10}$$

By taking into account relation (10), from the representation (9) we obtain the estimate

$$\left\|\frac{d^n}{d\lambda^n}\left(\lambda R^{1+k/2}\left(\lambda^2\right)\right)v(x)\right\|^2 \le \frac{\pi\ \gamma_p^2}{2^{k+1}\ \Gamma^2(k/2+1)} \quad\times$$

$$\times\quad \int_0^\infty s^{2p-k+2}\left|\int_0^\infty t^{(k+1)/2+n}e^{-\lambda t}J_{(k-1)/2}(ts)\,dt\right|^2|\hat{v}(s)|^2\,ds =$$

$$= \frac{\pi\ \gamma_p^2}{2^{k+1}\ \Gamma^2(k/2+1)}\int_0^\infty s^{2p-2k-2n-1}\left|\int_0^\infty \tau^{k+n}e^{-\lambda t/s}\tau^{(1-k)/2}J_{(k-1)/2}(\tau)\,d\tau\right|^2|\hat{v}(s)|^2\,ds \le$$

$$\leq \frac{M_0\,\pi\,\gamma_p^2}{2^{k+1}\,\Gamma^2(k/2+1)}\int_0^\infty s^{2p-2k-2n-1}\left|\int_0^\infty \tau^{k+n}e^{-\lambda t/s}\,d\tau\right|^2 |\hat{v}(s)|^2\,ds \leq$$

$$\leq \frac{M_1\,\Gamma^2(k+n+1)}{(\mathrm{Re}\,\lambda)^{2(k+n+1)}}\|v(x)\|^2,\quad n=0,\ 1,\ 2,\ldots$$

Therefore, the estimates (7) hold. Theorem 5 is true for the considered operator $\overline{A}$, and $\overline{A}\in G_k$ for each $k\geq 0$. In particular, $\overline{A}\in G_0$ and the corresponding cosine operator function has the form

$$C(t)v(x)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma-i\infty} e^{\lambda t}\,\lambda\,R(\lambda^2)v(x)\,d\lambda=$$

$$=\frac{\gamma_p}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\lambda e^{\lambda t}\int_0^\infty \frac{s^{2p+1}}{s^2+\lambda^2}j_p(sx)\hat{v}(s)\,dsd\lambda=\gamma_p\int_0^\infty s^{2p+1}j_p(sx)\cos st\;\hat{v}(s)\,ds.$$

for $\sigma>0$

It is convenient to use relation (5) to find the function $Y_k(t)$. For $k>0$, we have the representation

$$Y_k(t)v(x)=\frac{2}{B(1/2,k/2)}\int_0^1(1-\tau^2)^{k/2-1}C(t\tau)v(x)\,d\tau=$$

$$=\frac{2}{B(1/2,k/2)}\int_0^1(1-\tau^2)^{k/2-1}\gamma_p\int_0^\infty s^{2p+1}j_p(sx)\cos st\tau\;\hat{v}(s)\,dsd\tau=$$

$$=\gamma_p\int_0^\infty s^{2p+1}j_p(sx)\;j_{(k-1)/2}(st)\;\hat{v}(s)\,ds.$$

Example 2 Let $m>0$ and let $E=L^2_{x^m}\left(R_2^+\right)$ be the Hilbert space of complex-valued functions $v(x,y)$, $(x,y)\in R_2^+$ that are square integrable with weight x^m and with the norm

$$\|v(x,y)\|^2=\int_{-\infty}^\infty\int_0^\infty x^m\,|v(x,y)|^2\,dxdy.$$

Consider the set

$$S_2=\left\{v(x,y)\in C^\infty(R_2),\ v(-x,y)=v(x,y),\ \left|\frac{\partial^n}{\partial x^n}\frac{\partial^j}{\partial y^j}v(x,y)\right|\leq\frac{M_{n,j}}{(1+x^2+y^2)^N}\right\},$$

where $n,\ j \geq 0,\ N \geq 0$ are arbitrary integers and the $M_{n,j}$ are constants independent of $x,\ y$, and define the operator A by the differential expression

$$A = \frac{\partial^2}{\partial x^2} + \frac{m}{x}\frac{\partial}{dx} + \frac{\partial^2}{\partial y^2}$$

on functions in the set S_2 considered on R_2^+. Obviously, $\overline{D(A)} = L_{x^m}^2\left(R_2^+\right)$ and A is a symmetric upper semibounded operator; i.e., $(Av, v) \leq 0$. By the Friedrichs theorem, its closure $\overline{A}$ is a selfadjoint operator.

In addition to the Fourier–Bessel transform on functions in the set S_2, we define the Fourier transform (with respect to the variable y) by the formulas

$$\tilde{w}(x,\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\xi y}\, w(x,y)\, dy, \quad w(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\xi y}\, \tilde{w}(x,\xi)\, d\xi.$$

The Fourier–Bessel and Fourier transforms are one-to-one mappings of S_2 onto S_2. For $\operatorname{Re}\lambda > 0$, the operator $\overline{A}$ has the resolvent $R(\lambda)$ defined by the formula

$$R(\lambda)v(x,y) = \frac{\gamma_p}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_0^{\infty} e^{i\xi y}\,\frac{s^{2p+1}}{s^2+\xi^2+\lambda}\, j_p(xs)\tilde{\hat{v}}(s,\xi)\, ds d\xi,$$

in addition, by virtue of the Parseval relation, we have the estimate

$$\|R(\lambda)v(x,y)\|^2 \leq \frac{\gamma_p^2}{|\lambda|^2}\|v(x,y)\|^2, \quad \operatorname{Re}\lambda > 0.$$

Consequently, the operator $\overline{A} \in G$, i.e., it is the generator of the analytic semigroup

$$T_{2p+1}(t)v(x,y) = \frac{\gamma_p}{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty} \exp(-s^2t - \xi^2t + i\xi y)\, s^{2p+1}\, j_p(sx)\, \tilde{\hat{v}}(s,\xi)\, ds d\xi =$$

$$= \frac{1}{4\pi\sqrt{2}\, t\sqrt{t}\, x^p}\int_0^{\infty} \tau^{p+1}\exp\left(-\frac{x^2+\tau^2}{4t}\right) I_p\left(\frac{x\tau}{2t}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{(\eta-y)^2}{4t}\right) v(\tau,\eta)\, d\eta d\tau.$$

By analogy with Example 1, one can prove the estimates

$$\left\|\frac{d^n}{d\lambda^n}\left(\lambda R^{1+k/2}\left(\lambda^2\right)\right)v(x)\right\|^2 \leq \frac{M_1\,\Gamma^2(k+n+1)}{(\operatorname{Re}\lambda)^{2(k+n+1)}}\|v(x)\|^2, \quad n = 0,\ 1,\ 2,\ldots,$$

and consequently, $\overline{A} \in G_k$ for any $k \geq 0$, in addition,

$$C(t)v(x, y) = \frac{\gamma_p}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_0^{\infty} e^{i\xi y} s^{2p+1} j_p(sx) \cos\left(t\sqrt{s^2 + \xi^2}\right) \tilde{\hat{v}}(s, \xi)\, ds d\xi,$$

$$Y_k(t)v(x, y) = \frac{\gamma_p}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_0^{\infty} e^{i\xi y} s^{2p+1} j_p(sx)\ j_{(k-1)/2}\left(t\sqrt{s^2 + \xi^2}\right) \tilde{\hat{v}}(s, \xi)\, ds d\xi.$$

Example 3 Let $E = L_\infty(0, \infty)$ is a space of measurable functions $v(x)$ of variable $x \in (0, \infty)$ with norm $\|v(x)\| = \operatorname{ess} \sup\limits_{(0,\infty)} |v(x)|$.

Operator A is the Bessel differential expression for $m = 2$ on considered on the semiaxis $[0, \infty)$ even functions $v(x)$ from $L_\infty(-\infty, \infty)$ such that $v''(x) + 2/x\ v'(x) \in E$. Then A is closed operator with a dense domain of definition and the problem

$$\frac{\partial^2 u}{\partial t^2} + \frac{2}{t}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{2}{x}\frac{\partial u}{\partial x}, \quad t,\ x > 0, \quad u(0, x) = v(x), \quad \frac{\partial u(0, x)}{\partial t} = 0$$

has the unique solution of the form

$$u(t, x) = T_x^t v(x) = \frac{1}{2} \int_0^{\pi} v\left(\sqrt{x^2 + t^2 - 2xt\cos\varphi}\right) \sin\varphi\, d\varphi.$$

Function $u(t, x)$ for each $t \geq 0$ belongs to E and estimates (3), (4) with $\omega = 0$ are valid. Therefore, $A \in G_2$. We show that the operator A is not a generator COF, i.e. $A \notin G_0$. Indeed, the unique solution to the problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{2}{x}\frac{\partial u}{\partial x}, \quad t,\ x > 0, \quad u(0, x) = v(x) \in D(A), \quad \frac{\partial u(0, x)}{\partial t} = 0$$

is

$$u(t, x) = \frac{(x + t)v(x + t) + (x - t)v(x - t)}{2x} = \frac{v(x + t) + v(x - t)}{2} + \frac{t}{2x} \int_{t-x}^{t+x} v'(s)\, ds. \tag{11}$$

Obviously for defined by equality (11) function $u(t)$ evaluation (3) for $k = \omega = 0$ is not valid and, therefore, $A \notin G_0$. Based on this example, it can be argued that the statement is the opposite of Theorem 1 is, generally speaking, false, i.e. for $k > 0$ enclosure $G_0 \subset G_k$ is strict.

Here are some more properties of OFB $Y_k(t)$. Let $u_0 \in D(A)$ then for OFB $Y_k(t)$ the relations

$$Y_k'(t)u_0 = \frac{t}{k+1} Y_{k+2}(t) A u_0, \qquad \lim_{t\to 0} Y_k''(t)u_0 = \frac{1}{k+1} A u_0,$$

$$Y_k(t)Y_k(s) = T_s^t Y_k(s)$$

are valid. Here T_s^t is generalized translation corresponding to Eq. (1), defined by the equality (see [25])

$$T_s^t H(s) = \frac{1}{B(k/2, 1/2)} \int_0^\pi H\left(\sqrt{s^2+t^2-2st\cos\varphi}\right) \sin^{k-1}\varphi \, d\varphi.$$

Along with Eq. (1) for $m > 0$ we consider the equation perturbed by the operator coefficient B:

$$u''(t) + \frac{m}{t}u'(t) + Bu(t) = Au(t), \quad t > 0. \tag{12}$$

In [27] investigated the question of belonging of the operator $A - B$ to the correctness class G_m when $A \in G_k$ and $B \in L(E)$ is bounded operator and it is established that $A - B \in G_m$, $m \geq k$.

Theorem 6 ([27]) *Let for some $k > 0$ $A \in G_k$, B is bounded operator, $Y_k(t; A)$ and B commute. Then $A - B \in G_m$ for any $m \geq k$ and*

$$Y_k(t; A-B) = Y_k(t; A) + \frac{(-1/2)^{N+1}\Gamma(k/2+1/2)t^2 B}{\Gamma(k/2+1)\Gamma(N+1/2)} \times$$

$$\times \int_0^1 s^{2N} \left(\frac{1}{s}\frac{d}{ds}\right)^N (1-s^2)^{k/2} {}_1F_2\left(1; k/2+1, 2; t^2(s^2-1)B/4\right) Y_{2N}(ts; A)\, ds,$$

where N is the smallest integer such that $2N \geq k$, ${}_1F_2(\alpha; \beta, \gamma; \cdot)$ is generalized hypergeometric function and $Y_m(t; A-B)$ for $m > k$ determined through $Y_k(t; A-B)$ by the formula (5), written for the operator $A - B$.

If $(-B) \in G_p$, then ib [28] it is established that the closure of the operator $A-B$ belongs to G_m, $m \geq k+p+1$.

Theorem 7 ([28]) *Let for some $k \geq 0$ $A \in G_k$ and $(-B) \in G_{m-k-1}$ for $m \geq k+1$, $Y_k(t; A)$. Let operators $Y_{m-k-1}(t; -B)$ commute on $D = D(A) \bigcap D(B)$, $\overline{D} = E$.*

Then the closure of the operator $A - B$ *belongs to* G_m *and*

$$Y_m(t; \overline{A - B}) = \frac{2\Gamma(m/2 + 1/2)}{\Gamma(k/2 + 1/2)\Gamma(m/2 - k/2)} \times$$

$$\times \int_0^1 s^k (1 - s^2)^{(m-k)/2-1} Y_{m-k-1}\left(t\sqrt{1 - s^2}; -B\right) Y_k(ts; A)\, ds.$$

In the general case of the sum of n operators, the following theorem is established.

Theorem 8 ([28]) *Let* $A_j \in G_{k_j}$, $k_j \geq 0$, $j = 1, \dots, n$. *If for* $i \neq j$ A_i *and* A_j *commute on* $D = \bigcap_{j=1}^{n} D(A_j)$ *and* $\overline{D} = E$, *then operator closure* $A = \sum_{j=1}^{n} A_j$ *belongs to* G_k *for* $k = n - 1 + \sum_{j=1}^{n} k_j$ *and*

$$Y_k(t; \overline{A}) = \frac{2^{n-1}\Gamma(k/2 + 1/2)}{\prod_{j=1}^{n} \Gamma(k_j/2 + 1/2)} \int_{\Omega} \prod_{j=1}^{n} y_j^{k_j}\, Y_{k_j}(ty_j; A_j)\, dy,$$

where $\Omega = \{|y| = 1,\ y_1, \dots, y_n \geq 0\}$.

Theorem 3 established that OFB $Y_k(t; B)$, $B \in G_k$ *generates a semigroup* $T(t; B)$, *which allows us to solve the corresponding Dirichlet problem.*

Theorem 9 ([29]) *Let* $u_0 \in D(B)$, *in Eq. (12)* $A = 0$ *and operator B is a generator of a uniformly bounded* C_0*-semigroup* $T(t; B)$. *Then for* $m < 1$ *the function*

$$u(t) = \frac{(t/2)^{1-m}}{\Gamma(1/2 - m/2)} \int_0^{\infty} s^{m/2-3/2} \exp\left(-\frac{t^2}{4s}\right) T(s; B) u_0\, ds$$

is the unique limited solution to Eq. (12) for $A = 0$, *satisfying to condition* $u(0) = u_0$.

Weakening requirements for resolving operators of the Cauchy problem for abstract differential equations of the first and second orders led (see [30–33]) to the concept of an integrated semigroup and an integrated cosine operator function (ICOF).

Lower bound of the resolvent $R(\lambda^2, A)$ *of the operator A of the form*

$$\left\| \frac{d^n}{d\lambda^n} \left(\lambda^{1-\alpha} R(\lambda^2, A)\right) \right\| \leq \frac{M\, n!}{(\lambda - \omega)^{n+1}},\quad \lambda > \omega,\quad n = 0, 1, \dots$$

is the criterion for existence of the generator of ICOF $C_\alpha(t)$ (see, for example, theorem 2.2.5 from [33]).

Let $P_\nu(t)$ is the Legendre spherical function (see [25], p. 205). In papers [34], [35] formulas that associate ICOF with a resolving operator $Y_k(t)$ of (1), (2) are established and the following theorem is proved.

Theorem 10 *Let $k = 2\alpha > 0$ and operator A is a α-times generator ICOF $C_\alpha(t)$, $u_0 \in D(A)$. Then the problem (1), (2) uniformly correct, i.e., $A \in G_k$, and corresponding OFB has a form*

$$Y_k(t)u_0 = \frac{2^\alpha \Gamma(\alpha + 1/2)}{\sqrt{\pi}\, t^\alpha} \left(C_\alpha(t)u_0 - \int\limits_0^1 P'_{\alpha-1}(\tau) C_\alpha(t\tau)u_0\, d\tau \right).$$

In the end of this section we note that if $0 < k < 1$ then OFB $Y_k(t)$ can be used to solve the weighted Cauchy problem for the EPD equation (1) with conditions

$$u(0) = u_0, \quad \lim_{t\to 0} t^k u'(t) = u_1. \tag{13}$$

For $u_0,\ u_1 \in D(A)$ and $A \in G_k \subset G_{2-k}$ the unique solution to the Cauchy problem (1)–(13) is (see [36])

$$u(t) = Y_k(t)u_0 + \frac{1}{1-k} t^{1-k} Y_{2-k}(t)u_1.$$

3 Euler–Poisson–Darboux Equation: Bessel Operator Function with Negative Index

In this section for EPD equation (1) for $k < 0$ we consider the initial problem

$$u(0) = 0, \quad \lim_{t\to 0+} t^k u'(t) = u_1, \tag{14}$$

which, due to the presence of a factor in front of the derivative in the second initial condition, will be called the weighted Cauchy problem.

Correct setting of initial conditions depending on the parameter $k \in \mathbb{R}$ for the EPD equation (1) in the case when A is the Laplace operator with respect to spatial variables is given in Ch. 1 of [3] and the initial conditions for the abstract EPD equation are considered in [36]. We also note that for $k < 0$ Cauchy problem for EPD equation (1) with conditions

$$u(0) = 0, \quad u'(0) = u_1$$

is not correct due to loss of uniqueness (see [37]).

Definition 3 The problem (1), (14) is called uniformly correct if there exists a commuting on $D(A)$ with the A operator function $Z_k(\cdot) : [0, \infty) \to B(E)$ and numbers $M \geq 1, \omega \geq 0$ such that for any $u_1 \in D(A)$ function $Z_k(t)u_1$ is its unique solution and at the same time

$$\|Z_k(t)\| \leq M\, t^{1-k} \exp(\omega t),$$

$$\left\|Z_k'(t)u_1\right\| \leq M\, t^{-k} \exp(\omega t)\left(\|u_1\| + t\|Au_1\|\right).$$

Operator function $Z_k(t)$ for $k < 0$ we will call the Bessel operator function with a negative index (OBFNI) of the problem (1), (14). Set of operators for which the problem (1), (14) is uniformly correct we will denote by H_k. In addition, we denote $H_0 = G_2$ and $Z_0(t) = tY_2(t)$.

Here we present the main statements about OFBNI from the article [38], which are analogues of the corresponding properties OFB.

Theorem 11 *Let the problem (1), (14) is uniformly correct, i.e., $A \in H_k$ and $u_1 \in D(A)$. Then this problem is uniformly correct and for $m < k \leq 0$, i.e., $A \in H_m$. The corresponding Bessel operator function with a negative index $Z_m(t)$ has the form*

$$Z_m(t)u_1 = \mu_{k,m}\, t^{k-m} \int_0^1 s(1-s^2)^{(k-m)/2-1}\, Z_k(ts)u_1\, ds,$$

$$\mu_{k,m} = \frac{2(1-k)}{(1-m)\, B(3/2-k/2, k/2-m/2)},$$

where $B(\cdot, \cdot)$ is Euler beta-function.

Theorem 12 *If the problem (1), (14) is uniformly correct and Re $\lambda > \omega$, then λ^2 belongs to the resolvent set $\rho(A)$ and for any $x \in E$ the representation*

$$\lambda^{(k-1)/2} R(\lambda^2)x = \frac{2^{(k-1)/2}(1-k)}{\Gamma(3/2-k/2)} \int_0^\infty K_\nu(\lambda t) t^{(k+1)/2} Z_k(t)x\, dt$$

is valid.

Theorem 13 *Let the problem (1), (14) is uniformly correct and let $Z_k(t)$ is the Bessel operator function with a negative index for this problem. Then operator A is generator of C_0-semigroups $T(t)$ and for this semigroup, the representation*

$$T(t)x = \frac{1-k}{2^{2-k}\, \Gamma(3/2-k/2)\, t^{3/2-k/2}} \int_0^\infty s \exp\left(-\frac{s^2}{4t}\right) Z_k(s)x\, ds, \quad x \in E$$

is valid.

Theorem 14 *If the problem (1), (14) is uniformly correct and Re* $\lambda > \omega$, *then* λ^2*belongs to the resolvent set* $\rho(A)$ *of the operator A and for the fractional degree of the resolvent the representation*

$$R^{2-k/2}(\lambda^2)x = \frac{1-k}{\Gamma(3-k)\,\lambda}\int\limits_0^\infty t\exp(-\lambda t)Z_k(t)x\,dt, \quad x \in E$$

is valid. Also inequalities

$$\left\|\frac{d^n}{d\lambda^n}\left(\lambda R^{2-k/2}\left(\lambda^2\right)\right)\right\| \le \frac{M\,\Gamma(n-k+3)}{(Re\,\lambda-\omega)^{n-k+3}}, \quad n = 0,\ 1,\ 2,\ldots \tag{15}$$

are true.

Theorem 15 (Criterion for Uniform Correctness of the Weighted Cauchy Problem) *Let operator A is a generator of the analytic* C_0*–semigroup. In order to the problem (1), (14) was uniformly correct, it is necessary and sufficient that for some constants* $M \ge 1,\ \omega \ge 0$ *the number* λ^2 *with Re* $\lambda > \omega$ *belonged to the resolvent set of the operator A and for the fractional degree of the resolvent of the operator A the estimates (15) were valid.*

Theorem 16 *Suppose that the conditions of Theorem 16 are satisfied, then for* $k \le 0$ *the equality* $H_k = G_{2-k}$ *holds true and, moreover,* $Z_k(t) = \dfrac{1}{1-k}t^{1-k}Y_{2-k}(t)$.

Note that examples of operators belonging to G_{2-k}, *and, therefore, and* H_k, *are given in Sect. 2.*

Theorem 17 *Let* $\alpha < 0$ *and the operator A a generator of* $1-\alpha$*–timesintegrated COF* $C_{1-\alpha}(t)$. *Then* $A \in H_{2\alpha}$, *wherein the corresponding Bessel operator function with a negative index* $Z_{2\alpha}(t)$ *has the form*

$$Z_{2\alpha}(t) = \frac{2^{1-\alpha}\Gamma(3/2-\alpha)}{\sqrt{\pi}(1-2\alpha)t^\alpha}\left(C_{1-\alpha}(t) - \int\limits_0^1 P'_{-\alpha}(\tau)C_{1-\alpha}(t\tau)\,d\tau\right).$$

If the operator A is a generator of $(-\alpha)$*–times integrated COF* $C_{-\alpha}(t)$, *then*

$$Z_{2\alpha}(t) = \frac{2^{-\alpha}\Gamma(1/2-\alpha)t^{1-\alpha}}{\sqrt{\pi}}\int\limits_0^1 P_{-\alpha}(\tau)C_{-\alpha}(t\tau)\,d\tau.$$

4 The Bessel-Struve Equation: Operator Function Struve

In this section, for $k > 0$, we consider the equation

$$u''(t) + \frac{k}{t}\left(u'(t) - u'(0)\right) = Au(t), \quad t > 0, \tag{16}$$

which, unlike Eq. (1), contains the value of the derivative of an unknown function at the point $t = 0$.

Scalar equation of the form (16) is called the Bessel-Struve equation and it was previously met in [39–42]. Equation (16), following to [43, 44], can also be called a lightly loaded EPD equation. The growing interest in studying loaded differential equations is explained by the expanding scope of their applications and the fact that loaded equations constitute a special class of functional differential equations with their own specific tasks. A review of publications on loaded differential equations can be found in monographs [43, 44].

It is important to note that the presence in Eq. (16) given at $t = 0$ load changes the formulation of the initial problem. In contrast to the weighted problem (1), (13) for $k > 0$ we establish the well-posedness of the Cauchy problem

$$u(0) = u_0, \quad u'(0) = u_1 \tag{17}$$

for the Bessel-Struve equation (16) and we indicate the explicit form of the resolving operator.

First, we make a remark about the point $t = \tau,\ \tau \geq 0$, at which load value, i.e. the value of an unknown function or its derivative entering the equation.

Let consider the equation

$$u''(t) + \frac{k}{t}u'(t) = Au(t) + B_0 u(\tau), \quad t > 0 \tag{18}$$

with bounded operator B_0 and $A \in G_k$.

For $0 < k < 1$ solution to the problem (18), (2) satisfies equality (see [36])

$$u(t) = Y_k(t)u_0 + \frac{1}{1-k}\left(t^{1-k}Y_{2-k}(t)\int_0^t s^k Y_k(s)B_0 u(\tau)\,ds - Y_k(t)\int_0^t sY_{2-k}(s)B_0 u(\tau)\,ds\right). \tag{19}$$

Putting in (19) $t = \tau$ in order to find $u(\tau)$ we get the equation

$$(I - \Theta(\tau))u(\tau) = Y_k(\tau)u_0, \tag{20}$$

where

$$\Theta(\tau) = \frac{B_0}{1-k}\left(\tau^{1-k}Y_{2-k}(\tau)\int\limits_0^\tau s^k Y_k(s)\,ds - Y_k(\tau)\int\limits_0^\tau sY_{2-k}(s)\,ds\right).$$

In particular, if the inverse operator A^{-1} exists then $\Theta(\tau) = B_0(Y_k(\tau) - 1)A^{-1}$ (see [45]).

For sufficiently small τ, the norm of the bounded operator $\Theta(\tau)$ satisfies the inequality $\|\Theta(\tau)\| < 1$ and, therefore, from Eq. (20) can be determined

$$u(\tau) = (I - \Theta(\tau))^{-1}Y_k(\tau)u_0,$$

after which the solution to the problem (18), (2) found by the formula (19).

A similar situation arises if, in the EPD equation, instead of the load $B_0u(\tau)$, a load of the form $B_1u'(\tau)$ or $B_2u''(\tau)$ is introduced.

The operator $(I - \Theta(\tau))^{-1}$ in the formula (19) makes it difficult to find explicit representations for the resolving operator of initial problems. Finding such a representation is simplified if the equation contains a load at the point $\tau = 0$, then the problem with a given load is actually solved. Here are some examples.

Let consider two equations

$$u''(t) + \frac{k}{t}u'(t) = A(u(t) - b_0u(0)), \quad t > 0 \;\; b_0 \neq 0, \tag{21}$$

$$u''(t) + b_2u''(0) + \frac{k}{t}u'(t) = Au(t), \quad t > 0, \;\; b_2 \neq 0. \tag{22}$$

It is easy to verify that for $0 \le k < 1$, $A \in G_k$ (note that if $b_0 = 1$ or $b_2 = k+1$, then the condition on the operator A can be changed and require that $A \in G_{2-k} \supset G_k$) the unique solution to the Cauchy weighted problem (21), (13) is

$$u(t) = (1 - b_0)Y_k(t)u_0 + \frac{1}{1-k}t^{1-k}Y_{2-k}(t)u_1 + b_0u_0$$

and the unique solution to the (22), (13) unloaded is

$$u(t) = \frac{k - b_2 + 1}{k+1}Y_k(t)u_0 + \frac{1}{1-k}t^{1-k}Y_{2-k}(t)u_1 + \frac{b_2}{k+1}u_0.$$

Also note that in the paper [21] an explicit formula for a solution to a Cauchy problem for a weekly stressed Malmsteen equation was found in the form

$$u''(t) + \frac{k}{t}u'(t) + \frac{l}{t^2}\,(u(t) - u(0)) = Au(t), \quad t > 0. \tag{23}$$

If $A \in G_m$ for some $m \geq 0$ and $k > m,\ l \leq (k-1)^2/4$ then a function

$$u(t) = \frac{2\,\Gamma(p+1)\Gamma(q+1)}{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{k-m}{2}\right)} \int\limits_0^1 s^k \left(1-s^2\right)^{(k-m)/2-1} {}_2F_1\left(p, q; \frac{k-m}{2}; 1-s^2\right) Y_m(ts)u_0\, ds, \tag{24}$$

${}_2F_1\,(p, q; r; z)$—Gauss hypergeometric function, $p,\ q$—real roots of quadratic equation

$$x^2 + \frac{1-k}{2}x + \frac{l}{4} = 0, \quad l \leq \frac{(k-1)^2}{4},$$

is the unique solution of (23) satisfying conditions (2).

If $p = (k-m)/2,\ q = (m-1)/2,\ l = (k-m)(m-1) \leq (k-1)^2/4$ then (24) has a form

$$u(t) = (k-m) \int\limits_0^1 s \left(1-s^2\right)^{(k-m)/2-1} Y_m(ts)u_0\, ds. \tag{25}$$

More interesting is a problem of finding explicit solution to the Cauchy problem (16), (17), which leads to a new notion of operator function—Struve operator function. Let go to its introduction.

Consider the Cauchy problem (16), (17) in case $u_0 = 0$.

Theorem 18 ([46]) *Let $u_0 = 0,\ u_1 \in D(A),\ k = 2\alpha > 0$ and operator A is a generator of the operator cosine function α times of $C_\alpha(t)$. Then a function $u(t) = L_k(t)u_1$, with*

$$L_k(t)u_1 = \frac{2^\alpha\ \Gamma(\alpha+1)}{t^{\alpha-1}} \int\limits_0^1 P_{\alpha-1}(\tau)C_\alpha(t\tau)u_1\, d\tau,$$

is a solution to a problem (16), (17).

In formulations of theorems 10, 17 and 18 some integral operators are involved with spherical Legendre functions in kernel $P_\nu(t)$. These are Buschman–Erdélyi transmutations, they are extensively studied cf. [1, 2, 47–51].

Remark 1 If A is an operator of multiplication by a number then

$$Y_k(t) = \Gamma(k/2+1/2)\sum_{j=0}^{\infty}\frac{\left(t^2A/4\right)^j}{j!\ \Gamma(j+k/2+1/2)} = \Gamma(k/2+1/2)\left(t\sqrt{A}/2\right)^{1/2-k/2} I_{k/2-1/2}\left(t\sqrt{A}\right),$$

with $I_\nu(z)$ being a modified Bessel function,

$$L_k(t) = \frac{\sqrt{\pi}}{2}\,\Gamma\,(k/2+1)\sum_{j=0}^{\infty}\frac{t\ \left(t^2A/4\right)^j}{\Gamma\,(j+3/2)\,\Gamma\,(j+k/2+1)} =$$

$$= \frac{2^{k/2-1/2}\sqrt{\pi}\ \Gamma(k/2+1)}{A^{k/4+1/4}\ t^{k/2-1/2}}\,\mathbf{L}_{k/2-1/2}\left(t\sqrt{A}\right),$$

with $\mathbf{L}_\nu(z)$ being a Struve function. Due to it we call $Y_k(t)$ as operator Bessel function (OBF) and $L_k(t)$ operator Struve function (OSF).

Remark 2 Let $u_0 = 0$, then a condition on operator A in the theorem 18 to existence of only OCF $L_k(t)$ may be weakened. If operator A is a generator $\alpha+1$ times COF $C_{\alpha+1}(t)$, then the next representation is valid

$$L_k(t)u_1 = \frac{2^\alpha\ \Gamma(\alpha+1)}{t^\alpha}\left(C_{\alpha+1}(t)u_1 - \int\limits_0^1 P'_{\alpha-1}(\tau)C_{\alpha+1}(t\tau)u_1\,d\tau\right).$$

Also it is interesting to find formulas representing COF via OFB and These formulas follows from theorem 17 [47] and have the next form

$$C_\alpha(t) = \frac{\sqrt{\pi}}{2^\alpha\Gamma(\alpha+1/2)}\left(t^\alpha Y_{2\alpha}(t) + \int\limits_0^t P'_{\alpha-1}(t/\xi)\xi^{\alpha-1}Y_{2\alpha}(\xi)\,d\xi\right),$$

$$C_{\alpha+1}(t) = \frac{1}{2^\alpha\Gamma(\alpha+1)}\left(t^\alpha L_{2\alpha}(t) + \int\limits_0^t P'_{\alpha-1}(t/\xi)\xi^{\alpha-1}L_{2\alpha}(\xi)\,d\xi\right).$$

For OSF $L_k(t)$ and also OBF $Y_k(t)$ (cf. theorem 1) the next shift parameter formula is valid.

Theorem 19 ([46]) *Let $k = 2\alpha$ and operator A is a generator $\alpha+1$ times OCF $C_{\alpha+1}(t)$ and $m > k \geq 0$. Then operator function*

$$L_m(t) = \frac{2}{B(k/2+1, m/2-k/2)}\int\limits_0^1 s^k\ (1-s^2)^{(m-k)/2-1}\ L_k(ts)\,ds$$

is an OSF for a problem (16), (17) for a parameter choice m.

OBFs $Y_k(t)$ and OCFs $L_k(t)$ give solving operator to a problem (16), (17).

Theorem 20 ([46]) *Let $u_0,\ u_1 \in D(A)$, $k = 2\alpha > 0$ and operator A is a generator α times OCF $C_\alpha(t)$. Then a function $u(t) = Y_k(t)u_0 + L_k(t)u_1$ with OSF $Y_k(t)$ and OCF $L_k(t)$, which are defined in 10 and 18, is a unique solution to the Cauchy problem (16), (17).*

Let $u_1 \in D(A)$ then for OCF $L_k(t)$ the next is valid

$$L_k'(t)u_1 = \frac{t}{k+2}L_{k+2}(t)Au_1 + u_1, \quad \lim_{t\to 0+} L_k''(t)u_1 = 0.$$

OBF and OSF give solutions to a Cauchy problem for the stressed Malmsteen equation (23) for $l = -k$ and $A \in G_{k+2}$. From properties of these function we find a solution to

$$u''(t) + \frac{k}{t}u'(t) - \frac{k}{t^2}\left(u(t) - u(0)\right) = Au(t), \quad t > 0, \tag{26}$$

satisfying (17), and it has a form

$$u(t) = tY_{k+2}(t)u_1 + \frac{t}{k+2}L_{k+2}(t)Au_0 + u_0,$$

and equality (25) for $m = 0$ has a form

$$u(t) = \frac{t}{k+2}L_{k+2}(t)Au_0 + u_0.$$

So it may be stated that a pass from abstract wave equation $u''(t) = Au(t)$ to Euler–Poisson–Darboux (EPD) equation (1) with coefficient $k > 0$ a set of admissible operators A for which an initial problem with a condition (2) is correct, is expanded from G_0 to G_k, $G_0 \subset G_k$, and a further pass from EPD equation (1) to Eq. (26) expand this set to G_{k+2}, $G_k \subset G_{k+2}$.

We also note the relations

$$L_k(t)x = \int_0^t \frac{\xi}{\sqrt{t^2-\xi^2}}Y_{k+1}(\xi)x\,d\xi, \quad A \in G_{k+1}, \quad x \in E,$$

$$L_k(t)x = \frac{\sqrt{\pi}\,\Gamma(k/2+1)}{\Gamma(k/2+1/2)}\int_0^t {}_2F_1\left(\frac{1}{2}, \frac{k}{2}; 1; 1-\frac{t^2}{\tau^2}\right)Y_k(\tau)x\,d\tau, \quad A \in G_k, \quad x \in E.$$

If the problem (1), (2) is uniformly correct, i.e., $A \in G_k$ and $Y_k(t)$ is OFB of this problem then operator A is a generator of a strongly continuous semigroup $T(t)$ and for this semigroup the representation through OFB is valid (see Theorem 3).

We also indicate a formula that allows us to express this semigroup in terms of OFS $L_k(t)$

$$T(t)x = \frac{1}{\sqrt{\pi}\, 2^k\, \Gamma(k/2+1)\, t^{k/2+1}} \int\limits_0^\infty s^k \exp\left(-\frac{s^2}{4t}\right) \Psi\left(-\frac{1}{2}, \frac{k+1}{2}; \frac{s^2}{4t}\right) L_k(s)x\, ds,$$

where $\Psi(a, b; \cdot)$ *is confluent hypergeometric Tricomi function (see. ([52], p. 365 or [25], p. 309).*

5 The Legendre Equation: Legendre Operator Function

The study of many physical processes is based on solving equations containing the Laplace operator. Using the separating of variables in curvilinear coordinate systems one can lead to differential equations containing a singularity. If there is a certain symmetry, these equations turn into the Euler–Poisson–Darboux and Legendre equations. The initial problem for the abstract EPD equation was considered in Sect. 2. In this section, we study the Cauchy problem for another abstract singular equation, namely for the Legendre equation.

For $k > 0$ we consider the Legendre equation

$$L_k u(t) \equiv u''(t) + k \coth t\, u'(t) + (k/2)^2 u(t) = Au(t), \quad t > 0. \tag{27}$$

Differential operator L_k in the left part of (27) occurs when solving the Laplace equation in coordinates of an elongated ellipsoid of revolution [53], p. 138. If A is scalar multiplication operator then for $k = 2$ spherical functions (considered in [54], p. 53) satisfy to Eq. (27).

Also note papers [5, 55–58], in which partial differential equations containing a singular operator of the type under consideration were studied.

As follows from the results of the paper [59], correct statement of the initial conditions for the abstract Legendre equation (27) consists in setting the initial conditions at the point $t = 0$

$$u(0) = u_0, \quad u'(0) = 0, \tag{28}$$

in this case, if $k \geq 1$ then initial condition $u'(0) = 0$ removed. The definition of uniform correctness of the problem (27), (28) formulated similarly to the Definition 2.

In [59] found that set of operators A with which the problem (27), (28) correct uniformly coincides with the set G_k introduced in Section. The resolving operator of this problem is denoted by $P_k(t)$ and called operator Legendre function (OLF).

OLF can also be used and for solving the weighted Cauchy problem for the Legendre equation. If $0 < k < 1$ then more general then in (28) initial conditions are correct. Let consider the initial conditions of the form

$$u(0) = u_0, \quad \lim_{t\to 0} \left(\frac{\sinh t}{t}\right)^k u'(t) = u_1. \tag{29}$$

For $u_0,\ u_1 \in D(A)$ and $A \in G_k \subset G_{2-k}$ the unique solution to the Cauchy problem (27), (29) has the form (see [59])

$$u(t) = P_k(t)u_0 + \frac{1}{1-k}\left(\frac{\sinh t}{t}\right)^{1-k} P_{2-k}(t)u_1.$$

Note that if $A \in G_k$ and $k \geq 1$ then the problem (27), (29) is not correct.

Theorem 21 ([59]) *Let the problem (27), (28) uniformly correct when parameter $m \geq 0$ ($A \in G_m$) then this problem uniformly correct and for $k > m \geq 0$ ($A \in G_k \supset G_m$). While corresponding OLF $P_k(t)$ is*

$$P_k(t) = \Upsilon_{k,m} P_m(t) = \frac{2^{(k-m)/2} \sinh^{1-k} t}{B(k/2 - m/2, m/2 + 1/2)} \int_0^t (\cosh t - \cosh s)^{(k-m)/2-1} \sinh^m y \, P_m(s)\, ds. \tag{30}$$

The equality (30) written on the initial element u_0 is called the formula of a shift by the parameter k of the solution of the Cauchy problem for Eq. (27) and $\Upsilon_{k,m}$ is transmutation operator transmuting differential operators L_m and L_k and preserving initial conditions (28).

In addition, the equality

$$P_k'(t)u_0 = \frac{\sinh t}{k+1}\, P_{k+2}(t)\left(A - \frac{k^2}{4}I\right)u_0$$

is valid. From this equality follows that the first and the second producing operators of OLF $P_k(t)$ are equal to zero and to $\dfrac{1}{k+1}\left(A - \dfrac{k^2}{4}I\right)$, respectively.

In the particular case when the operator $A = (\delta + 1/2)^2,\ \delta \in \mathbb{R}$ is the operator of multiplication by a number then OLF $P_k(t)$ is expressed through the associated Legendre function of the first kind $\mathbf{P}_\delta^\beta(\cdot)$ (see [52], p. 661)

$$P_k(t) = \Gamma(1-\beta)\left(\frac{1}{2}\sinh t\right)^\beta \mathbf{P}_\delta^\beta(\cosh t), \quad \beta = \frac{1-k}{2}.$$

As indicated in Theorem 3, the operator $A \in G_k$ is a generator of the semigroup $T(t)$ which in case of even k can be represented (see [59]) through the OLF $P_k(t)$ (see [59])

$$T(t) = \frac{1}{\Gamma(k/2+1/2)\sqrt{t}} \int_0^\infty \sinh^k s \left(-\frac{1}{2\sinh s}\frac{d}{ds}\right)^{k/2} \exp\left(-\frac{s^2}{4t}\right) P_k(s)\, ds.$$

In the case of integer $k/2$ semigroup $T(t)$ can be represented through OLF $P_k(t)$ using for

$$\left(-\frac{1}{2\sinh s}\frac{d}{ds}\right)^{k/2},$$

the definition of a fractional derivative.

In conclusion of this section, we note that the OFL $P_k(t)$ was used by the author in [60] to establish the criterion for stabilizing the solution of the Cauchy problem for an abstract differential equation of the first order.

6 The Loaded Legendre Equation

In this section, we consider the equation

$$u''(t)+k\coth t\left(u'(t)-\frac{\cosh^{2-k}(t/2)}{\cosh t}u'(0)\right)+\frac{k^2}{4}u(t) = Au(t), \quad t>0, \tag{31}$$

which, unlike Eq. (27), contains the value of the derivative of the unknown function at the point $t = 0$ and which we will call the weakly loaded Legendre equation.

The presence in Eq. (31) given at $t = 0$ load changes the setting of the initial problem. Unlike the weighted problem (27)–(29) for $k > 0$ we will establish the correctness of the Cauchy problem

$$u(0) = u_0, \quad u'(0) = u_1 \tag{32}$$

for a lightly loaded equation (31) and indicate the explicit form of the resolving operator.

In this section, we will further assume $g(t) = \cosh t$ and

$$\mu_k = \frac{2^{k/2}\Gamma(k/2+1/2)}{\sqrt{\pi}\,\Gamma(k/2)}.$$

To prove the following statements, it is convenient to use the concept of a fractional integral of a function $f(t)$ by the function $g(t) = \cosh t$ (see [20], p. 248)

$$I_g^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (\cosh t - \cosh s)^{\alpha-1} \sinh s \; f(s)\, ds.$$

Let an operator A is a generator COF $C(t)$, $u_0 \in D(A)$. Then from Theorem 21 the following representation follows for OFL $P_k(t)$

$$P_k(t)u_0 = \mu_k \sinh^{1-k} t \int_0^t (\cosh t - \cosh s)^{k/2-1} C(s)u_0 \; ds = \mu_k \; \Gamma(k/2) \sinh^{1-k} t \; I_g^{k/2} \left[\frac{C(t)}{\sinh t} \right] u_0 \tag{33}$$

is valid.

Next, we consider the Cauchy problem (31)–(32) in case when $u_0 = 0$. Let $\nu_k = k2^{k/2-1}$ and

$$S(t) = \int_0^t C(s)\, ds$$

is a sine operator function (SOF).

Theorem 22 ([61]) *If $u_0 = 0, \; u_1 \in D(A)$ and the operator A is a generator COF $C(t)$, then function $u(t) = Q_k(t)u_1$, where*

$$Q_k(t)u_1 = \nu_k \sinh^{1-k} t \int_0^t (\cosh t - \cosh \tau)^{k/2-1} S(\tau)u_1 \; d\tau = \nu_k \; \Gamma(k/2) \sinh^{1-k} t \; I_g^{k/2} \left[\frac{S(t)}{\sinh t} \right] u_1 \tag{34}$$

is the solution to the problem (31)–(32), and wherein

$$Q_k'(t)u_1 = \frac{\sinh t}{k+2} \, Q_{k+2}(t) \left(A - \frac{k^2}{4} I \right) u_1 + \frac{u_1}{\cosh^k(t/2)}.$$

Theorem 23 ([61]) *Let $u_0, \; u_1 \in D(A)$ and operator A is a generator COF $C(t)$. Then function $u(t) = P_k(t)u_0 + Q_k(t)u_1$, where operator functions $P_k(t)$ and $Q_k(t)$ are given by (33), (34), is the unique solution to the Cauchy problem (31)–(32).*

7 Nonlocal Problems

Opposite to Sect. 1 of this paper let find a solution $u(t) \in C^2([0,1],E) \cap C((0,1],D(A))$ to EPD equation (1), with nonlocal integral condition

$$\lim_{t\to 1} I_{\nu,\beta}\, u(t) = u_1 \tag{35}$$

and condition

$$u'(0) = 0, \tag{36}$$

with $\nu = (k-1)/2,\ \beta > 0,\ I_{\nu,\beta}$ being an Erdélyi–Kober operator defined by (cf. [20], p. 246)

$$I_{\nu,\beta}\, u(t) = \frac{2}{\Gamma(\beta)\, t^{2(\beta+\nu)}} \int\limits_0^t s^{2\nu+1}(t^2-s^2)^{\beta-1} u(s)\, ds.$$

The problem (1), (35), (36) with nonlocal condition (35) in general is not correct. Many ill-posed problems for differential–operator equations may be reduced to operator equations of the first kind $Bx = y,\ x, y \in E$ and the main problem is to prove its solvability. We formulate conditions for an operator A and element $u_1 \in E$ which are sufficient for unique solvability.

Let refer to papers on solvability of nonlocal problems with integral condition for abstract first order equation [62] and [63]. Necessary and sufficient condition for solution's uniqueness was found in [64].

As it follows from the results of the first section of this work correct initial problem for EPD equation (1) include given values at $t = 0$ and a condition (36), which is dropped for $k \geq 1$,

$$u(0) = u_0 \in D(A). \tag{37}$$

Further let fix a condition $A \in G_k$ as valid, it means uniform correctness of the problem (1), (37), (36), and below we consider a determination of initial element u_0 in condition (37) by nonlocal condition (35). This nonlocal problem is reduced to an operator equation of the first kind $Y_k(1)u_0 = y$ which we solve on a subset $D(A)$.

Let introduce an entire function

$$\cosh\, i_{k,\beta}(\lambda) = \frac{\Gamma((k+1)/2)}{\Gamma((k+1)/2+\beta)}\, {}_0F_1\left(\frac{k+1}{2}+\beta; \frac{\lambda}{4}\right),$$

which is called characteristic function for nonlocal condition (35).

Theorem 24 ([65]) *Let A being a bounded operator and $u_1 \in E$. For unique solvability of the problem (1), (35), (36) is necessary and sufficient for the next condition being valid on a spectrum $\sigma(A)$ of operator A*

$$\cosh\, i_{k,\beta}(\lambda) \neq 0, \quad \lambda \in \sigma(A). \tag{38}$$

From the Theorem 24 it follows that position of zeroes of the function $\cosh\, i_{k,\beta}(\lambda)$ *is responsible for the unique solvability of the problem (1), (35), (36) with a bounded operator A. For EPD equation with unbounded operator A the condition (38) will not be sufficient for the unique solvability, though position of zeroes is also important.*

Now let find necessary condition for the uniqueness of a solution for the inverse problem (1), (35), (36) with an unbounded operator A.

Theorem 25 ([65]) *Let A being a linear closed operator in E. Propose that nonlocal problem (1), (35), (36) has a solution $u(t)$. Then for this solution being unique it is necessary that all zeroes $\mu_j,\ j = 1, 2, \ldots$ of the entire function* $\cosh\, i_{k,\beta}(\lambda)$ *do not belong to the set of eigenvalues of operator A.*

In contrast to Theorem 24, the proof of a sufficient condition for unique solvability requires additional conditions.

Theorem 26 ([65]) *Let the operator $A \in G_k$ and each zero $\mu_j,\ j = 1, 2, \ldots$ of function* $\cosh\, i_{k,\beta}(\lambda)$ *belongs to the resolvent set $\rho(A)$. Let also exists such $d > 0$ that $\sup\limits_{j=1,2,\ldots} \|R(\mu_j)\| \leq d$. If $u_1 \in D(A^{n+1})$, where $n \in \mathbb{N}$ chosen so that the inequality $n > \max\{(k + \beta + 1)/2, (k/2 + \beta + 2)/2\}$ is true then the problem (1), (35), (36) has a unique solution.*

A similar nonlocal problem for the abstract Malmsten equation, which is a generalization of the EPD equation, was considered in [66].

We also point out that the nonlocal problem for the Legendre equation (27) with conditions

$$\lim_{t\to 1} I_g^{\beta}\left(\sinh^{k-1} t\, u(t)\right) = u_1, \quad u'(0) = 0$$

and the boundary control problem for a lightly loaded Legendre equation (31) with conditions

$$u(1) = u_2, \quad u'(1) = u_3$$

were studied in [61]. Results on the solvability of a nonlocal problem for the Bessel-Struve equation (16) with two nonlocal conditions containing Erdeyi-Kober operators were announced in [67].

8 Dirichlet Problem for the Bessel-Struve Equation

Boundary problems for Eq. (16) for $A \in G_k$ (hyperbolic case), generally speaking, they are not correct, but the need to solve these incorrect problems is now generally recognized (see introduction [68–70] and their bibliography). The second chapter of the monograph [68] explores the correctness of general boundary value problems for a first-order differential-operator equation and for an abstract wave equation $u''(t) = Au(t)$.

We will look for a solution $u(t) \in C^2([0, 1], E) \cap C((0, 1], D(A))$ of Eq. (16) for $t \in [0, 1]$, satisfying to the boundary conditions

$$u(0) = u_0, \quad u(1) = u_1. \tag{39}$$

Dirichlet Problem (16), (39) can be reformulated as the inverse problem of finding a function $u(t)$ and an element $p \in D(A)$ which is the second initial condition in (17). So $u(t)$ and p should be found from the equation

$$u''(t) + \frac{k}{t}u'(t) = Au(t) + \frac{k}{t}p \tag{40}$$

by initial and final conditions from equality (39). A detailed review of the work on various inverse problems can be found in [71].

Returning to the problem we are considering (40), (39), note that, taking into account the Theorem 20, we should define an element $p \in D(A)$ from the operator equation

$$L_k(1)p = u_2, \tag{41}$$

where $u_2 = u_1 - Y_k(1)u_0$.

To establish the solvability of Eq. (41) we impose an additional condition to the resolvent of the operator A. An important role will be played by the entire function

$$\cosh i_k(\lambda) = \frac{2^{k/2-1/2}\sqrt{\pi}\,\Gamma(k/2+1)}{\lambda^{k/4+1/4}}\,\mathbf{L}_{k/2-1/2}\left(\sqrt{\lambda}\right), \tag{42}$$

Condition 1 Each zero $\mu_j,\ j = 1, 2, \dots$ defined by equality (42) of entire function $\cosh i_k(\lambda)$ belongs to the resolvent set $\rho(A)$ and there is such $d > 0$ then

$$\sup_{j=1,2,\dots} \|R(\mu_j)\| \le d.$$

Note that in the general case for $k > 0$ distribution of zeros μ_j of function $\cosh i_k(\lambda)$ we do not know, but in special cases for $k = 0$ and $k = 2$ zeros μ_j are calculated explicitly. In these particular cases, respectively, we have:

$$\cosh\, i_0(\lambda) = \frac{\sinh\sqrt{\lambda}}{\sqrt{\lambda}}, \quad \mu_j = -\pi^2 j^2, \quad j \in \mathbb{N},$$

$$\cosh\, i_2(\lambda) = \frac{2\left(\cosh\sqrt{\lambda} - 1\right)}{\lambda}, \quad \mu_j = -4\pi^2 j^2, \quad j \in \mathbb{N}.$$

Let the condition 1 is valid. Since each zero μ_j, $j = 1, 2, \ldots$ of the function $\cosh\, i_k(\lambda)$ belongs to $\rho(A)$, then it belongs to $\rho(A)$ together with a circular neighborhoodΩ_j with the radius $\frac{1}{d}$, whose boundary is traversed along clockwise, we denote γ_j.

Condition 2 For some n, such that

$$n > \frac{1}{4}\left(k + 7 - \max\{3 - k, 1\}\right),$$

series

$$\sum_{j=1}^{\infty} \int\limits_{\gamma_j} \frac{R(z)\; dz}{\cosh\, i_k(z)\;(z - \lambda_0)^n}, \qquad \lambda_0 \in \rho(A), \quad \mathrm{Re}\lambda_0 > \sigma$$

absolutely converges.

We formulate a theorem on the solvability of the Dirichlet problem for the Bessel-Struve equation, which was announced in [72].

Theorem 27 *Let $A \in G_k$ and conditions 1, 2 are valid. If $u_0, u_1 \in D(A^{n+1})$ then the problem (16), (39) has a unique solution.*

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