УДК 515.17

LOGARITHMIC DIFFERENTIAL FORMS AND ANALYSIS OF COMPLEX DYNAMICAL SYSTEMS

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Abstract. A wide class of complex dynamical systems can be described by evolutionary processes given by a vector field with polynomial, analytic or smooth coefficients in phase space. Such systems are investigated by perturbation analysis of the control and behavioral spaces together with associated bifurcation sets and discriminants. Our approach is based essentially on the theory of logarithmic differential forms, deformations theory and integrable connections associated with deformations. Such a connection can be represented as a holonomic system of differential equations of Fuchsian type whose coefficients have logarithmic poles along the bifurcation set or discriminant of a deformation. In addition we also describe another interesting application, a new method for computing the topological index of a complex vector field on hypersurfaces with arbitrary singularities.

Keywords: logarithmic differential forms, hypersurface singularity, torsion differentials, regular meromorphic differential forms, residue-map, index of vector fields.

Introduction

Let us consider a complex dynamical system given by an evolutionary process described by a vector field in phase space. A point of phase space defines the state of such system. The vector at this point indicates the velocity of change of the state. The points where the vector field vanishes are called equilibrium points, equilibrium positions or singularities of the vector field.

It was shown by [9] that the typical phase portraits in the neighbourhood of an equilibrium point of a generic system can be classified so that the corresponding list consists of the five simple types: two stable (focus, node) and three unstable (saddle, focus, node).

Of course, generic systems or, in other words, systems which are in general position correspond to real evolutionary processes and vice versa. Such a system always depends on parameters that are never known exactly. A small generic change of parameters transforms a non-generic system into a generic one. Thus, at the first sight, more complicated cases might not be considered since they turn into combinations of the above types after a small generic perturbation of the system.

However, if one is interested not in an individual system but in systems depending on parameters the situation is quite different and more complex. Thus, let us consider the space

A.G. Aleksandrov partially supported by the Russian Foundation of Basic Research (RFBR) and by the National Natural Science Foundation of China (NSFC) in the framework of the bilateral project "Complex Analysis and its applications"(project No. 08-01-92208), and by the National Counsel of Technological and Scientific Development CNPq of Brazil (project No. 454594/2007-0)

A.A. Castro, Jr. partially supported by the National Counsel of Technological and Scientific Development CNPq of Brazil (project No. 454594/2007-0)

of all systems divided into domains of generic systems. The dividing sets (hypersurfaces) correspond to degenerate systems. Under a small change of the parameters a degenerate system becomes non-degenerate. A one-parameter family of systems is presented by a curve which can intersect transversely the boundary separating different domains of nondegenerate systems.

Hence, although for each fixed value of the parameter the system can be always transformed by a small perturbation into a nondegenerate one, it is impossible to do this simultaneously for *all* values of the parameter. In fact, every curve closed to the one considered intersects the boundary of the separate hypersurface at a close enough value of the parameter.

Thus, if one studies not an individual system only but the whole family, the degenerate cases are not *removable*. If the family depends on a one parameter than the simplest degeneracies are unremovable one, those represented by boundaries of codimension one (that is, boundaries given by one equation) in the space of all systems. The more complicated degenerate systems, forming a set of codimension two in the space of all systems, may be gotten rid of by a small perturbation of the one-parameter family.

If one analyzes two-parameter families then one needs not to consider degenerate systems forming a set of codimension three and so on. Therefore at first it ought to analyze all generic systems, then degeneracies of codimension one, then – two and so on (see [4]). Herewith one must not restrict the study of degenerate systems to the picture at the moment of degeneracy, but must also include a description of the reorganizations that take place when the parameter passes through the degenerate value.

1 Control space and parameters

Let us consider a family of *smooth* functions

$$f : \mathbf{R}^n \times \mathbf{R}^r \to \mathbf{R},$$

describing a certain process happening in various copies of $\mathbf{R}^n$ governed by the function f and affected by the point in $\mathbf{R}^r$. The coordinate space $\mathbf{R}^n$ is usually called the space of *internal* variables while $\mathbf{R}^r$ the space of *external* variables over which each copy sits. Such terminology is suitable when the variables in $\mathbf{R}^r$ label in physical space as in mechanics, optics, biology or ecology, and so on.

For systems which one alters something and then to observe what happens the variables in $\mathbf{R}^r$ are called the *control* parameters while the variables in $\mathbf{R}^n$ are called the *behavioral* parameters. Accordingly the space $\mathbf{R}^r$ is referred to as the *control* space while $\mathbf{R}^n$ as the *behavior* space. In the strictly mathematical context it natural to call the space $\mathbf{R}^r$ the *deformation space* while its points (or their coordinates) the *parameters of a deformation*. The number r is correspondingly the *external* or *control* dimension, or the *dimension* of deformation.

Suppose that a submanifold $M \subset \mathbf{R}^n \times \mathbf{R}^r$ is given by the equation

$$\mathcal{D}f_u(x) = 0,$$

where $f_u(x) = f(x, u)$, $(x, u) \in \mathbf{R}^n \times \mathbf{R}^r$, and $\mathcal{D}$ is the usual differential of the image

$$f_u : \mathbf{R}^n \longrightarrow \mathbf{R}.$$

In other words, the manifold M is the set of all *critical points* of all the potentials f_u in the family f. Denote by ξ the restriction to M of the natural projection

$$\pi : \mathbf{R}^n \times \mathbf{R}^r \longrightarrow \mathbf{R}^r, \qquad \pi(x, u) = u.$$

The *critical set* is identified with the subset $\mathcal{C} \subset M$ consisting of *singular* points of the image ξ. In other words, $\mathcal{C}$ consists of points in which the map ξ is *singular*, that is, the rank of the derivative $\mathcal{D}\xi$ is less than r. The image of the critical set $\xi(\mathcal{C}) \subset \mathbf{R}^r$ is called the *bifurcation set* $\mathcal{B}$.

It is not difficult to see, by computing $\mathcal{D}\xi$, that $\mathcal{C}$ is the set of points $(x, u) \in M$, at which $f_u(x)$ has a *degenerate critical* point. It follows that $\mathcal{B}$ is the locus where the number and nature of critical points *change* (that is, it happens jump changes in the state of a control system); for by structural stability of Morse functions such changes can only occur by *passing* through a degenerate critical point. In most applications (for instance, in problems of stability, optimization, in studying caustics, wave fronts and so on) it is the *bifurcation* set that is the most important, for it lies in the control space, hence is "observable and all delay convention jumps occur in it.

Investigations show that a bifurcation set as variety possesses highly complicated topological, analytic, algebraic and geometric *structures*. Herewith it appears that characteristics of such a variety depend mainly on the structure of its subvariety of *singularities*, which, in turn, also can possess singularities and so on. This observation directly leads to the notion of a *stratification* variety, but in the most general context the study of bifurcation sets is reduced to the study of *stratified* varieties (see [11]).

Remark that in virtue of the well-known splitting lemma a *smooth* function f can be represented around a point, where it has corank k, in the form:

$$\widetilde{f}(x_1, \ldots, x_k) \pm x_{k+1}^2 \pm \ldots \pm x_n^2$$

(perhaps with parameters in $\mathbf{R}^r$ for $\widetilde{f}$). Herewith the variables $x_1, \ldots, x_k$ are called *essential* while $x_{k+1}, \ldots, x_n$ – *unessential*. Certainly, such presentation is very far from unique. It should be also noted that most singularities met by an r-dimensional family will even when not regular or Morse, have codimension less than r. However, it is possible to write an r-parameter family f, around a point, where it meets transversely a singularity of codimension ν, in a way in which only ν control parameters appear. When one has done so, one may call the coordinates on $\mathbf{R}^r$ that no longer appear, *disconnected* or *dummy* control parameters.

2 Deformations

In fact, general evolutionary processes can be described with the help of polynomial, analytical or smooth functions and systems of equations as well as in a wider context by systems of differential equations. In particular, using properties of associated bifurcation sets, the discriminants or, more generally, singular loci, basic properties of the corresponding systems are investigated. One of the most efficient tools of the investigation is a general notion of integrable connection associated naturally with any deformation of a system. Let us shortly discuss basic ideas of the theory. Consider the system of polynomial or analytic equations

$$\begin{cases} f_1(z_1, z_2, \ldots, z_m) &= 0 \\ \quad\vdots & \\ f_k(z_1, z_2, \ldots, z_m) &= 0 \end{cases} \tag{2.1}$$

given in a neighbourhood U of the origin in $\mathbf{C}^m$. For simplicity we shall assume that $k = m - 1$ and the set X_0 of the solutions of our system in the neighbourhood U is one-dimensional. We

shall say that a point, laying on the curve X_0, is *nonsingular* if the differential form $df_1 \wedge \ldots \wedge df_{m-1}$ does not vanish on it. Otherwise, this point (and the curve) refers to as *singular*, or shortly a *singularity*. Without loss of generality one can suppose that X_0 has the only singularity at the origin $\{0\} \in X_0 \subset \mathbf{C}^m$, that is, X_0 is the germ of a reduced *curve*.

We shall assume now that the equations of the system (2.1) can be *perturbed:*

$$\begin{cases} f_1(z_1, z_2, \ldots, z_m) &= t_1 \\ \vdots & \\ f_{m-1}(z_1, z_2, \ldots, z_m) &= t_{m-1} \end{cases} \tag{2.2}$$

in such a manner that at each sufficiently small value of parameters $t = (t_1, \ldots, t_{m-1}) \in \mathbf{C}^{m-1}$ in the chosen neighbourhood U the set X_t of the solutions of the system (2.2) is also *one-dimensional*. In other words, we shall consider the *principal (flat) deformation* of the curve singularity X_0 given by the holomorphic map:

$$f\colon \mathcal{X} \longrightarrow \mathbf{C}^{m-1}. \tag{2.3}$$

Let X be the intersection of a ball of a small radius $\varepsilon > 0$ centered at the distinguished point $\{0\} \in X_0$ with $f^{-1}(T)$, where $T \subset \mathbf{C}^{m-1}$ is a punctured ball of a radius $0 < \delta \ll \varepsilon$ centered at the origin $0 = f(0)$. Consider the natural restriction $f\colon X \longrightarrow T$ of the mapping (2.3). Then for some values of parameters $t \in T$ the fibres X_t are non-singular curve germs, for other ones the corresponding fibres may have singular points called the *critical points* of map f.

3 Period integrals

Denote by $C \subset X$ the set of critical points of f and by D its image $f(C) \subset T$. Thus, parameters corresponding to the fibres with singularities form the set D which refers to as the discriminant set, or the *discriminant* of the principal deformation X_0. In many important cases the discriminant is the zero-set of the only equation $h(t_1, \ldots, t_k) = 0$, that is, D is a *hypersurface*. Set

$$T' = T - D, \quad X' = X - C.$$

The restriction $f\colon X' \to T'$ is a local trivial differentiable fibre bundle called the *Milnor fibration* of f, that is, fibres $X_t = f^{-1}(t) \cap X$ (of *real* dimension two) form a smooth fibre bundle over T'. Fix a point $t_0 \in T'$. Then for each *smooth closed path* $\gamma_0 \subset X_{t_0}$, corresponding to the 1-cycle in $H^1(X_{t_0}, \mathbf{C})$, it is possible to construct a family of 1-cycles $\gamma(t) \subset X_t$, $t \in T'$, such that $\gamma(t_0) = \gamma_0$.

If one takes a holomorphic differential form $\omega = g(z)dz_1 \wedge \ldots \wedge dz_m$ of the maximal degree in a neighbourhood of the origin in $\mathbf{C}^m$, then using the identity $df_1 \wedge \ldots \wedge df_{m-1} \wedge \psi = \omega$ one can find a differential form ψ, which is the result of the division of ω by $df_1 \wedge \ldots \wedge df_{m-1}$. The form ψ is not determined uniquely, but up to the summands containing differentials of the functions $f_1, \ldots, f_{m-1}$. It is easy to prove, that for all parameters t, rather close to zero, the integral

$$I(t) = \int_{\gamma(t)} \psi = \int_{\gamma(t)} \frac{\omega}{df_1 \wedge \ldots \wedge df_{m-1}} \tag{3.1}$$

is determined correctly. Moreover, the integral $I(t)$ is an *analytic* function in the variable t. Integrals of such type are called the *period integrals*.

Replacing the differential form ω with another, the integral (3.1), generally speaking, will also change. However, it is possible to prove that the set of all such integrals contains a finite number of the elements $I_1(t), \ldots, I_\mu(t)$ so that any integral of the type (3.1) may be expressed by means of these generators as a linear combination with *holomorphic* coefficients. In the present context μ is the *Milnor number* which is a topological invariant of the singularity X_0.

The same observation holds, if one fixes the form ω and takes various families $\gamma(t)$. For definiteness, we shall fix a family of vanishing cycles $\gamma(t)$ and consider μ independent *period integrals* of the following type

$$I_j(t) = \int_{\gamma(t)} \frac{\omega_j}{df_1 \wedge \ldots \wedge df_{m-1}},$$

where $1 \leq j \leq \mu$. The period integrals $I_j(t)$ can be differentiated with respect to the parameter t. Between integrals and their derivatives there arose linear relations *(syzygies)* with polynomial coefficients in t. These relations generate a system of differential equations for the integrals $I_j(t)$ expressed through a finite number of independent integrals.

4 Connection

In such a way a system of differential equations in the variable t is associated with the germ X_0; this system is defined correctly outside of the discriminant and refers to as *Gauss-Manin connection,* or *Gauss-Manin system,* associated with the principal deformation of X_0. The main problem is to describe a system of differential equations defined on the *whole* space of parameters, which is equivalent to the initial one outside of the discriminant (or, in other words, to *extend* the initial system to the discriminant set). It is possible to show that the solution of this problem depends mainly on properties of the discriminants as well as on properties of fibres of the deformation.

It turns out that the connection in question can be represented in a quite elegant form. In order to explain this idea we need the following notion. Let ω be a meromorphic differential form on S having poles along a reduced divisor $D \subset S$. Then ω is called the *logarithmic* along D differential form if and only if ω and its total differential $d\omega$ have poles along D at worst of the first order. That is, $h\omega$ as well as $hd\omega$ are *holomorphic* differential forms on S where h is a local equation of the hypersurface $D \subset S$.

The $\mathcal{O}_S$-module of logarithmic differential q-forms is usually denoted by $\Omega^q_S(\log D)$. Logarithmic differential forms have many remarkable analytic and algebraic properties (for example, see [1]).

Following [10] denote by $\mathrm{Der}_S(\log D)$ the $\mathcal{O}_S$-module of *logarithmic vector fields* along D on S. This module consists of germs of holomorphic vector fields η on S for which $\eta(h)$ belongs to the principal ideal $(h)\mathcal{O}_S$. In particular, the vector field η is tangent to D at its smooth points. The inner multiplication of vector fields and differential forms induces a natural pairing of $\mathcal{O}_S$-modules

$$\mathrm{Der}_S(\log D) \times \Omega^q_S(\log D) \longrightarrow \Omega^{q-1}_S(\log D).$$

For $q = 1$ this $\mathcal{O}_S$-bilinear mapping is a *non-degenerate* pairing so that $\mathrm{Der}_S(\log D)$ and $\Omega^1_S(\log D)$ are $\mathcal{O}_S$-dual.

Let $\mathcal{H}$ be a free $\mathcal{O}_S$-module. Then a connection ∇ on $\mathcal{H}$ with *logarithmic* poles along $D \subset S$ is a $\mathbf{C}$-linear morphism

$$\nabla_{X/S}\colon\ \mathcal{H} \longrightarrow \mathcal{H} \otimes_{\mathcal{O}_S} \Omega^1_S(\log D) \tag{4.1}$$

satisfying the following conditions:

$$1)\ \nabla(\omega+\omega')=\nabla(\omega)+\nabla(\omega'),$$
$$2)\ \nabla(f\omega)=\omega\otimes df+f\nabla(\omega),\quad f\in\mathcal{O}_S.$$

Consider the case where $\Omega^1_S(\log D)$ is a *free* $\mathcal{O}_S$-module of rank m. Obviously, in such a case $\Omega^p_S(\log D)=\bigwedge^p\Omega^1_S(\log D)$, $p\geq 1$. It is often said that the divisor D is *free* or, equivalently, D is a *Saito free* divisor. The following characteristic property of such divisors was discovered by [10].

Proposition 4.1 *Suppose that there exist m logarithmic vector fields $\mathcal{V}^1,\dots,\mathcal{V}^m\in \mathrm{Der}_S(\log D)$ such that for the $(m\times m)$-matrix $\mathcal{M}$ whose entries are the coefficients of $\mathcal{V}^i$, $i=1,\dots,m$, one has $\det(\mathcal{M})=ch$, where c is a unit. Then $\mathcal{V}^1,\dots,\mathcal{V}^m$ form a basis of the free $\mathcal{O}_S$-module $\mathrm{Der}_S(\log D)$. In particular, $\Omega^1_S(\log D)$ is a free $\mathcal{O}_S$-module with the dual basis $\omega_1,\dots,\omega_m$.*

For example, $\Omega^1_S(\log D)$ is free when D is the discriminant of the *minimal versal* deformation of the system defined by a function with isolated singularity.

Now let D be a Saito free divisor. Then we can describe the *logarithmic connection* (4.1) on $\Omega^1_S(\log D)$ itself. In other words, let us consider the case when $\mathcal{H}=\Omega^1_S(\log D)$:

$$\nabla\colon\ \Omega^1_S(\log D)\to\Omega^1_S(\log D)\otimes_{\mathcal{O}_S}\Omega^1_S(\log D).$$

Let $\omega_1,\dots,\omega_m$ be free generators of the module $\Omega^1_S(\log D)$. Then the connection ∇ can be expressed in terms of Christoffel symbols in the following way:

$$\nabla\omega_i=\sum_{j=1}^m\omega_j\otimes\omega_i^j,\quad \omega_i^j=\sum_{k=1}^m\Gamma_i^{jk}\omega_k.$$

The connection ∇ is called *torsion free* if

$$d\,\omega_i=\sum_{j=1}^m\omega_i^j\wedge\omega_j=\sum_{k,j=1}^m\Gamma_i^{jk}\omega_k\wedge\omega_j,$$

and ∇ is called *integrable* if

$$d\omega_i^j=\textstyle\sum_{k=1}^m\omega_i^k\wedge\omega_k^j,\quad \text{that is,}\quad d\nabla=\nabla\wedge\nabla,$$

where $\nabla=\|\,\omega_i^j\|$ is the *coefficient matrix* of the connection ∇. In particular, it means that the composition

$$\mathcal{H}\xrightarrow{\nabla}\mathcal{H}\otimes\Omega^1_S(\log D)\xrightarrow{\nabla}\mathcal{H}\otimes\wedge^2\Omega^1_S(\log D)$$

is zero.

5 Holonomic systems

It is possible to associate with any integrable and torsion free connection ∇ on the module $\Omega^1_S(\log D)$ a holonomic system of Fuchsian type in the following way.

It is known (see [1]) that the multiplication by h induces the surjection

$$\Omega^1_S(\log D) \xrightarrow{\cdot h} \operatorname{Tors}\Omega^1_D \longrightarrow 0, \tag{5.1}$$

whose kernel coincides with an $\mathcal{O}_S$-module

$$\mathcal{O}_S\frac{dh}{h} + \Omega^1_S.$$

Here Ω^1_S is the module of holomorphic differential 1-forms on S generated by the differentials $dz_1, \ldots, dz_m$ over $\mathcal{O}_S$,

$$\Omega^1_D = \Omega^1_S/(h\,\Omega^1_S + \mathcal{O}_S dh)$$

is the module of regular differential 1-forms on the divisor D, and $\operatorname{Tors}\Omega^1_D$ is the *torsion* submodule of Ω^1_D. The support of $\operatorname{Tors}\Omega^1_D$ is contained in the singular locus $\operatorname{Sing} D$ of the hypersurface D. The torsion $\mathcal{O}_D$-module $\operatorname{Tors}\Omega^1_D$ has a system of generators containing at least $m-1$ elements.

By definition, the generalized Fuchsian system is a holonomic system of linear differential equations on S with *meromorphic* coefficients containing in $\Omega^1_S(\log D)$:

$$d\,\mathbf{I} = \Omega\,\mathbf{I}, \tag{5.2}$$

where $\mathbf{I} = {}^{\mathrm{tr}}(I_1, \ldots, I_k)$ is a vector-column of unknown functions and the matrix differential form Ω is defined as follows:

$$\Omega = A_0\frac{dh}{h} + \sum_{i=1}^{\ell} A_i\frac{\vartheta_i}{h}.$$

Here the differential 1-forms $\vartheta_i \in \Omega^1_S$, $i = 1, \ldots, \ell$, correspond via (5.1) to non-zero elements of the torsion submodule $\operatorname{Tors}\Omega^1_D$, and $A_i \in \operatorname{End}(\mathbf{C}^k) \otimes \mathcal{O}_S$, $i = 0, 1, \ldots, \ell$, are coefficient matrices with holomorphic entries such that the integrability condition $d\,\Omega = \Omega \wedge \Omega$ holds.

It is not difficult to show that one can associate to any integrable and torsion free connection ∇ on the module $\Omega^1_S(\log D)$ the generalized Fuchsian system of type (5.2) (see [3]). Moreover, using the Christoffel symbols of such connection, it is possible to express the integrability condition in terms of commuting relations of the coefficient matrices A_i, $i = 1, \ldots, \ell$.

Under some additional assumptions on entries of the coefficient matrices A_i it is possible to investigate the system of type (5.2) and to describe its explicit solutions. In fact, such solutions are quite useful in describing the control of evolutionary processes, perturbations of multidimensional systems, and many applications in dynamical systems, bifurcation theory, etc. (for example, see [8], [4]).

6 Topological index

The index of a vector field is one of the very first concepts in topology and geometry of smooth manifolds, and its properties underlie important results of the theory, including the Poincaré-Hopf theorem, which states that the total index of a vector field on a closed smooth orientable manifold is independent of the field and coincides with the Euler-Poincaré characteristic of the manifold. When studying singular varieties such as bifurcation sets, discriminants, etc., it is natural to ask whether there exists a similar invariant in a more general context. One possible

generalization of this type, which originally arose in topology of foliations, turned out to be well suited for use in the theory of singular varieties. In this section, we shortly describe a new method for the calculation of the index of vector fields on a hypersurface on the basis of the theory of logarithmic differential forms and vector fields. The main idea of our approach is to describe the index in terms of meromorphic differential forms defined on the ambient variety and having logarithmic poles along the hypersurface (see [2]). We shall see that the systematic use of the theory of logarithmic forms permits one not only to simplify the calculations dramatically but also to clarify the meaning of the basic constructions underlying many papers on the subject (for example, see [6]).

6.1 Regular differential forms

Let S be a complex manifold of dimension $m = n + 1$, $n \geq 1$, and let Ω^q_D be the $\mathcal{O}_D$-module of germs of regular (Kähler) differentials of order q on D, so that

$$\Omega^q_{D,x} = \Omega^q_{S,x} / \big(h \cdot \Omega^q_{S,x} + dh \wedge \Omega^{q-1}_{S,x}\big), \quad q \geq 0,$$

where $x \in S$. By analogy with smooth case, elements of $\Omega^q_{D,x}$ are usually called germs of regular holomorphic forms on D. Now let $\mathrm{Der}(D) = \mathrm{Hom}_{\mathcal{O}_D}(\Omega^1_D, \mathcal{O}_D)$ be the sheaf of germs of regular vector fields on D and let us consider an element $V \in \mathrm{Der}(D)$. By $\mathcal{V} \in \mathrm{Der}(S)$ we denote a holomorphic vector field on S such that $\mathcal{V}|_D = V$. Then the interior multiplication (contraction) $\iota_{\mathcal{V}} \colon \Omega^q_S \longrightarrow \Omega^{q-1}_S$ of vector fields and differential forms defines the structure of a complex on $\Omega^\bullet_S$, since $\iota^2_{\mathcal{V}} = 0$. The contraction $\iota_{\mathcal{V}}$ induces a homomorphism $\iota_V \colon \Omega^q_D \longrightarrow \Omega^{q-1}_D$ of $\mathcal{O}_D$-modules and also the structure of a complex on $\Omega^\bullet_D$. The corresponding ι_V-homology sheaves and groups are denoted by $H_\star(\Omega^\bullet_D, \iota_V)$ and $H_\star(\Omega^\bullet_{D,x}, \iota_V)$, respectively.

6.2 Homological index

If the vector field $\mathcal{V}$ has an *isolated* singularity at a point $x \in D$, then ι_V-homology groups of the complex $\Omega^\bullet_{D,x}$ are finite dimensional vector spaces, so that the Euler characteristic

$$\chi\big(\Omega^\bullet_{D,x}, \iota_V\big) = \sum_{i=0}^{n+1} (-1)^i \dim H_i\big(\Omega^\bullet_{D,x}, \iota_V\big),$$

of the complex of regular differentials is well-defined. It is called the *homological* index of the vector field V at the point $x \in D$ and denoted by $\mathrm{Ind}_{\mathrm{hom},D,x}(V)$ (see [7]). At *nonsingular* points of D the homological index coincides with the *topological* index, or, equivalently, with the Poincaré-Hopf local index.

6.3 Logarithmic index

Let us consider a vector field $\mathcal{V} \in \mathrm{Der}_S(\log D)$. The interior multiplication $\iota_{\mathcal{V}}$ defines the structure of a complex on $\Omega^\bullet_S(\log D)$.

Lemma 6.1 *If all singularities of the vector field $\mathcal{V}$ are isolated, then $\iota_{\mathcal{V}}$-homology groups of the complex $\Omega^\bullet_S(\log D)$ are finite dimensional vector spaces.*

Proof. Assume that $S \cong \mathbf{C}^m$, $m = n+1$, and the point $x_0 = 0 \in D \subset S$ is an isolated singularity of the field $\mathcal{V}$, so that $\mathcal{V}(x_0) = 0$. Then $\mathcal{V}(x) \neq 0$ at each point x in a sufficiently small punctured neighbourhood of x_0. In a suitable neighbourhood of x there exists a coordinate system $(t, z_1', \ldots, z_n')$ such that $\mathcal{V} = \partial/\partial t$. Since $\mathcal{V}(h) \subseteq (h)\mathcal{O}_{S,0}$, it follows that $D \cong T \times D_0$, where T is a small disc in the variable t and D_0 is a hypersurface in $\mathbf{C}^n$. It is easy to show that

$$\Omega^q_{\mathbf{C}^{n+1},0}(\log D) \cong \left(\Omega^q_{\mathbf{C}^n,0}(\log D_0) \oplus \Omega^{q-1}_{\mathbf{C}^n,0}(\log D_0) \wedge dt\right) \otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{C},0}.$$

Indeed, for germs of holomorphic forms one has the isomorphism $\Omega^q_{D,0} \cong \left(\Omega^q_{D_0,0} \oplus \Omega^{q-1}_{D_0,0} \wedge dt\right) \otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{C},0}$ which can readily be obtained by considering of the canonical projections of the analytic set $T \times D_0$ onto the first and second factors and the definition of $\Omega^q_{D,0}$. The desired isomorphism for germs of logarithmic forms can be obtained by a similar argument with the use of the exact sequence

$0 \to \Omega^q_{\mathbf{C}^{n+1},0} + \frac{dh}{h} \wedge \Omega^{q-1}_{\mathbf{C}^{n+1},0} \to \Omega^q_{\mathbf{C}^{n+1},0}(\log D) \xrightarrow{\cdot h} \Omega^q_{D,0} \to \Omega^q_{D,0}/\operatorname{Tors}\Omega^q_{D,0} \to 0$, which follows from the exact sequence expressing the torsion subsheaves $\operatorname{Tors}\Omega^q_D$ in terms of logarithmic differential forms (see [2]).

Further, in the q-th piece of the complex $(\Omega^\bullet_{S,0}(\log D), \iota_{\mathcal{V}})$ one has

$$\operatorname{Ker}(\iota_{\partial/\partial t}) \cong \operatorname{Im}(\iota_{\partial/\partial t}) \cong \left(\Omega^q_{\mathbf{C}^n,0}(\log D_0) \oplus (0)\right) \otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{C},0}.$$

That is, the corresponding homology groups vanish for all q. The same conclusion readily follows for the point $x_0 \in S \setminus D$. Consequently the $\iota_{\mathcal{V}}$-homology groups of the complex $\Omega^\bullet_S(\log D)$ may be non-trivial only at singular points of the field. Since the sheaves of logarithmic forms as well as their cohomology are coherent, we arrive at the statement of the Lemma.

Thus if the vector field $\mathcal{V}$ has *isolated singularities,* then the Euler characteristic

$$\chi\left(\Omega^\bullet_{S,x}(\log D), \iota_{\mathcal{V}}\right) = \sum_{i=0}^{n+1} (-1)^i \dim H_i\left(\Omega^\bullet_{S,x}(\log D), \iota_{\mathcal{V}}\right)$$

of the complex of logarithmic differential forms is well defined for any point $x \in S$. It is called the *logarithmic* index of the field $\mathcal{V}$ at the point x and denoted by $\operatorname{Ind}_{\log D,x}(\mathcal{V})$. It follows from the preceding that $\operatorname{Ind}_{\log D,x}(\mathcal{V}) = 0$ whenever $\mathcal{V}(x) \neq 0$.

6.4 The index of vector fields on hypersurfaces

To study the $\iota_{\mathcal{V}}$-homology of the complex $\Omega^\bullet_D$, one can use an approach based on a representation of regular holomorphic differential forms on the hypersurface D via meromorphic forms with logarithmic poles along D (see [2]). Recall [loc. cite] that for all $q = 0, 1, \ldots, n+1$, there exist exact sequences

$$0 \to \Omega^{q-1}_{S,x}/h \cdot \Omega^{q-1}_{S,x}(\log D) \xrightarrow{\wedge dh} \Omega^q_{S,x}/h \cdot \Omega^q_{S,x} \to \Omega^q_{D,x} \to 0$$

of $\mathcal{O}_{S,x}$-modules, where $\wedge dh$ is the homomorphism of exterior multiplication. Hence one obtains the exact sequence

$$0 \to \left(\Omega^\bullet_{S,x}/h\Omega^\bullet_{S,x}(\log D), \iota_{\mathcal{V}}\right)[-1] \xrightarrow{\wedge dh} \left(\Omega^\bullet_{S,x}/h\Omega^\bullet_{S,x}, \iota_{\mathcal{V}}\right) \to \left(\Omega^\bullet_{D,x}, \iota_V\right) \to 0 \qquad (6.1)$$

of complexes. Indeed, the fact that the multiplication by $\wedge dh$ induces a morphism of complexes follows from the identity

$$\iota_V(\omega) \wedge dh = \iota_V(\omega \wedge dh) + (-1)^{q-1}\omega \wedge V(h),$$

since the second summand from the right-hand side vanishes in the quotient complex $\Omega^\bullet_S/h\Omega^\bullet_S$ in view of the condition $V(h) \in (h)\mathcal{O}_{S,x}$. Now note that from the exact sequence

$$0 \longrightarrow (\Omega^\bullet_{S,x}, \iota_V) \xrightarrow{\cdot h} (\Omega^\bullet_{S,x}, \iota_V) \longrightarrow (\Omega^\bullet_{S,x}/h\Omega^\bullet_{S,x}, \iota_V) \longrightarrow 0$$

of complexes it follows that $\chi\big(\Omega^\bullet_{S,x}/h\Omega^\bullet_{S,x}, \iota_V\big) = 0$. Thus from the exact sequence (6.1) one obtains

$$\mathrm{Ind}_{\mathrm{hom},D,x}(V) = -\chi((\Omega^\bullet_{S,x}/h\Omega^\bullet_{S,x}(\log D), \iota_V)[-1]) = \chi(\Omega^\bullet_{S,x}/h\Omega^\bullet_{S,x}(\log D), \iota_V).$$

Proposition 6.2 *Suppose that $x \in D$ is an isolated singularity of a vector field $V \in \mathrm{Der}(\log D)$, the germs $v_i \in \mathcal{O}_{S,x}$ are determined by the expansion $V = \sum_i v_i\partial/\partial z_i$, and $J_xV = (v_1, \ldots, v_m)\mathcal{O}_{S,x}$. Then*

$$\mathrm{Ind}_{\mathrm{hom},D,x}(V) = \dim \mathcal{O}_{S,x}/J_xV - \mathrm{Ind}_{\log D,x}(V).$$

Let us consider the case when D is a Saito free divisor. Then the complex $\big(\Omega^\bullet_{S,x}(\log D), \iota_V\big)$ is naturally isomorphic to the *Koszul* complex $\mathbf{K}_\bullet\big((\alpha_1, \ldots, \alpha_m); \mathcal{O}_{S,x}\big)$ on the generators $e_i = \omega_i$, $i = 1, \ldots, m$, where the germs $\alpha_i \in \mathcal{O}_{S,x}$ are determined as coefficients of the expansion $V = \sum_i \alpha_i V^i$ of V in the basis of logarithmic vector fields. In this case one readily obtains the following identity:

$$\mathrm{Ind}_{\log D,x}(V) = \chi\big(\mathbf{K}_\bullet\big((\alpha_1, \ldots, \alpha_m); \mathcal{O}_{S,x}\big)\big).$$

Corollary 6.3 *Let $J_{\log D,x}V = (\alpha_1, \ldots, \alpha_m)\mathcal{O}_{S,x}$. Suppose that the coefficients $(\alpha_1, \ldots, \alpha_m)$ form a regular $\mathcal{O}_{S,x}$-sequence. Then*

$$\mathrm{Ind}_{\mathrm{hom},D,x}(V) = \dim \mathcal{O}_{S,x}/J_xV - \dim \mathcal{O}_{S,x}/J_{\log D,x}V.$$

6.5 Normal hypersurfaces

Let $Z = \mathrm{Sing}\, D$ be the singular locus of a reduced divisor D, and let $c = \mathrm{codim}\,(Z, D)$ be the codimension of Z in D. It is well-known (see [1]) that $c = 1$ for Saito free divisors, that is, in a sense, the singularities of D form the maximal possible subset of the divisor. For $c \geq 2$, Serre's criterion implies that the hypersurface D is a *normal* variety. For further analysis of this case we use the following reformulation due to [10] of the notion of logarithmic forms.

Lemma 6.4 *The germ ω of a meromorphic differential q-form at a point $x \in S$ with poles along D is the germ of a logarithmic form (that is, $\omega \in \Omega^q_{S,x}(\log D)$) if and only if when there exists a holomorphic function germ $g \in \mathcal{O}_{S,x}$, a holomorphic $(q-1)$-form germ $\xi \in \Omega^{q-1}_{S,x}$ and a holomorphic q-form germ $\eta \in \Omega^q_{S,x}$, such that* $\begin{array}{l}(i)\ \dim_{\mathbf{C}} D \cap \{z \in M\colon g(z) = 0\} \leq n-1,\\ (ii)\ g\omega = \frac{dh}{h} \wedge \xi + \eta.\end{array}$

Let $\varrho\colon \Omega^q_{S,x}(\log D) \longrightarrow \Omega^q_{S,x}$ be the homomorphism of multiplication by h, and let $\omega \in \Omega^q_{S,x}(\log D)$. Then there exists an element $g \in \mathcal{O}_{S,x}$ in Lemma 6.4 such that $gh\omega \in h\Omega^q_{S,x} + dh \wedge \Omega^{q-1}_{S,x}$, that is, $gh\omega = 0$ in $\Omega^q_{D,x}$. Since the germ g defines a zero non-divisor in $\mathcal{O}_{D,x}$, in particular, this means that $h\omega \in \operatorname{Tors}\Omega^q_{D,x}$, where the torsion submodule of the sheaf of regular q-differentials is denoted by $\operatorname{Tors}\Omega^q_{D,x}$. Thus, $\operatorname{Im}(\varrho) \subseteq \operatorname{Tors}\Omega^q_{D,x}$ (actually one has the equality). If $\operatorname{Tors}\Omega^q_{D,x} = 0$, then the germ g in (ii) can only be an invertible element; consequently,

$$\Omega^q_{S,x}(\log D) \cong \Omega^q_{S,x} + \frac{dh}{h} \wedge \Omega^{q-1}_{S,x}. \tag{6.2}$$

In fact, this isomorphism can be obtained without the preceding argument if one directly makes use of the exact sequence for the torsion submodules $\operatorname{Tors}\Omega^q_{D,x}$ (for example, see [1]).

Theorem 6.5 *Let D be a normal hypersurface. Then*

$$\operatorname{Ind}_{\mathrm{hom},D,x}(V) = \dim \mathcal{O}_{S,x}/(h, J_x\mathcal{V}) + \textstyle\sum_{i=c'}^{n+1}(-1)^i \dim H_i(\Omega^\bullet_{D,x}, \iota_V),$$

where $c' = 2[\frac{c+1}{2}] + 1$, the square brackets denote the integer part of rational numbers, and the sum is zero by convention if the lower limit is greater than the upper limit.

Proof. It is well-known that $\operatorname{Tors}\Omega^q_D = 0$ if $0 < q < c$. Hence, together with the isomorphism (6.2), this means that $\Omega^q_{S,x}/h\Omega^q_{S,x}(\log D) \cong \Omega^q_{D,x}$ for all such q. Therefore, it follows from the exact sequence (6.1) that

$$H_i(\Omega^q_D, \iota_V) \cong H_{i-1}(\Omega^q_D[-1], \iota_V) = H_{i-2}(\Omega^q_D, \iota_V)$$

for all $i = 3, \ldots, c+1$. In particular, in this range the dimensions of the ι_V-homology groups of the complex $\Omega^\bullet_{D,x}$ in the two series H_{2i} and H_{2i-1} coincide. Further, one can readily see that the dimensions of groups H_1 and H_2 also coincide (see [2]), whence the desirable formula follows. The integer part in the lower limit of the sum is needed in order to distinguish between the cases of even and odd codimension.

Corollary 6.6 *Suppose that a point $x \in D$ is an isolated singularity of the hypersurface D as well as of a vector field $\mathcal{V} \in \operatorname{Der}(\log D)$, $\mathcal{V}(h) = \varphi h$ and $\varphi \in \mathcal{O}_{S,x}$. Then*

$$\operatorname{Ind}_{\mathrm{hom},D,x}(V) = \dim \mathcal{O}_{S,x}/(h, J_x\mathcal{V}) + \varepsilon \dim \operatorname{Ann}_{\mathcal{B}_x}(h)/(\varphi)\mathcal{B}_x,$$

where $\varepsilon = -1$ if n is even and $\varepsilon = 0$ otherwise, and $\mathcal{B}_x$ is the local ring $\mathcal{O}_{S,x}/J_x\mathcal{V}$.

7 Conclusion

In many applications (say, in the theory of dynamical systems, bifurcation theory, in economic, biology, chemistry, etc.) a stable equilibrium state describes the established conditions in the corresponding real system (see [8], [5]). When it merges with an unstable equilibrium state the system must jump to a completely different state: as the parameter is changed the equilibrium condition in the neighbourhood considered suddenly disappears. The described results allow one to investigate in detail jumps of this kind with the use of invariants of bifurcation sets and discriminants associated with deformations of a complex system.

Bibliography

1. A.G. Aleksandrov. Nonisolated hypersurface singularities. In V.I. Arnold, editor, Theory of singularities and its applications, Advances in Soviet Mathematics, volume 1, pages 211–246. Amer. Math. Soc., Providence, RI, 1990.
2. A.G. Aleksandrov. Index of vector fields and logarithmic differential forms. Funct. Anal. and Appl., 39(4):1–13, 2005.
3. A.G. Aleksandrov. Generalized Fuchsian systems of differential equations. Journal of Mathematical Sciences (N.Y.), 39(6): 689–699, 2006.
4. V.I. Arnol'd. Catastrophe theory. Znanie, Moscow, 1981.
5. D.K. Arrowsmith, C.M. Place. Ordinary differential equations. A qualitative approach with applications. Chapman and Hall, London New York, 1982.
6. N.A. Bobylëv. Topological index of extremals of multidimensional variational problems. Funct. Anal. and Appl., 20(2):8–13, 1986.
7. X. Gómez-Mont. An algebraic formula for the index of a vector field on a hypersurface with an isolated singularity. Journal of Algebraic Geometry, 7(4):731–752.
8. J.E. Marsden, M. McCracken. The Hopf bifurcation and its applications. Springer-Verlag, New York, 1976.
9. H. Poincaré. Sur les propriétés des fonctions définies par les équations aux différences partielles. Thèse inaugurale, Paris, 1879.
10. K. Saito. On the uniformization of complements of discriminant loci. In: Hyperfunctions and Linear partial differential equations, RIMS Kōkyūroku, volume 287, pages 117–137. RIMS Kyoto University, Kyoto, 1977.
11. H. Whitney. On singularities of mappings of Euclidean spaces I. Mappings of the plane into the plane. Annals of Mathematics (2), 62:374–410, 1956.

ЛОГАРИФМИЧЕСКИЕ ДИФФЕРЕНЦИАЛЬНЫЕ ФОРМЫ И КОМПЛЕКСНЫЕ ДИНАМИЧЕСКИЕ СИСТЕМЫ

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Аннотация. Широкий класс сложных динамических систем может быть описан как эволюционный процесс, заданный векторным полем с полиномиальными, аналитическими или гладкими коэффициентами в фазовом пространстве. Такие системы исследуются методом возмущений и анализом пространств управления и поведения вместе с соответствующим бифуркационным множеством и дискриминантом. Описывается подход к изучению таких систем, основанный на методах теории логарифмических дифференциальных форм, теории деформаций и интегрируемых связностей, ассоциированных с деформациями. Такая связность может быть представлена в виде голономной системы дифференциальных уравнений фуксового типа, коэффициенты которой обладают логарифмическими полюсами вдоль бифуркационного множества или дискриминанта деформации. Кратко обсуждается и другое интересное приложение – новый метод вычисления топологического индекса комплексного векторного поля на гиперповерхности с произвольными особенностями.

Ключевые слова: логарифмические дифференциальные формы, гиперповерхность с особенностями, кручение дифференциалов, регулярные мероморфные дифференциальные формы, форма-вычет, индекс векторного поля.