Necessary Condition for the Existence of an Intertwining Operator and Classification of Transmutations on Its Basis

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Abstract The authors study second-order ordinary differential operators with functional coefficients for all derivatives and the Volterra integral operator with a definite kernel. Results of the paper establish a hyperbolic equation and additional conditions that allow one to construct a kernel according to the ODE. The statements of the paper show the possibility of splitting the ODE into classes according to the type of the kernel of the Volterra operator. Examples are considered related to ODE with Pöschl-Teller type potentials, Bessel functions with complex arguments and Euler's relation for hypergeometric functions.

1 Introduction

The transmutation operator (the intertwining operator) [1–3] is a Volterra integral operator associated with other mathematical structures, which imposes a restriction on its construction. The article proves a theorem on conditions that interlaced ordinary differential operators of second order with variable coefficients for all derivatives impose on the form of the kernel of the Volterra operator. The inverse statement is also presented that for a given kernel, interlaced structures cannot be arbitrary, but are divided into classes of feasible functions, largely determined by the structure of the core, and the ODE coefficients of the highest derivative.

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2 Problem Definition

Historically, the first intertwining operators, rebounding from generalized translation operators [4, 5], appeared in the form of a Volterra type II integral operator [6, ch I, лемма 1.1.1], [7, ch. I, (1.4)]. However, according to the traditional approach to integral equations, it is more natural to take the Volterra type I integral operator in a one-dimensional space ($T : L^2(I) \to L^2(I)$) defined by the formula

$$f_1(x) = Tf_0(x) = \int_0^x K(x,t) f_0(t)dt \tag{1}$$

which reduced initial class of functions $f_0 \in E_0$ into reduced class $f_1 \in E_1$, при $I = [0, b]$, $K \in L^2(I \times I)$. Transition function $K(x, t)$ is called the kernel of the transmutation operator.

If in (1) kernel $K(x, x) = \gamma \neq 0$, then by differentiation (1) it traditionally turns into

$$f_1'(x) = \gamma f_0(x) + \int_0^x \frac{dK(x,t)}{dx} f_0(t)dt$$

Due to this fact, only transformations of the first kind will be investigated in the future.

Comment The features of the kernel and the coefficients of the subsequent differential equations involved in the construction of $K(x, t)$, require a more correct record of the proposed definition. Exactly

$$Tf(x) = f_1(x) = \int_\varepsilon^{x-\delta} K(x,t) f_0(t)dt$$

with the subsequent passage to the limit $\varepsilon \to 0$ and $\delta \to 0$. These clarifications will be clearly spelled out when installing the conditions imposed on the kernal of the Volterra operator.

The Transmutation Operator (Intertwining operator) (in the work [1]—the transformation operator for entities A and B) is a triplet of $\{A, B, T\}$ objects that satisfy the condition

$$TA = BT \tag{2}$$

where A, B are ordinary differential operators, traditionally defined by differential expressions and initial conditions

$$A = \begin{cases} \frac{d}{dt}\left(a_0(t)\frac{df_0(t)}{dt}\right) + \frac{d}{dt}\left(b_0(t)f_0(t)\right) + c_0(t)f_0(t) = 0 \\ \left.\frac{df_0(t)}{dt}\right|_{t=0} - h_0 * f_0(t)|_{t=0} = 0 \quad \text{или} \\ f_0(0) = 0; \quad \left.\frac{df_0(t)}{dt}\right|_{t=0} = H_0; \end{cases} \tag{3}$$

$$B = \begin{cases} a_1(x)\frac{d^2 f_1(x)}{dx^2} + b_1(x)\frac{df_1(x)}{dx} + c_1(x)f_1(x) = 0 \\ f_1(0) = 0; \ \left.\frac{df_1(x)}{dx}\right|_{x=0} = H_1; \end{cases} \tag{4}$$

and T is an integral operator, represented in (1). Note that the Sturm-Liouville operator (3) is written in the generally accepted divergent form (see Sturm-Liouville theory, Wikipedia) for favorable integration in parts, which is necessary in proving the following theorem. The transition from the divergent form to the usual one is not difficult and is, for example, registered in [8, Ch. 9]

The initial conditions for determining the entity (4) are associated with the tendency of the Volterra operator of the first kind to zero for $x \to 0$. Very often, when specifying the initial ratio (3), one of the standard constructions is used [9, ch. 8]

$$\begin{aligned} &f_0(0) = 1; \quad f_0'(0) = 0; \quad or \\ &f_0(0) = 0; \quad f_0'(0) = 1; \end{aligned}$$

which contributes to the selection of the even or odd part of the solution $f_0(t)$. Then, after applying differentiation to transform (1) into a Volterra mapping of the second kind, the introduced transmutation operator corresponds to the transformation operators $\mathbf{K}_h$ and $\mathbf{K}_\infty$ used in Sturm-Liouville spectral theory [10, 11].

The proposed definition of the intertwining operator admits a generalization by modifying the operators A and B (for example, increasing the order of differential equations), as well as changing the form of the integral transform (1), but this extension is not intended.

In papers [1, 10, 11]) for $\{a_0(t) = a_1(x) = 1;\ b_0(t) = b_1(x) = 0,\ c_0(t) = q_0(t),\ -c_1(x) = q_1(x)\}$, [12]—for Bessel operators, [13]—in general, a relationship is established between the coefficients of differential operators and the type of the $K(x, t)$ transformation operator.

Theorem 1 *A necessary and sufficient conditions that the Volterra integral operator (1) be the transmutation operator for ordinary differential (3, 4) operators is:*

(a) The kernel of the transformation operator (1) must be a solution to the hyperbolic equation

$$\begin{cases} L\left[K(x,t)\right] = \left[\dfrac{\partial}{\partial t}\left(a_0(t)\dfrac{\partial K(x,t)}{\partial t}\right) - b_0(t)\dfrac{\partial K(x,t)}{\partial t} + c_0(t)K(x,t)\right] - \\ \qquad - \left[a_1(x)\dfrac{\partial^2 K(x,t)}{\partial x^2} + b_1(x)\dfrac{\partial K(x,t)}{\partial x} + c_1(x)K(x,t)\right] = 0 \end{cases} \tag{5a}$$

(b) On the characteristic t = x, the kernel K (x, t) and its first derivative with respect to t exist; at $t \to x - \delta$ и $\delta \to 0$

$$b1)\quad a_0(x) = a_1(x) = a(x)$$

$$b2)\quad 2a(x)\frac{dK(x,x-\delta)}{dx} + (b_1(x) - b_0(x))K(x,x-\delta) = 0 \tag{5b}$$

(c) With initial condition $t = \varepsilon \to 0$

$$\left\{a(\varepsilon)\left[\frac{dK(x,t)}{dt}\right]_{t=\varepsilon} - b_0(\varepsilon)K(x,\varepsilon) - h_0 a(\varepsilon)K(x,\varepsilon)\right\} f_0(\varepsilon) \to 0 \tag{5c}$$

(d) Condition at the edge. Under $\delta < \varepsilon, \quad \delta \to 0, \quad \varepsilon \to 0$

$$K(\varepsilon, \varepsilon - \delta) * f_0(\varepsilon) \to 0; \tag{5d}$$

Note that in (b) and (c) it is not necessary to know the explicit form of the function $f_0(x)$. What is important is the tempo of striving $f_0(\varepsilon)$ to zero with $\varepsilon \to 0$ to compensate for the singularity of the coefficients a, b, c at the origin point.

The proof of the theorem is based on the definition of the transmutation operator (2), which with respect to ordinary differential operators looks like

$$T(Af_0)(x) = B(Tf_0)(x); \quad \forall x;$$

The integral in the left component of the equality is taken in parts, and in the right component differentiation takes place according to its variable upper limit.

Proof of Theorem 1 Let us prove the assertions of the theorem, generalizing the method [1, 9–12]. For convenience and brevity of the record, we introduce the notation

$$K_0(x) = K(x, x-\delta); \quad f(x) = f_0(x);$$

$$\partial_t = \frac{d}{dt}; \quad \partial_{tt} = \frac{d^2}{dt^2}; \quad \partial_x = \frac{d}{dx}; \quad \partial_{xx} = \frac{d^2}{dx^2};$$

The first operation will be TA.

$$TA(f(x)) = \int_{\varepsilon}^{x-\delta} K(x,t)\{\partial_x(a_0(x)\partial_x f(x)) + \partial_x(b_0(x)f(x)) + c_0(x)f(x)\}dt$$

The integral with the first term is taken two times in parts. It is precisely at this moment that the record of the operator (3) in a divergent form is highly desirable. Similarly, in parts, the second addend will be transformed only once. This leads to the following result

$$\begin{aligned}
&\mathrm{TA}(f(x)) = a_0(x)K_0(x)\partial_x f(x) + \\
&+ \left[K_0(x)b_0(x) - a_0(x)\ \{\partial_t K(x,t)\}|_{t=x-\delta}\right] f(x) - a_0(\varepsilon)K(x,\varepsilon)\ \{\partial_t f(t)\}|_{t=\varepsilon} + \\
&+ a_0(\varepsilon)\ \{\partial_t K(x,t)\}|_{t=\varepsilon}\ f(\varepsilon) - b_0(\varepsilon)K(x,\varepsilon)f(\varepsilon) + \\
&+ \int_{\varepsilon}^{x-\delta} \{\partial_t[a_0(t)\partial_t K(x,t)] - b_0(t)\partial_t K(x.t) + c_0(t)K(x,t)\}\ f(t)dt
\end{aligned}$$

Further action is the study of the relationship BT

$$B(Tf(x)) = \left\{a_1(x)\partial_{x,x}(\circ) + b_1(x)\partial_x(\circ) + c_1(x)(\circ)\right\} \left\{ \int\limits_{\varepsilon}^{x-\delta} K(x,t)f(t)dt \right\}$$

The calculation of the derivative of the integral over a variable upper limit generates the equality

$$\begin{aligned}
&B(Tf(x)) = \{a_1(x)\partial_x K_0(x) + a_1(x)\ [\partial_x K(x,t)\rfloor|_{t=x-\delta} + \\
&b_1(x)K_0(x)\}f(x) + a_1(x)K_0(x)\partial_x f(x) + \\
&+ \int_{\varepsilon}^{x-\delta} K(x,t)\{a_1(x)\partial_{xx}K(x,t) + b_1(x)\partial_x K(x,t) + c_1(x)K(x,t)\}f(x)
\end{aligned}$$

Comparison of integrands implies (5a). Due to the arbitrariness of $f(x)$, the coefficients in front of the function and its first derivative should be separately equal to zero. Comparing the elements before the first derivative gives (5b.1). If we take into account this fact in the coefficient adjacent to f (x), as well as for $\delta \to 0$, use equality

$$\frac{dK(x, x-\delta)}{dx} = \left.\frac{\partial K(x,t)}{\partial x}\right|_{t=x} + \left.\frac{\partial K(x,t)}{\partial t}\right|_{t=x}$$

then the grouping of elements before $f(x)$ establishes a correspondence (5b.2). It remains to group the initial conditions string when $t = \varepsilon \to 0$. All its elements are entirely in the TA operator. There will be an expression

$$-a_2(\varepsilon)K(x,\varepsilon)\ \{\partial_t f(t)\}|_{t=\varepsilon} + a(\varepsilon)\ \{\partial_t K(x,t)\}|_{t=\varepsilon}\ f(\varepsilon) - b_2(\varepsilon)K(x,\varepsilon)f(\varepsilon)$$

The final result is fixed in the condition (5c). To formulate the condition at the vertex we take the derivative of (1)

$$\partial_x Tf(x) = K(x, x-\delta)f(x-\delta) + \int\limits_{\varepsilon}^{x-\delta} \partial_x K(x,t)f(t)dt$$

At the point $x = \delta + \varepsilon$

$$\partial\ Tf(x)|_{x=\delta+\varepsilon} = K(\delta+\varepsilon, \varepsilon) * f(\varepsilon)$$

In the end we take into account the initial conditions

$$\partial\ f_1(x)|_{x=\varepsilon} - h_1\ f_1(x)|_{x=\varepsilon} = 0; \quad \partial\ f_0(x)|_{x=\varepsilon} - h_0\ f_0(x)|_{x=\varepsilon} = 0;$$

what gives (5d).

The presented conditions refer to an arbitrary form of the kernel, but even they impose substantial restrictions on it and on the articulated operators A and B. First, the coefficients of the highest derivative in (3) and (4) must coincide with the accuracy of the free variable, that is, $a_0(t) = a_1(x)$ with $t = x$. The type of ordinary differential equation is largely determined by these coefficients, so often transmutation occurs between A and B with similar properties. Secondly, the absence of singularity of the kernel $K(x,t)$ and its derivative with respect to the argument t leaves outside the scope of this consideration intertwining transformations with special points, for example, the integral Mohler-Fock representation for Legendre functions and their generalizations [14]

$$P_{-\frac{1}{2}+\imath\nu}(\cosh x) = \frac{2}{\pi}\int\limits_0^x \frac{\cos(xt)}{\sqrt{2\,(\cosh x - \cosh t)}}\,dt$$

It is possible to overcome this difficulties with the help of integrals in the sense of the Hadamard finite part [15], but it requires a more detailed consideration of the presented structures.

An Example of the Theorem 1 Let us show that the Volterra operator performing the transformation for Gegenbauer polynomials

$$\int_0^x \left(x^2 - t^2\right)^{\beta-1} C_{2n}^{2\nu}(t)dt = \frac{2\sqrt{\pi}\Gamma(\beta)}{\Gamma\left(\beta + \frac{1}{2}\right)} x^{2\beta-1} C_n^{\beta}\left(2x^2 - 1\right); \quad Re(\beta) > 2; \tag{6}$$

is a transmutation operator. The presented identity follows from [16, Vol II, 16.3, (19)] after replacing the variable and modifying the indices. It is easy to check that the function

$$f_0(t) = C_{2n}^{2\beta}(t)$$

turns out to be a solution of a differential operator (3) with coefficients

$$a_0(t) = (1 - t^2); \quad b_0(t) = n(1 - 4\beta)t; \quad c_0(t) = 4n(n + 2\beta) + (4\beta - 1);$$

For even lower symbols (2n), the derivative of the Gegenbauer polynomials vanishes when t = 0, so the middle row is used as the initial condition in the operator (3) for $h_0 = 0$. Right part

$$f_1(x) = \frac{2\sqrt{\pi}\Gamma(\beta)}{\Gamma\left(\beta + \frac{1}{2}\right)} x^{2\beta-1} C_n^{\beta}\left(2x^2 - 1\right);$$

satisfies the operator (4) with coefficients

$$a_1(x) = (1 - x^2); \quad b_1(x) = \frac{2(1 - \beta) - 3x^2}{x}; \quad c_1(x) = 4(n + \beta)^2 - 1;$$

If we substitute the kernel

$$K(x,t) = \left(x^2 - t^2\right)^{\beta-1} \tag{7}$$

into a hyperbolic equation (5a) with the above groups of coefficients a, b, c, then it will turn it into a true equality. The core exponent ensures that the condition on the characteristic is met.

The left part of the initial condition (5c) is expanded in a series with the first member

$$h_0 \frac{4^n \sqrt{\pi}\Gamma(m+2\beta)}{\Gamma\left(\frac{1}{2}-n\right)\Gamma(2n+1)\Gamma(2\beta)} x^2 + O(\varepsilon)$$

However, it was previously noted that $h_0 = 0$, and, therefore, is realized (5c). The condition at the vertex (5d) is an identity due to the type of kernel. As a result of the fulfillment of all conditions, the Volterra operator of the first kind becomes a transformation operator for ordinary differential operators (3) and (4).

3 Formulation and Specification of Reverse Statement

It can be seen from Theorem 1 that the kernal construction of the transmutation operator can be determined on the basis of the coefficients of intertwined ordinary differential operators (3–4). In this article, we make following inverse statement the cornerstone—'*The kernals of the $K(x,t)$ transmutation operator split the intertwined operators A and B into classes, causing the appearance of their coefficients.*'

This position is related to the conditions on the characteristic of the hyperbolic operator (5a). The work [3] noted that "the content of the Copson lemma is that the initial data on the characteristics cannot be specified arbitrarily, they must be connected by Bushman-Erdeyi operators of the first kind. The main point of the proposed current article is the opposite and extended statement".

Statement 1 Conditions on the characteristic of a hyperbolic equation (5a) together with (5b) are necessary to classify the linked operators A and B by classes of kernels $K(x,t)$.

Example for Statement 1 Let us find the classes of intertwined operators A and B for an already familiar kernel (7), but with a different coefficient in the main part. Exactly,

$$K(x,t) = \left(x^2 - t^2\right)^{\beta-1}; \quad a_0(t) = 1; \quad a_1(x) = 1;$$

Substituting the specified kernel into the hyperbolic equation (5a) leads to the relation

$$L[K(x,t)] = -\left(x^2 - t^2\right)^{\beta-2}$$
$$\left(-4(\beta-1)^2 + 2(\beta-1)tb_0(t) - 2(\beta-1)xb_1(x) + (x^2-t^2)(c_0(t)-c_1(x))\right)$$

in this embodiment, the result can be obtained directly, without using the ratio on the characteristic. It is easy to see that the right-hand side vanishes at constant and equal values of the free members of the ‘c’ and coefficients of the ‘b’, inversely proportional to their arguments

$$b_0(t) = \frac{b_0}{t}; \quad b_1(x) = \frac{b_1}{x};$$

In this case, the next identity must be satisfied

$$b_1 = b_0 + 2 - 2\beta$$

with arbitrary b_0. A change in b_0 leads to an extensive one-parameter class of possible representations of the operators A and B, but the most attractive results are obtained for $b_0 = -(2\, nu + 1)$. Then

$$\begin{aligned}
&a_0(t) = 1; \quad a_1(x) = 1;\\
&b_0(t) = -\frac{2\nu + 1}{t}; \quad b_1(x) = -\frac{2(\beta + \nu) - 1}{x}; \quad c_0(t) = \omega^2; \quad c_1(x) = \omega^2;
\end{aligned}$$

The solutions of ordinary differential operators (3) and (4) with $h_0 = 0$ are Bessel functions, which makes it possible to write the transmutation operator [17, Vol II, No 2.12.4 (6)]

$$\int_0^x \left(x^2 - t^2\right)^{\beta-1} t^{\nu+1} J_\nu(\omega t)dt = \frac{2^{\beta-1} x^{\beta+\nu}}{\omega^\beta} \Gamma(\beta) J_{\beta+\nu}(\omega x); \tag{8}$$

Thus, we arrive at the following conclusion: *intertwined operators with a known form of the kernel $K(x, t)$ are not constructed in an arbitrary way and are largely determined by the type of this kernel*. Most often, the main factor in partitioning differential operators (3) and (4) into classes that are consistent with the kernel $K(x, t)$, is the main part of these operators $a(x) = a_0(x) = a_1(x)$. Recall the generality of the principal parts, up to a free variable, written in (5a).

With respect to the ad hoc kernels $K(x, t)$, statement 1 is strictly impossible to prove strictly, but it is well formalized for specific categories of $K(x, t)$. We introduce auxiliary expressions

$$\begin{aligned}
\Upsilon_1(x) &= 4a_0(x)\phi'(x)\Omega'(x) + \Omega(x)\left(\phi'(x)a_0'(x) + 2a_0(x)\phi''(x)\right)\\
\Upsilon_2(x) &= -\Omega(x)\phi'(x)\left(b_1(x) - b_0(x)\right)
\end{aligned}$$

Lemma 1 *For operator class*

$$K(x, t) = K\left(\Omega(x)\sqrt{\phi(x) - \phi(t)}\right) \tag{9}$$

with kernel satisfying the requirements ((5a)–(5d)), the conditions on the characteristic impose the following restrictions on the coefficients of the intertwined operators A and B

$$\Psi_1(x) = K'(0)\,(2\Upsilon_2(x) - \Upsilon_1(x)) = 0 \tag{10a}$$

$$\Psi_2(x) = -2K(0)\,(c_1(x) - c_0(x)) + K''(0)\Omega(x)\,(\Upsilon_2(x) - \Upsilon_1(x)) \tag{10b}$$

For even functions, the first equality is automatically fulfilled, for odd functions—the second one. The proof of the lemma is carried out by substituting (9) into a hyperbolic operator (5a). As a result, when $t \to x$, an expression appears that contains singular and regular parts

$$\frac{\Psi_1(x)}{4\sqrt{\phi'(x)(x-t)}} + \Psi_2(x) + O(x-t)$$

In fact, a parametrix is constructed modulo smoothing operators used recently in hyperbolic equations [18], although the study of relations on characteristics has a rich history [19, Ch. 4]

Example 2 to Lemma 1 Consider the class of kernals of the form

$$K(x,t) = J_0\left(\Omega(x)\sqrt{\cosh(\mu x) - \cosh(\mu t)}\right) \tag{11}$$

under $a_0(t) = 1$; $a_1(x) = 1$. The condition (5b) immediately leads to the equality $b_1(x) = b_0(x)$, moreover, due to the parity of the Bessel function of zero index $J_0(\ xi)$, the first line in the condition (10a) is performed automatically. The second generates identity

$$\frac{1}{2}\mu\Omega(x)\left(\mu\Omega(x)\cosh(\mu x) + 2\sinh(\mu x)\Omega'(x)\right) = 0$$

Selection of $\Omega(x)$ in the form of an exponent makes possible the following kind of coefficients

$$b_0(t) = \frac{b}{\sinh(\mu t)}; \quad b_1(x) = \frac{b}{\sinh(\mu x)};$$
$$\Omega(x) = \exp\left(-\frac{\mu}{2}x\right);$$
$$c_1(x) = c_0(x) + \frac{1}{2}\mu^2\beta^2\exp(-2\mu x);$$

If these values are substituted into (5a), then we get an expression that includes two linearly independent terms, one of which contains the factor ‘b’, the second—the factor $c_0(t) - c_0(x)$. Equating $b = 0$; $c_0(\ xi) = \omega^2$, we arrive at the transmutation operator

$$f_1(x) = Tf_0(x) = \int_0^x J_0\left(\exp\left(-\frac{\mu}{2}x\right)\sqrt{\cosh(\mu x) - \cosh(\mu t)}\right) f_0(t)dt \tag{12}$$

intertwining ordinary differential operators

$$\frac{d^2}{dt^2} f_0(t) + \omega^2 f_0(t) = 0 \tag{13a}$$

$$\frac{d^2}{dx^2} f_1(x) + \left(\omega^2 + \frac{1}{2}\mu^2\beta^2 \exp(-2\mu x)\right) f_1(x) = 0 \tag{13b}$$

Earlier Sergey M. Sitnik obtained the kernel (11) by the method of fixed-point iteration, solving the integral equation given in the work of Marchenko [11, Ch. I].

If the initial condition is written in (3) in the traditional form with $h_0 = 0$, and a solution that satisfies the zero initial condition is selected in (4), then the transmutation operator taking into account [20, No.2.37 b] will give the following result

$$\int_0^x J_0\left(e^{-\frac{\mu}{2}x}\sqrt{\cosh(\mu x) - \cosh(\mu t)}\right)\cos(\omega t)dt =$$

$$= -\frac{\imath\pi}{2\mu}\frac{1}{\sinh\left(\frac{\pi\omega}{\mu}\right)}\left[J_{\frac{\imath\omega}{\mu}}\left(\frac{\beta}{\sqrt{2}}\right)J_{-\frac{\imath\omega}{\mu}}\left(\frac{\beta e^{-\mu x}}{\sqrt{2}}\right) - J_{-\frac{\imath\omega}{\mu}}\left(\frac{\beta}{\sqrt{2}}\right)J_{\frac{\imath\omega}{\mu}}\left(\frac{\beta e^{-\mu x}}{\sqrt{2}}\right)\right] \tag{14}$$

For $\mu \to 0$, the relation presented is reduced to the Vekua transformation operator [21, Ch. I, Par.12], created at the time to solve elliptic equations of mathematical physics. Its feature is the shift in spectral parameter

$$\int_0^x J_0\left(\beta\sqrt{x^2 - t^2}\right)\cos(\omega t)dt = \frac{\sin\left(\sqrt{\omega^2 + \beta^2}\,x\right)}{\sqrt{\omega^2 + \beta^2}} \tag{15}$$

Equalities (10a) lead to another class of transmutation operators.

An isolated class with respect to intertwined second-order operators are the Bushman-Erdei transformations, which include the Legendre functions [3]. It suffices to look at the tables [17, vol II, No 2.17-2.18] to see in most of the options the record of the transformed component of $f_1(x)$ by means of the generalized hypergeometric series ${}_pF_q$ with $p + q > 3$. Thus, a very significant set of second-order differential operators ‘B’ do not fit into the construction (4). But in rare exceptions, the method of studying a hyperbolic operator on the characteristic admits cases of finding new versions of the Bushman-Erdeia OP.

Let’s start with the traditional core of the Bushman-Erdeyi operator

$$K(x,t) = P_\nu\left(\frac{t}{x}\right) \tag{16}$$

where the Legendre function $P_\nu(z)$ is a solution of a differential equation [16, vol I, Ch. III]. The singularity in calculating $L[K(x,t)]$ with $t \to x$ is

$$\frac{1}{2x^2}\Big[-2\nu(\nu+1) + \nu(\nu+1)(b_1(x) - b_0(x))x - 2x^2(c_1(x) - c_0(x))\Big] + O(t-x)$$

For its elimination it is enough to put

$$a_0(t) = a_1(x) = 1; \quad b_0(t) = b_1(x) = 0; \quad c_0 = \omega^2; \quad c_(x) = \omega^2 - \frac{\nu(\nu+1)}{x^2}; \tag{17}$$

With such coefficients, the relations (3) and (4) taking into account the initial conditions in (3) and the finiteness of the solution at the origin for (4) are given for integer values the index ν [17, Vol II, No 2.17.7 (1)]. Exactly,

$$\int_0^x P_{2n+1}\left(\frac{t}{x}\right)\sin(\omega t)dt = (-1)^n\sqrt{\frac{\pi x}{2\omega}}J_{2n+\frac{3}{2}}(\omega x) \tag{18a}$$

and

$$\int_0^x P_{2n}\left(\frac{t}{x}\right)\cos(\omega t)dt = (-1)^n\sqrt{\frac{\pi x}{2\omega}}J_{2n+\frac{1}{2}}(\omega x) \tag{18b}$$

We show that the kernel (18) can be extended to wider classes of functions. In this case, we obtain the original solutions of hyperbolic equations and representations for some hypergeometric functions, including composite arguments. Consider the Bushman-Erdeia transmutation operators with kernels

$$K(x,t) = P_\nu\left(\frac{\sinh(\mu t)}{\sinh(\mu x)}\right) \tag{19}$$

The study of the relation on the characteristic $L[K(x,t)]$ with $t \to x$ leads to an estimate

$$L[K(x,t)] = -\nu(\nu+1)\mu^2 Csch^2(\mu x) + \frac{1}{2}\nu(\nu+1)(b_1(x) - b_0(x)) \\ - (c_1(x) - c_0(x) + O(t-x) \tag{20}$$

The selection of coefficients in the equations is not complicated. Initially, they are located so as to nullify the final term in (20), and then the final sorting takes place to turn (5a) into an identity. Finally, it found that the kernel (19) satisfies the hyperbolic equation

$$\frac{\partial^2 K(x,t)}{\partial t^2} + \omega^2 K(x,t) = \frac{\partial^2 K(x,t)}{\partial x^2} + \left(\omega^2 - \mu^2 \frac{\nu(\nu+1)}{\sinh^2(\mu x)}\right) K(x,t) \tag{21}$$

The steps involved in transforming an ordinary differential equation for the operator 'B' are well known from books on quantum mechanics [22, Problem 39]. Replace variables and the function sought are sequentially performed

$$y = -\sinh^2(\mu x); \quad f_1(y) = y^{\frac{\nu+1}{2}} v(y)$$

and also, parameter designation is introduced

$$a = \frac{\nu}{2} + \imath\frac{\omega}{2\mu}; \quad b = -\frac{\nu}{2} + \imath\frac{\omega}{2\mu};$$

The solution consists of a linear combination of the regular part tending to zero for $x \to 0$

$${}_2F_1\left(-a, b, \frac{1}{2} - \nu, -\sinh^2(\mu x)\right)(-\sinh^2(\mu x))^{\frac{\nu+1}{2}}$$

and singular part

$${}_2F_1\left(\frac{1}{2} - b, \frac{1}{2} + a, \frac{3}{2} + \nu, -\sinh^2(\mu x)\right)(-\sinh^2(\mu x))^{-\frac{\nu}{2}}$$

Clearly, the transmutation operator

$$\int_0^x P_\nu\left(\frac{\sinh(\mu t)}{\sinh(\mu x)}\right)\cos(\omega t)dt \tag{22}$$

correlates only with the regular component, however, due to the complexity of the parameters of the hypergeometric function, it is very difficult to trace the exact match. Nevertheless, the finite number of components in the Legendre polynomials for integer $\nu = 2n$ allows us to express and investigate the result in a much simpler form.

When $nu = 2$, the integral (22) takes the value

$$f_1(x) = -\frac{\sin(\omega x)}{2\omega} + \\ + \frac{3}{4}\frac{1}{\sinh^2(\mu x)}\left[-\frac{\sin\omega x}{\omega} + \frac{\omega\cosh(2\mu x)\sin(\omega x) + 2\mu\cos(\omega x)\sinh(2\mu x)}{\omega^2 + (2\mu)^2}\right]$$

When $\mu \to 0$, this representation completely coincides with the right-hand side of Eq. (18b) for $n = 1$. The presented examples with different integer indices describe a certain set of Bargman potentials [23, Ch. VI.I] and can be used for their construction and study. Calculations with kernels of the type (22) are carried out similarly. We present their results in the following lemma.

Lemma 2 *Bushman-Erdei transmutation operators with kernels*

$$K(x,t) = P_\nu\left(\frac{\sinh(\mu t)}{\sinh(\mu x)}\right); \qquad K(x,t) = P_\nu\left(\frac{\cosh(\mu x)}{\cosh(\mu t)}\right);$$
$$K(x,t) = P_\nu\left(\frac{\sin(\mu t)}{\sin(\mu x)}\right); \qquad K(x,t) = P_\nu\left(\frac{\cos(\mu x)}{\cos(\mu t)}\right); \tag{23}$$

connect the solution to the equation

$$\frac{d^2 f_0(t)}{dt^2} + \omega^2 f_0(t) = 0$$

with solutions of equations

$$\frac{d^2 f_1(x)}{dx^2} + \left(\omega^2 + V(x)\right) f_1(x) = 0$$

for potentials

$$V(x) = \mu^2\nu(\nu+1)U(x) \tag{24}$$

where respectively

$$U(x) = \frac{1}{\sinh^2(\mu x)}; \qquad U(x) = \frac{1}{\cosh^2(\mu x)};$$
$$U(x) = \frac{1}{\sin^2(\mu x)}; \qquad U(x) = \frac{1}{\cos^2(\mu x)}; \tag{25}$$

In quantum mechanics, the potentials presented are called Peschl-Teller potentials (modified and ordinary) [22, Problems No 38, 39]. Their use for integer $\nu = n$

is important when considering eigenvalues and eigenfunctions that are consistent with boundary or other quantization conditions [24]. The group of transmutation operators presented in Lemma 3 is an essential addition to the set of Bushman-Erdei operators given in the work [3].

4 Some Convolutions as Transmutation Operators and Their Modifications

Convolution type transformation operators have been extensively studied in the literature (see, for example, [25]), so we will only touch on those that are important from a transmutation point of view.

Lemma 3 *By definition, each transmutation operator is a Volterra operator of the first or second kind, the converse is false.*

Let us give an example of the last statement—the Kapteyn trigonometric integral [26, 12.21]

$$\int_0^x \cos(x-t)J_0(t)dt = xJ_0(x)$$

Here the kernel is $K(x,t) = cos(x-t)$, and the coefficients in the ordinary differential operators (3) and (4) are

$$\begin{array}{l} a_0(t)=1;\quad b_0(t)=\frac{1}{t};\quad c_0(t)=1+\frac{1}{t^2}; \\ a_1(x)=1;\ b_1(x)=-\frac{1}{x};\ c_1(x)=1+\frac{1}{x^2}; \end{array}$$

It is easy to check the impracticability of the hyperbolic equation (5a) with a similar combination of elements necessary for the transmutation operator.

At the same time, extensive combinations of $K(x,t);\ f_0(t);\ f_1(x)$ associated with hypergeometric functions for which there is a possibility of linking. Imagine an initially simple illustration. It is easy to check that the coefficients

$$\begin{array}{lll} a_0(t)=t; & b_0(t)=(1-\beta)-t; & c_0(t)=\beta-\alpha; \\ a_1(x)=x; & b_1(x)=(2-\beta-\gamma)-x; & c_1(x)=\beta+\gamma-\alpha-1; \end{array}$$

substituted into Eqs. (3) and (4) lead to Kummer intertwined functions. In this case, the kernal

$$K(x,t)=(x-t)^{\gamma-1}$$

Replacing the variables $t \to xt$ gives the well-known integral relation [27, Vol II, No 20.3 (2)]

$$\int_0^x (x-t)^{\gamma-1} t^{\beta-1} \, {}_1F_1(\alpha, \beta, t) dt = \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)} x^{\beta+\gamma-1} \, {}_1F_1(\alpha, \beta+\gamma, x); \tag{26}$$

A degenerate hypergeometric function with an integral nonpositive first argument is a generalized Laguerre polynomial

$${}_1F_1(-n, \beta, z) = L_n^{\beta}(x)$$

Together with (26), this leads to the transmutation operator [27, Vol II, No 16.6 (5)]

$$\int_0^1 (1-t)^{\beta-\alpha-1} t^{\alpha} L_n^{\alpha}(xt) dt = \frac{\Gamma(\alpha+n+1)\Gamma(\beta-\alpha)}{\Gamma(\beta+n+1)} L_n^{\beta}(x); \tag{27}$$

5 Euler Transformation for Hypergeometric Functions as a Transmutation Operator

For the basis of further intertwining operators, we take the Euler transformation [28, Ch. 4]

$${}_{p+1}F_{q+1}\left(\begin{matrix} a_1 \dots a_p \ c \\ b_1 \dots b_q \ d \end{matrix}; z\right) =$$

$$= \frac{\Gamma(d)}{\Gamma(c)\Gamma(d-c)} \int_0^1 \xi^{c-1}(1-\xi)^{d-c-1} \, {}_pF_q\left(\begin{matrix} a_1 \dots a_p \ c \\ b_1 \dots b_q \ d \end{matrix}; z\xi\right) d\xi$$

There are two directions in which it can develop. In the first case, this is a transition to the standard transformation operator, by replacing $t = z\xi$.

$$z^{d+1} \, {}_{p+1}F_{q+1}\left(\begin{matrix} a_1 \dots a_p \ c \\ b_1 \dots b_q \ d \end{matrix}; z\right) =$$

$$= \frac{\Gamma(d)}{\Gamma(c)\Gamma(d-c)} \int_0^z (z-t)^{d-c-1} t^{c-1} \, {}_pF_q\left(\begin{matrix} a_1 \dots a_p \ c \\ b_1 \dots b_q \ d \end{matrix}; t\right) d\xi \tag{28}$$

The second option is more interesting. The Euler transformation initially relies on $z = \kappa x^2$, where $\kappa = \pm 1$. Then the integral follows the replacement $\xi = \eta^2$ with the following substitution $t = z\eta$. The final ratio is as follows.

$$z^{2(d-1)}\,{}_{p+1}F_{q+1}\left(\begin{matrix} a_1 \dots a_p \; c \\ b_1 \dots b_q \; d \end{matrix}; \kappa z^2\right) =$$

$$= \frac{2\Gamma(d)}{\Gamma(c)\Gamma(d-c)} \int\limits_0^z (z^2 - t^2)^{d-c-1} t^{2c-1}\,{}_pF_q\left(\begin{matrix} a_1 \dots a_p \; c \\ b_1 \dots b_q \; d \end{matrix}; \kappa t^2\right) d\xi \tag{29}$$

We emphasize that the integral relations (28) and (29) in this article are only postulated as Euler transformation operators and their modifications. The proof that they turn out to be intertwining operators in the general version is difficult, if only by replacing the standard hyperbolic equation (5a) with its generalized analogue. One of the works, highlighting the path of development in this direction [29].

Since the preimage ${}_0F_1$ satisfies the operator with the second, and, accordingly, the image of the Euler transformation ${}_1F_2$ to the operator with the third derivative, in the framework of second-order differential equations, only two types of hypergeometric functions [30, 31]:

$${}_0F_0(t) = F(;\,;t) = e^t; \quad {}_1F_0(t) = F(a;\,;t) = (1-t)^{-a};$$

The Euler transformation for ${}_0F_0$ leads to an integral representation of the Kummer function [16, раздел 6.5], [2, 32, 33]

$$\Theta(c, d, x) = x^{d-1}\,{}_1F_1(c; d; x) = \frac{\Gamma(d)}{\Gamma(d-c)\Gamma(c)} \int\limits_0^x (x-t)^{d-c-1} t^{c-1}\,{}_0F_0(t) dt \tag{30}$$

provided that x is a real variable and $Re(d) > Re(c) > 0$. We note an important fact: the resulting transformation operator covers a smaller set of parameters than the series

$${}_1F_1(c; d; x) = \sum_{k=0}^{\infty} \frac{(c)_k}{(d)_k k!} x^k$$

where $(q)_k$ is a Pohgammer symbol, since the inequalities $Re(d) > Re(c) > 0$ impose significant restrictions on the domains of parameter changes.

We prove that the transformation operator (30) is a transmutation operator. The function ${}_0F_0(t) = e^t$, which is present under the integral sign, is a solution of a first order differential equation, but the conjugate form (3) allows you to artificially add another differentiation digit. Exactly if

$$f_0(t) = t^{c-1}\,{}_0F_0(t) = t^{c-1} e^t$$

$$a_0(t) = t; \quad b_0(t) = 1 - c - t; \quad c_0(t) = 0;$$

then the equality (3) takes the form

$$\frac{d}{dt}\left[a_0(t)\frac{df_0(t)}{dt}+b_0(t)f_0(t)\right]=0$$

For the transformed function $f_1(x) = x^{d-1}\,{}_1F_1(c,d,x)$, the identity (4) with the coefficients

$$a_1(x)=x; \quad b_1(x)=2-d-x; \quad c_1(x)=d-c-1;$$

It is easy to verify that with the coefficients indicated above, and $K(x,t) = (x-t)^{d-c-1}$, the hyperbolic equation (5a) holds.

Thus, the formula (30) is a two-parameter family of intertwining operators. According to the definition [34, Ch I, Def 2.1], it simultaneously belongs to the class of fractional integrals. The enumeration of the permissible values of the parameters [33, Ch 3] leads to many interesting results illustrating the significance of transmutation operators. For example, ratio

$$\Theta(1,2,x)=e^x-1; \quad \text{при} \quad x>0$$

with the help of the OP it turns out much easier to expand the Kummer function in a series. On the other hand, much less elementary results are possible. With real $x > 0$

$$\Theta(\frac{3}{4},\frac{3}{2},x)=\sqrt{2}\sqrt[4]{x}e^{\frac{x}{2}}\Gamma\left(\frac{5}{4}\right)I_{\frac{1}{4}}\left(\frac{x}{2}\right)$$

with a modified Bessel function, a fractional argument—and this is not the highest bar of complexity.

At one time, the identity (28) was used by Leonard Euler to determine the traditional hypergeometric function. Because of the literal following (28), the definition will take on a different look.

$$x^{d-1}\frac{\pi}{\sin(\pi c)\Gamma(c)\Gamma(1-c)}\,{}_2F_1(a,c,d,x)=$$
$$=\frac{\Gamma(d)}{\Gamma(c)\Gamma(d-c)}\int\limits_0^x(x-t)^{d-c-1}t^{c-1}\,{}_1F_0(a,t)dt \tag{31}$$

or

$$x^{d-1}\frac{\pi}{\sin(\pi c)\Gamma(c)\Gamma(1-c)}\,{}_2F_1(a,c,d,x)=\frac{\Gamma(d)}{\Gamma(c)\Gamma(d-c)}\int\limits_0^x\frac{(x-t)^{d-c-1}t^{c-1}}{(1-t)^a}dt \tag{32}$$

The fact that the presented formula has a three-century history does not save it from checking for the agreement of all the equations for the transition from the variety of transformations to the class of intertwining operators. We have

$$\begin{aligned}
&a_0(t) = t(1-t); \quad b_0(t) = 2 - c - (3 + a - c) - \tfrac{da_0(t)}{dt};\\
&a_1(x) = x(1-x); \quad b_1(x) = 2 - d - (3 + a + c - 2d)x;\\
&c_0(t) = c - a - 1 - \tfrac{db_0(t)}{dt}; \quad c_1(x) = -(1 + a - d)(1 + c - d).
\end{aligned}$$

If, as before, $K(x,t) = (x-t)^{d-c-1}$, then the hyperbolic equation (5a) turns into an identity. Note again that the transformation is valid only for real $0 < x < 1$ and $Re(d) > Re(c) > 0$. In addition, the parameter c should not be an integer. The number of representatives of this transmutation operator with different variants of the coefficients is almost innumerable [33, Ch. 2, Section 2.4]

Lemma 4 *The Euler transformation (32) is the intertwining operator for the hypergeometric functions when selecting the coefficients in (3) and (4) mentioned above.*

We will not check the relations (29), but instead show how knowing the values on the characteristic of a hyperbolic equation helps to find a rather complicated integral that is close in some parameters, for example, 2 paragraph 1 of [35].

$$\int_0^x (x^2 - t^2)^\beta \cos(\omega t)dt \tag{33}$$

Here, the coefficients for the input function $f_0(t) = \cos(\omega t)$ are obvious

$$a_0(t) = 1; \quad b_0(t) = 0; \quad c_0(t) = \omega^2;$$

We substitute them in (5a), taking into account simultaneously (5b). For $t \to x$, a relation arises on the characteristic of a hyperbolic operator for the kernel $K(x,t) = (x^2 - t^2)^\beta$

$$L\left[K(x,t)\right]_{t\to x} = 2^\beta \beta(2\beta + b_1(x))(x(x-t))^{\beta-1} + (x(x-t))^\beta O(x-t);$$

The remaining coefficient $c_1(x)$ is easily chosen. As a result

$$a_1(x) = 1; \quad b_1(x) = -\frac{2\beta}{x}; \quad c_1(x) = \omega^2;$$

The solution of an ordinary differential equation (4) is a linear combination

$$x^{\frac{2\beta+1}{2}}\left[C_1 J_{\frac{2\beta+1}{2}}(\omega x) + C_2 Y_{\frac{2\beta+1}{2}}(\omega x)\right]$$

with Bessel and Neumann functions as components. The singularity at zero eliminates the coefficient C_2. For different interpretations of the result, it is convenient to use the relationship between the Bessel function and the hypergeometric function. The integral (33) takes the form

$$\int_0^x (x^2-t^2)^\beta \cos(\omega t)dt = \frac{\sqrt{\pi}}{2}\Gamma(\beta+1)x^{\frac{2\beta+1}{2}} J_{\frac{2\beta+1}{2}}(\omega x) =$$

$$= \frac{\sqrt{\pi}}{2}\Gamma(\beta+1)_0F_1\left(;\beta+\frac{3}{2};-\left(\frac{\omega x}{2}\right)^2\right)$$

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