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ON SOME EXPANSIONS OF THE NUMBER $\zeta(3)$ IN CONTINUED FRACTIONS

L.A. Gutnik

Moscow State Institute of Electronics and Mathematics,
Bol'shoj Trekhsvjatitel'skii per., 1-3/12, Moscow, 109028, Russian Federation, e-mail:
gutnik@gutnik.mccme.ru

Abstract. In this note new expansions of $\zeta(3)$ in continued fractions are obtained.
Keywords: zeta-function, $\zeta(3)$, expansion in continued fractions.

Foreword

This article is a brief account of my talk given at Moscow session of China-Russia Symposium "Complex Analysis and its applications"on October 24, 2009.

Preliminaries

Given two sequences of variables

$$\{a_\nu\}_{\nu=1}^{+\infty} \text{ and } \{b_\nu\}_{\nu=0}^{+\infty} \tag{1.1}$$

we can produce the following sequences of fractions:

$$R_0(b_0) = b_0,\ R_1(b_0, a_1, b_1) = b_0 + \frac{a_1}{R_0(b_1)}, \dots \tag{1.2}$$

$$R_\nu(b_0, a_1,\ b_1,\ \dots,\ a_\nu,\ b_\nu) = b_0 + \frac{a_1}{R_{\nu-1}(b_1, a_2,\ b_2,\ \dots,\ a_\nu,\ b_\nu)} \tag{1.3}$$

for a positive integer ν. They are called the fraction R_ν by the finite continued fraction generated by sequences (1.1). Below we use the following standard notation:

$$R_\nu = b_0 + \frac{a_1|}{|b_1} + \dots + \frac{a_\nu|}{|b_\nu} \tag{1.4}$$

If all elements of sequences (1.1) are complex numbers (but not variables), all fractions (1.4) are well defined for these complex numbers and there exists the limit

$$\lim_{\nu\to\infty} R_\nu = \alpha,$$

then it is said that α has expansion in continued fraction

$$\alpha = b_0 + \frac{a_1|}{|b_1} + \dots + \frac{a_\nu|}{|b_\nu} + \dots \tag{1.5}$$

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They are called the finite continued fractions (1.4) as convergents of the continued fraction (1.5). Let us consider further the following difference equation

$$x_{\nu+1} - b_{\nu+1}x_\nu - a_{\nu+1}x_{\nu-1} = 0, \tag{1.6}$$

with nonnegative ν. Let $\{P_\nu\}_{\nu=-1}^{+\infty}$ and $\{Q_\nu\}_{\nu=-1}^{+\infty}$ be solutions of this equation with the following initial values

$$P_{-1} = 1,\ Q_{-1} = 0,\ P_0(b_0) = b_0,\ Q_0(b_0) = 1. \tag{1.7}$$

Then easy induction shows that P_ν and Q_ν are numerator and denominator of finite continuous fraction (1.4). Since we have the equality

$$\begin{pmatrix} P_{\nu+1} \\ Q_{\nu+1} \end{pmatrix} = b_{\nu+1}\begin{pmatrix} P_\nu \\ Q_\nu \end{pmatrix} + a_{\nu+1}\begin{pmatrix} P_{\nu-1} \\ Q_{\nu-1} \end{pmatrix}, \tag{1.8}$$

it follows that

$$\Delta_{\nu+1} = det\begin{pmatrix} P_{\nu+1} & P_\nu \\ Q_{\nu+1} & Q_\nu \end{pmatrix} = -a_{\nu+1}\Delta_\nu = (-1)^\nu \prod_{k=1}^{\nu+1} a_k,$$

and therefore

$$\frac{P_{\nu+1}}{Q_{\nu+1}} - \frac{P_\nu}{Q_\nu} = (-1)^\nu \frac{\prod\limits_{k=1}^{\nu+1} a_k}{Q_\nu Q_{\nu+1}}. \tag{1.9}$$

It follows from Apéry results that the number $\alpha = \zeta(3)$ has the expansion in the continued fraction (1.5) with

$$b_0 = 0,\ a_1 = 6, b_1 = 5,\ a_{\nu+1} = -\nu^6,\ b_{\nu+1} = \tag{1.10}$$

$$34\nu^3 + 51\nu^2 + 27\nu + 5,$$

where $\nu \in \mathbb{N}$. Yu.V. Nesterenko (1996) has offered the following expansion of the number $\zeta(3)$ in continuous fraction:

$$\zeta(3) = 1 + \frac{1|}{|4} + \frac{4|}{|4} + \frac{1|}{|3} + \frac{4|}{|2}..., \tag{1.11}$$

with a_ν and b_ν given by the following equalities

$$b_0 = 1,\ a_1 = 1,\ b_1 = a_2 = b_2 = 4. \tag{1.12}$$

$$b_{4k+1} = 2k + 2,\ a_{4k+1} = k(k+1),\ b_{4k+2} = \tag{1.13}$$

$$2k + 4,\ a_{4k+2} = (k+1)(k+2)$$

for $k \in \mathbb{N}$,

$$b_{4k+3} = 2k + 3,\ a_{4k+3} = (k+1)^2,\ b_{4k+4} = \tag{1.14}$$

$$2k + 2,\ a_{4k+4} = (k+2)^2$$

for $k \in \mathbb{N}_0$.

Let $P_\nu^\wedge$ and $Q_\nu^\wedge$ be numerator and denominator of Nesterenko fractions. It is easy to prove that numerator and denominator of Nesterenko fractions with subscript $4\nu - 2$ are equal to the numerator and denominator of Apéry fraction with subscript ν, respectively.

2 The main result

The goal of present work is to give some supplements to Apéry's and Nesterenko's results. Our research is based on the results about difference systems connected with Mejer's functions; I gave a talk about these results on conference in memory of professor N.M.Korobov.

Thus, we have found the following expansions of the number $\zeta(3)$ in continuous fractions:
Theorem A. *The number $\zeta(3)$ has the following two expansions in continued fraction: the first one is*

$$2\zeta(3) = b_0^{(*1)} + \frac{a_1^{(*1)}|}{|b_1^{(*1)}} + ... + \frac{a_\nu^{(*1)}|}{|b_\nu^{(*1)}} + ..., \tag{2.1}$$

with b_ν and a_ν given by the equalities

$$b_0^{(*1)} = 3,\ a_1^{(*1)} = -81,$$

$$a_\nu^{(*1)} = -(\nu-1)^3\nu^3(4\nu^2-4\nu-3)^3$$

for $\nu \in [2, +\infty) \cap \mathbb{N}$,

$$b_\nu^{(*1)} = 4(68\nu^6 - 45\nu^4 + 12\nu^2 - 1)$$

for $\nu \in \mathbb{N}$, and the second is

$$2\zeta(3) = b_0^{(*2)} + \frac{a_1^{(*2)}|}{|b_2^{(*2)}} + ... + \frac{a_\nu^{(*2)}|}{|b_\nu^{(*2)}} + ..., \tag{2.2}$$

with b_ν and a_ν given by the equalities

$$b_0^{(*2)} = 2,\ a_1^{(*2)} = 42,$$

$$a_\nu^{(*2)} = -(\nu-1)^3\nu^3(4\nu^2-4\nu-3)((\nu+1)^3-\nu^3)((\nu-1)^3-(\nu-2)^3)$$

for $\nu \in [2, +\infty) \cap \mathbb{N}$,

$$b_\nu^{(*2)} = 2(102\nu^6 - 68\nu^4 + 21\nu^2 - 3),$$

for $\nu \in \mathbb{N}$.

As a result we specify also a way to obtain many other expansions of the number $\zeta(3)$ in continued fractions.

The next three sections contain a sketch of proof of Theorem A.

3 Auxiliary functions

Suppose that z satisfies to the following conditions:

$$|z| > 1, -3\pi/2 < \arg(z) \le \pi/2, \log(z) = \ln(|z|) + i\arg(z), \tag{3.1}$$

let δ be the differentiation $z\frac{\partial}{\partial z}$, and let α be a nonnegative integer. My first auxiliary function is a finite sum

$$f_{\alpha,1}^{*\vee}(z,\nu) := f_{\alpha,1}^{*}(z,\nu) := \sum_{k=0}^{\nu+\alpha}(z)^k \binom{\nu+\alpha}{k}^2 \binom{\nu+k}{\nu}^2. \tag{3.2}$$

Let us consider the rational function given by the equality

$$R(\alpha, t, \nu) = \frac{\prod_{j=1}^{\nu}(t-j)}{\prod_{j=0}^{\nu+\alpha}(t+j)}. \tag{3.3}$$

My second and fourth auxiliary function are sums of the following series

$$f^{*}_{\alpha,2}(z,\nu) = \sum_{t=1}^{+\infty} z^{-t}\frac{(\nu+\alpha)!^2}{\nu!^2}(R(\alpha,t,\nu))^2, \tag{3.4}$$

$$f^{*}_{\alpha,4}(z,\nu) = -\sum_{t=1}^{+\infty} z^{-t}\frac{(\nu+\alpha)!^2}{\nu!^2}\left(\frac{\partial}{\partial t}(R^2)\right)(\alpha,t,\nu). \tag{3.5}$$

Finally my third auxiliary function is defined as follows:

$$f^{*}_{\alpha,3}(z,\nu) = (\log(z))f^{*}_{\alpha,2}(z,\nu) + f^{*}_{\alpha,0,4}(z,\nu). \tag{3.6}$$

We consider also the functions $f_{\alpha,k}(z,\nu)$, $k = 1, 2, 3, 4$, related with previous functions by means of the equalities

$$f_{\alpha,k}(z,\nu) = \frac{\nu!^2}{(\nu+\alpha)!^2}(z,\nu)f^{*}_{\alpha,k}(z,\nu), \tag{3.7}$$

where $k = 1, 2, 3, 4$, $\nu \in \mathbb{N}_0$. Making use of the expansion of the following rational function

$$\frac{(\nu+\alpha)!^2}{(\nu!)^2}(-t)^r(R(\alpha,t,\nu))^2$$

into partial fractions relatively to t, and some simple transformations we obtain the following equality

$$\delta^r f^{*}_{\alpha,2+j}(z,\nu) - j(\log(z))\delta^r f^{*}_{\alpha,2}(z,\nu) = \tag{3.8}$$

$$\left(\sum_{i=1}^{2}(1-j+ij)\beta^{*(r)}_{\alpha,i}(z;\nu)L_{i+j}(1/z)\right) - \beta^{(r)}_{\alpha,3+j}(z;\nu),$$

where $\delta = z\frac{\partial}{\partial z}$, $j = 0, 1$, $r = 0, 1, 2, 3$, $|z| > 1$, $\alpha \in \mathbb{N}$, $s \in \mathbb{Z}$,

$$L_s(1/z) = \sum_{n=1}^{\infty} 1/(z^n n^s) \tag{3.9}$$

are polylogarithms and $\beta^{*(r)}_{\alpha,0,i}(z;\nu)$, $\beta^{*(r)}_{\alpha,0,3+j}(z;\nu)$, are polynomials of z with rational coefficients. It is clear that

$$L_s(1) = \zeta(s), \quad s > 1. \tag{3.10}$$

4 Passing to a difference system

In fact, the auxiliary functions $f^{\vee}_{\alpha,k}(z,\nu)$ are generalized hypergeometric functions, so called Mejer's functions. They satisfy the following differential equation

$$D_\alpha(z,\nu,\delta) f^{\vee}_{\alpha,k}(z,\nu) = 0, \tag{4.1}$$

where $\nu \in [0,+\infty) \cap \mathbb{Z}$, $k \in \mathfrak{K}_0 = \{1, 2, 3\}$,

$$D_\alpha(z,\nu,\delta) = z(\delta - \nu - \alpha)^2(\delta + \nu + 1)^2 - \delta^4. \tag{4.2}$$

is differential operator, and $\delta := z\frac{\partial}{\partial z}$. It follows from general properties of the Mejer's functions that

$$(\delta + \nu + 1)^2 f_{\alpha,k}(z,\nu) = (\delta - \nu - 1 - \alpha)^2 f_{\alpha,k}(z,\nu+1), \tag{4.3}$$

where $\nu \in [0,+\infty) \cap \mathbb{Z}$, $k \in \mathfrak{K}_0$. Since,

$$(1 - 1/z)^{-1} D_\alpha(z,\nu,\delta) = \delta^4 - \sum_{k=1}^{4} b_{\alpha,k}\delta^{k-1},$$

we can obtain by standard considerations the differential system

$$\delta X_{\alpha,k}(z;\nu) = B_\alpha(z;\nu) X_{\alpha,k}(z;\nu), \tag{4.4}$$

where $k = 1, 2, 3$, $|z| > 1$, $\nu \in \mathbb{N}_0$,

$$B_\alpha(z;\nu) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \\ b_{\alpha,1}(z;\nu) & b_{\alpha,2}(z;\nu) & b_{\alpha,3}(z;\nu) & b_{\alpha,4}(z;\nu) \end{pmatrix},$$

$$X_{\alpha,k}(z;\nu) = \begin{pmatrix} f^{*}_{\alpha,k}(z,\nu) \\ \delta f^{*}_{\alpha,k}(z,\nu) \\ \delta^2 f^{*}_{\alpha,k}(z,\nu) \\ \delta^3 f^{*}_{\alpha,k}(z,\nu) \end{pmatrix},$$

where $k = 1, 2, 3$, $|z| > 1$. In view of (4.2),

$$D_\alpha(z, -\nu - \alpha - 1, \delta) = D_\alpha(z,\nu,\delta). \tag{4.5}$$

Therefore we can put

$$X_{\alpha,k}(z; -\nu - 1 - \alpha) = X_{\alpha,k}(z;\nu), \tag{4.6}$$

where $\nu \in \mathbb{N}_0$, and then consider $X_{\alpha,k}(z;\nu)$ on

$$\nu \in M^{***}_\alpha = ((-\infty, -1-\alpha] \cup [0,+\infty) \cap \mathbb{Z},$$

Finally, we use the equations (4.1), (4.3) and (4.4) to obtain the following difference system.
Theorem 1. *The column $X_{\alpha,k}(z;\nu)$ satisfies to the equation*

$$\nu^5 X_{\alpha,k}(z;\nu - 1) = A^{*}_\alpha(z;\nu) X_{\alpha,k}(z;\nu), \tag{4.7}$$

for $\nu \in M_\alpha^* = (-\infty, -1-\alpha] \cup [1, +\infty)) \cap \mathbb{Z}$, $k = 1, 2, 3$, $|z| > 1$; *moreover, the matrix* $A_\alpha^*(z;\nu)$ *has the following property:*

$$-\nu^5(\nu+\alpha)^5 E_4 = A_\alpha^*(z; -\nu-\alpha) A_{\alpha},^* (z;\nu), \tag{4.8}$$

where E_4 *is the* 4×4 *unit matrix,* $z \in \mathbb{C}$, $\nu \in \mathbb{C}$.

Although the matrix $A_\alpha^*(z;\nu)$ is a 4×4-matrix, its elements are huge polynomials in $\mathbb{Q}[z, \nu, \alpha]$. For example, if we put

$$\mu = \mu_\alpha(\nu) = (\nu+\alpha)(\nu+1),\ \tau = \tau_\alpha(\nu) = \nu + \frac{1+\alpha}{2}, \tag{4.9}$$

then the the matrix $A_\alpha^*(z;\nu)$ has on intersection of its first row and its first column the element

$$a_{\alpha,1,1}^*(z,\nu) = a_{\alpha,1,1}^\vee(z,\nu) + \tau a_{\alpha,1,1}^\wedge(z,\nu),$$

where

$$a_{\alpha,1,1}^\vee(z,\nu) = \frac{1}{2}(-1+2\alpha-\alpha^2-5\mu+3\alpha\mu-5\mu^2-\alpha\mu^2)+ \tag{4.10}$$

$$\frac{z}{2}(-4+12\alpha-13\alpha^2+6\alpha^3-\alpha^4)+$$

$$\frac{z}{2}\mu(-32+54\alpha-29\alpha^2+5\alpha^3-56\mu+20\alpha\mu),$$

$$a_{\alpha,1,1}^\wedge(z;\nu) = 1-\alpha+3\mu+\mu^2+ \tag{4.11}$$

$$z(4-8\alpha+5\alpha^2-\alpha^3+24\mu-22\alpha\mu+5\alpha^2\mu+16\mu^2),$$

So, the equality (4.8) was very helpful for us, when we have checked our calculations.

5 Reducing the obtained system to the difference system of the second order in the case $\alpha = 1$.

This is key point in our research, it leads to desirable results. In the case $\alpha = 1$ the situation simplifies since the above system is reducible and our problem can be reduced to the consideration of a system of the second order. To be more precise, in this case

$$\tau = \tau_1(\nu) = \nu+1,\ \mu = \mu_1(\nu) = (\nu+1)^2, \tag{5.1}$$

$$\frac{1}{z}D_\alpha(z,\nu,\delta) = (1-1/z)\delta^4 + \sum_{k=0}^{3} r_{\alpha,k+1}(\nu)\delta^k, \tag{5.2}$$

where

$$r_1(\nu) = \mu_1(\nu)^2 = (\nu+1)^4 = \tau^4,\ r_2(\nu) = 0,$$

$$r_3(\nu) = -2\mu_1(\nu) = -2(\nu+1)^2,\ r_4(\nu) = 0,$$

Let us consider the row

$$R(\nu) = (r_1(\nu), r_2(\nu), r_(\nu), r_4(\nu)). \tag{5.3}$$

Let E_4 be the 4×4-unit matrix, and let $C(\nu)$ be the result of replacement of 1-th row of the matrix E_4 by the row in (5.3). Let further $D(\nu)$ be the adjoint matrix to the matrix $C(\nu)$. Then

$$C(\nu)D(\nu) = \mu^2 E_4,\ C(-\nu-2) = C(\nu),\ D(-\nu-2) = D(\nu), \tag{5.4}$$

Set

$$A_1^{**}(1,\nu) = C(\nu-1)A_1^{*}(z,\nu)D(\nu). \tag{5.5}$$

and

$$Y_{1,k}(z;\nu) = \begin{pmatrix} y_{1,1,k}(z;\nu) \\ y_{1,2,k}(z;\nu) \\ y_{1,3,k}(z;\nu) \\ y_{1,4,k}(z;\nu) \end{pmatrix} = C(\nu)X_{1,k}(z;\nu), \tag{5.6}$$

where $k = 1,\, 2,\, 3,\, |z| > 1,\, \nu \in M_1^{***} = ((-\infty,-2] \cup [0,+\infty)) \cap \mathbb{Z}$. Then

$$Y_{1,k}(z;-\nu-2) = Y_{1,k}(z;\nu), \tag{5.7}$$

$$\mu_1(\nu)^2\nu^5 Y_{1,k}(z;\nu-1) = A_1^{**}(z,\nu)Y_{1,k}(z;\nu), \tag{5.8}$$

where $\kappa = 0,\, 1,\, k = 1,\, 2,\, 3,\, |z| > 1,\, \nu \in M_1^{*} = ((-\infty,-2] \cup [1,+\infty)) \cap \mathbb{Z}$. Replacing in the equality (5.8)

$$\nu \in M_1^{*} = ((-\infty,-2] \cup [1,+\infty)) \cap \mathbb{Z}$$

by

$$\nu := -\nu-2 \in M_1^{**} = ((-\infty,-3] \cup [0,+\infty)) \cap \mathbb{Z},$$

and taking in account (5.7) we obtain the equality

$$\mu_1(\nu)^2(\nu+2)^5 Y_{1,k}(z;\nu+1) = -A_1^{**}(z,-\nu-2)Y_{1,k}(z;\nu), \tag{5.9}$$

where $k = 1,\, 2,\, 3,\, |z| > 1,\, \nu \in M_1^{**} = ((-\infty,-3] \cup [0,+\infty)) \cap \mathbb{Z}$. The matrix $A_1^{**}(z,\nu)$ can be represented in the form

$$A_1^{**}(z;\nu) = A_1^{**}(1;\nu) + (z-1)V_1^{**}(\nu), \tag{5.10}$$

where the matrix $V_1^{**}(\nu)$ does not depend from z. We will tend $z \in (1,+\infty)$ to 1. Therefore we must study the behavior of our auxiliary functions, when tend $z \in (1,+\infty)$ to 1. Then

$$t^r R(1,t,\nu)^2 = \frac{\prod_{j=1}^{\nu}(t-j)^2}{\prod_{j=0}^{\nu+1}(t+j)^2} = t^{r-4} + t^{r-5}O(1)\ (t \to +\infty) \tag{5.11}$$

$$t^r \left(\frac{\partial}{\partial t}(R^2)\right)(1,t,\nu) = t^{r-5}O(1)\ (t \to +\infty) \tag{5.12}$$

for $r = 0,\, 1,\, 2 3,\, 4$. Therefore

$$(z-1)\delta^r f_{1,2}(z,\nu) = \tag{5.13}$$

$$\sum_{t=1}^{+\infty} z^{-t}(-t)^r(R(\alpha,t,\nu))^2 = (z-1)O(1)\ln(1-1/z) \to 0\ (z \to 1+0)$$

for $r = 0,\, 1,\, 2,\, 3,$

$$(z-1)\delta^4 f_{1,2}(z,\nu) = \tag{5.14}$$

$$\sum_{t=1}^{+\infty} z^{-t}(-t)^4(R(\alpha,t,\nu))^2 = 1 + (z-1)O(1)\ln(1-1/z) \to 1\ (z \to 1+0)$$

$$(z-1)\delta^r f_{1,4}(z,\nu) = \tag{5.15}$$

$$-\sum_{t=1}^{+\infty} z^{-t}(-t)^r \left(\frac{\partial}{\partial t}(R^2)\right)(1,t,\nu) = (z-1)O(1) \to 0 \ (z \to 1+0)$$

for $r = 0,\ 1,\ 2,\ 3,\ 4$ and

$$(z-1)\delta^r f_{1,3}(z,\nu) = (z-1)(\log(z))\delta^r f_{1,2}(z,\nu) + \tag{5.16}$$

$$(z-1)r\delta^{r-1} f_{1,2}(z,\nu) + (z-1)\delta^r f_{1,4}(z,\nu) \to 0 \ (z \to 1+0)$$

for $r = 0,\ 1,\ 2,\ 3,\ 4$. Further we have

$$y_{1,j+1,k}(z,\nu) = \delta^j f_{1,k}(z,\nu), \tag{5.17}$$

where $j = 1,\ 2,\ 3\, k = 1,\ 2,\ 3,\ |z| > 1,\ \nu \in \mathbb{N}_0$. Further we have

$$y_{1,1,k}(1,\nu) := \lim_{z \to 1+0} y_{1,1,k}(z,\nu) = \tag{5.18}$$

$-\lim\limits_{z \to 1+0}(1 - 1/z)\delta^4 f_{1,k}(z,\nu) = (k-1)(k-3)$, where $k = 1,\ 2,\ 3,\ ,\nu \in \mathbb{N}_0$,

$$A_1^{**}(1;\nu) = \begin{pmatrix} (\nu+1)^4\nu^5 & 0 & 0 & 0 \\ a^{**}_{1,2,1}(1;\nu) & a^{**}_{1,2,2}(1;\nu) & a^{**}_{1,2,3}(1;\nu) & 0 \\ a^{**}_{1,3,1}(1;\nu) & a^{**}_{1,3,2}(1;\nu) & a^{**}1,3,3(1;\nu) & 0 \\ a^{**}_{1,4,1}(1;\nu) & a^{**}_{1,4,2}(1;\nu) & a^{**}_{1,4,3}(1;\nu) & (\nu+1)^4\nu^5 \end{pmatrix} \tag{5.19}$$

with

$$a^{**}_{1,2,1}(1;\nu) = \tag{5.20}$$

$$-\tau^2(\tau-1)(2\tau-1)(6\tau^2-4\tau+1),$$

$$a^{**}_{1,2,2}(1;\nu) = \tau^5(\tau-1)(\tau^3+2(2\tau-1)^3), \tag{5.21}$$

$$a^{**}_{1,2,3}(1;\nu) = -3\tau^4(\tau-1)(2\tau-1)^3, \tag{5.22}$$

$$a^{**}_{1,3,1}(1;\nu) = \tag{5.23}$$

$$\tau^2(\tau-1)^2(2\tau-1)(4\tau^2-3\tau+1),$$

$$a^{**}_{1,3,2}(1;\nu) = \tag{5.24}$$

$$-2\tau^5(\tau-1)^2(2\tau-1)(\tau^3-(\tau-1)^3),$$

$$a^{*0*}_{1,3,3}(1;\nu) = \tag{5.25}$$

$$\tau^4(\tau-1)^2((\tau-1)^3+2(2\tau-1)^3),$$

$$a^{**}_{1,4,1}(1;\nu) = \tag{5.26}$$

$$-\tau^2(\tau-1)^3(2\tau-1)(2\tau^2-2\tau+1),$$

$$a^{**}_{1,4,2}(1;\nu) = \tag{5.27}$$

$$\tau^5(\tau-1)^3(2\tau-1)(4\tau^2-5\tau+3).$$

$$a^{**}_{1,4,3}(1;\nu) = \tag{5.28}$$

$$-\tau^4(\tau-1)^3(2\tau-1)(6\tau^2-8\tau+3).$$

If we consider the second and third equations in the system of equations (5.8) with $k = 1, 3$ and tend $z \in (1, +\infty)$ to 1, then, in view of (5.18) and (5.19), we obtain equations

$$\mu_1(\nu)^2\nu^5\delta^i f_{1,0,k}(1, \nu-1) = \tag{5.29}$$

$$\left(\sum_{j=1}^{2} a^{**}_{1,0,i+1,j+1}(1;\nu)(\delta^j f_{1,0,k})(1,\nu)\right),$$

where $i = 1, 2$, $k = 1, 3$ and $\nu \in M_1^* = ((-\infty, -2] \cup [1, +\infty)) \cap \mathbb{Z}$.

Let us take

$$F = \{F(\nu)\}_{nu=-\infty}^{+\infty} \text{ and } G = \{G(\nu)\}_{nu=-\infty}^{+\infty} \tag{5.30}$$

such that

$$F(-\nu-2) = F(\nu),\ G(-\nu-2) = G(\nu),\ F(\nu) \in \mathbb{Q},\ G(\nu) \in \mathbb{Q} \tag{5.31}$$

for $\nu \in \mathbb{Z}$. Then in view of (5.7),

$$y^{**}_{F,G}(z, -\nu-2) = y^{**}_{F,G}(z, \nu) = \tag{5.32}$$

for $\kappa = 0, 1$, $k = 1, 3$ and $\nu \in M_1^{***} = ((-\infty, -2] \cup [0, +\infty)) \cap \mathbb{Z}$. In view of (5.29)

$$\left(\sum_{j=1}^{2} a^{***}_{F,G,j+1}(1;\nu)(\delta^j f_{1,0,k})(1,\nu)\right) = \tag{5.33}$$

$$\mu_1(\nu)^2\nu^5 y^{***}_{F,G}(z, \nu-1),$$

where $k = 1, 3$ and $\nu \in M_1^* = ((-\infty, -2] \cup [1, +\infty)) \cap \mathbb{Z}$. Replacing in (5.33) $\nu \in M_1^*$ by

$$\nu := -\nu - 2 \in M_1^{**} = ((-\infty, -3] \cup [0, +\infty)) \cap \mathbb{Z},$$

and taking in account the equality (5.7) we obtain the equalities

$$\left(\sum_{j=1}^{2} a^{**\kappa*}_{F,G,j+1-\kappa}(1;-\nu-2)(\delta^{j-\kappa} f_{1,0,k})(1,\nu)\right) = \tag{5.34}$$

$$-\mu_1(\nu)^2(\nu+1)^5 y^{**}_{F,G}(z, \nu+1),$$

where $k = 1, 3$ and $\nu \in M_1^{**} = ((-\infty, -3] \cup [0, +\infty)) \cap \mathbb{Z}$. Set

$$\vec{w}^{(\kappa)}_{F,G,j}(\nu) = \begin{pmatrix} a^{***}_{F,G,j+1}(1;-\nu-2) \\ F(\nu)(2-j) + G(\nu)(j-1) \\ a^{***}_{F,G,j+1}(1;\nu-1) \end{pmatrix}, \tag{5.35}$$

where $j = 1, 2$, $\nu \in M_1^{****} = ((-\infty, -3] \cup [1, +\infty)) \cap \mathbb{Z}$,

$$W_{F,G}(\nu) = \begin{pmatrix} \vec{w}^{(\kappa)}_{F,G,1}(\nu) & \vec{w}^{(\kappa)}_{F,G,2}(\nu) \end{pmatrix} = \tag{5.36}$$

$$\begin{pmatrix} a^{***}_{F,G,2}(1;-\nu-2) & a^{***}_{F,G,3}(1;-\nu-2) \\ F(\nu) & G(\nu) \\ a^{***}_{F,G,2}(1;\nu) & a^{***}_{F,G,3}(1;\nu) \end{pmatrix},\ Y_k^{***}(\nu) =$$

$$\begin{pmatrix} (\delta f_{1,0,k})(1,\nu) \\ (\delta^2 f_{1,0,k})(1,\nu) \end{pmatrix},$$

$$Y^{*****}_{F,G,k}(\nu) = \tag{5.37}$$

$$\begin{pmatrix} -\mu_1(\nu)^2(\nu+2)^5 y^{**}_{F,G}(z,-\nu-2) \\ y^{**}_{F,G}(z,\nu) \\ \mu_1(\nu)^2\nu^5 y^{**}_{F,G}(z,\nu-1) \end{pmatrix},$$

where $k = 1, 3$, $\nu \in M_1^{****} = ((-\infty,-3] \cup [1,+\infty)) \cap \mathbb{Z}$. In view of (5.33) and (5.34)

$$Y^{*****}_{F,G,k}(\nu) = W_{F,G}(\nu) Y^{***}_k(\nu). \tag{5.38}$$

Let further

$$\vec{w}_{F,G,3}(\nu) = \begin{pmatrix} w_{F,G,3,1}(\nu) \\ w_{F,G,3,2}(\nu) \\ w_{F,G,3,3}(\nu) \end{pmatrix} = [\vec{w}_{F,G,1}(\nu), \vec{w}_{F,G,2}(\nu)] \tag{5.39}$$

be the vector product of $\vec{w}^{F,G,1}(\nu)$ and $\vec{w}_{F,2}(\nu)$, and let

$$\bar{w}_{F,G,3}(\nu) = (\vec{w}_{F,G,3}(\nu))^t$$

be the row conjugate to the column $\vec{w}_{F,G,3}(\nu)$. Then for scalar products

$$(\vec{w}^{(\kappa)}_{F,G,3}(\nu), \vec{w}^{(\kappa)}_{F,G,j}(\nu))$$

we have the equalities

$$\bar{w}^{(\kappa)}_{F,G,3}(\nu)\vec{w}^{\kappa}_{F,G,j}(\nu) =$$

$$(\vec{w}^{(\kappa)}_{F,G,3}(\nu), \vec{w}^{(\kappa)}_{F,G,j}(\nu)) = 0,$$

where $\kappa = 0, 1$, $j = 1, 2$ and

$$\nu \in M_1^{****} = ((-\infty,-3] \cup [1,+\infty)) \cap \mathbb{Z}.$$

Therefore

$$\bar{w}_{F,G,3}(\nu) W_{F,G}(\nu) = \begin{pmatrix} 0 & 0 \end{pmatrix}, \tag{5.40}$$

where $\nu \in M_1^{****} = ((-\infty,-3] \cup [1,+\infty)) \cap \mathbb{Z}$. In view of (5.29), (5.34) and (5.40),

$$\bar{w}(\kappa)_{F,G,3}(\nu) Y^{*\kappa****}_{F,G,k}(\nu) = \tag{5.41}$$

$$\bar{w}(\kappa)_{i,3}(\nu) W(\kappa)_i(\nu) Y^{***}_k(\nu) = 0,$$

where $k = 1, 3$ and $\nu \in M_1^{****} = ((-\infty,-3] \cup [1,+\infty)) \cap \mathbb{Z}$.

Thus, for given F and G we obtain a difference equation of the second order, which leads to desirable results. First, taking $F(\nu) = 1$ and $G(\nu) = 0$ for all $\nu \in \mathbb{Z}$, we then obtain the first expansion described in Theorem A. Further, taking $F(\nu) = 0$ and $G(\nu) = 1$ for all $\nu \in \mathbb{Z}$, we then obtain the second expansion from Theorem A.

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О НЕКОТОРЫХ РАЗЛОЖЕНИЯХ $\zeta(3)$ В НЕПРЕРЫВНЫЕ ДРОБИ

Л.А. Гутник
Московский государственный институт электроники и математики,
Большой Трехсвятителский пер., 1-3/12, Москва, 109028, Россия, e-mail: gutnik@gutnik.mccme.ru

Аннотация. В данной работе получены новые разложения $\zeta(3)$ в непрерывные дроби.
Ключевые слова: дзета-функция, $\zeta(3)$, разложения в непрерывные дроби.