2000 MSC 32A15, 32A20, 32A22, 32H30

UNIQUENESS THEOREMS OF MEROMORPHIC FUNCTIONS IN SEVERAL COMPLEX VARIABLES

Pei-Chu Hu[1], Chung-Chun Yang[2]

[1] Department of Mathematics, Shandong University,
Jinan 250100, Shandong, P. R. China e-mail: pchu@sdu.edu.cn
[2] Department of Mathematics, The Hong Kong University of Science & Technology
Clear Water Bay, Kowloon, Hong Kong e-mail: chungchun.yang@gmail.com

Abstract. In the survey, results on the existence, growth, uniqueness, and value distribution of meromorphic (or entire) solutions of homogeneous linear partial differential equations of the second order with polynomial coefficients that are similar or different from that of meromorphic solutions of linear ordinary differential equations have been obtained. We have characterized those entire solutions of a special partial differential equation that relate to Bessel functions and prove in general that meromorphic solutions that grow much faster than the coefficient have zero Nevanlinna's deficiency for each non-zero complex value. It's well-know result that if a nonconstant meromorphic function f on $\mathbb{C}$ and its l-th derivative $f^{(l)}$ have no zeros for some $l \geq 2$, then f is of the form $f(z) = \exp(Az + B)$ or $f(z) = (Az + B)^{-n}$ for some constants A, B. We have extended this result to meromorphic functions of several variables, by first extending the classic Tumura-Clunie theorem for meromorphic functions of one complex variable to that of meromorphic functions of several complex variables by utilizing Nevanlinna theory.

Keywords: meromorphic functions, homogeneous linear partial differential equation, holomorphic coefficients, Nevanlinna's value distribution theory.

Analytic properties or characterizations of meromorphic (or entire) solutions of some partial differential equations (or system) of the first order have been exhibited clearly by several authors (cf. [2], [13], [18], [19]). In this survey, we introduce a few results on meromorphic solutions of homogeneous linear partial differential equations of the second order in two independent complex variables

$$a_0 \frac{\partial^2 u}{\partial t^2} + 2a_1 \frac{\partial^2 u}{\partial t \partial z} + a_2 \frac{\partial^2 u}{\partial z^2} + a_3 \frac{\partial u}{\partial t} + a_4 \frac{\partial u}{\partial z} + a_6 u = 0, \tag{1.1}$$

where $a_k = a_k(t, z)$ are holomorphic functions for $(t, z) \in \Sigma$, where Σ is a region on $\mathbb{C}^2$. Basic idea comes from S. N. Bernšteĭn [3], H. Lewy [17], I. G. Petrovskiĭ[20]. For more detail, see [15]. To prove these results, we used some methods in [5], [7], [11], [14], [21], [23] and [26].

First of all, we examine the following special differential equation:

$$t^2 \frac{\partial^2 u}{\partial t^2} - z^2 \frac{\partial^2 u}{\partial z^2} + t \frac{\partial u}{\partial t} - z \frac{\partial u}{\partial z} + t^2 u = 0. \tag{1.2}$$

The work of Chung-Chun Yang was partially supported by Natural Science Foundation of China and second author was partially supported by a UGC Grant of Hong Kong: Project 604106.

Theorem 1.1 *The differential equation* (1.2) *has an entire solution* $f(t,z)$ *on* $\mathbb{C}^2$ *if and only if* f *is an entire function expressed by the series*

$$f(t,z) = \sum_{n=0}^{\infty} n! c_n J_n(t) z^n \tag{1.3}$$

such that

$$\limsup_{n\to\infty} |c_n|^{1/n} = 0, \tag{1.4}$$

where $J_n(t)$ *is the first kind of Bessel's function of order* n. *Moreover, the order* $\mathrm{ord}(f)$ *of the entire function* f *satisfies*

$$\rho \le \mathrm{ord}(f) \le \max\{1, \rho\},$$

where

$$\rho = \limsup_{n\to\infty} \frac{2\log n}{\log |c_n|^{-1/n}}. \tag{1.5}$$

By definition, the order of f is defined by

$$\mathrm{ord}(f) = \limsup_{r\to\infty} \frac{\log^+\log^+ M(r,f)}{\log r},$$

where

$$\log^+ x = \begin{cases} \log x, & \text{if } x \ge 1; \\ 0, & \text{if } x < 1, \end{cases}$$

and

$$M(r,f) = \max_{|t|\le r, |z|\le r} |f(t,z)|.$$

G. Valiron [25] showed that each transcendental entire solution of a homogeneous linear ordinary differential equation with polynomial coefficients is of finite positive order. However, Theorem 1.1 shows that Valiron's theorem is not true for general partial differential equations. Here we exhibit another example that the following equation

$$t^2\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial z^2} + t\frac{\partial u}{\partial t} = 0$$

has an entire solution $\exp(te^z)$ of infinite order.

If $0 < \lambda = \mathrm{ord}(f) < \infty$, we define the type of f by

$$\mathrm{typ}(f) = \limsup_{r\to\infty} \frac{\log^+ M(r,f)}{r^\lambda}.$$

For the type of entire solutions of the equation (1.2), we have an analogue of Lindelöf-Pringsheim theorem, its proof is essentially the same as that of the determining of the type for Taylor series of entire functions of one complex variable.

Theorem 1.2 *If* $f(t,z)$ *is an entire solution of* (1.2) *defined by* (1.3) *and* (1.4) *such that* $1 < \lambda = \mathrm{ord}(f) < \infty$, *then the type* $\sigma = \mathrm{typ}(f)$ *satisfies*

$$e\lambda\sigma = 2^{-\lambda/2} \limsup_{n\to\infty} 2n|c_n|^{\lambda/(2n)}.$$

Brosch [4] proved that if two nonconstant meromorphic functions f and g on $\mathbb{C}$ share three distinct values c_1, c_2, c_3 counting multiplicities, and if f is a solution of the differential equation

$$\left(\frac{dw}{dz}\right)^n = \sum_{j=0}^{2n} b_j(z)w^j := P(z, w)$$

such that $b_0, b_1, \cdots, b_{2n}$ ($b_{2n} \not\equiv 0$) are small functions of f (grow slower than f), furthermore if $P(z, c_i) \not\equiv 0$ for $i = 1, 2, 3$, then $f = g$. To state a generalization of Brosch's result to PDE, we abbreviate

$$u_t = \frac{\partial u}{\partial t}, \ u_{tz} = \frac{\partial^2 u}{\partial t \partial z}, \ u_{tt} = \frac{\partial^2 u}{\partial t^2},$$

and so on, and set

$$Du = a_0 u_t^2 + 2a_1 u_t u_z + a_2 u_z^2,$$

$$Lu = a_0 u_{tt} + 2a_1 u_{tz} + a_2 u_{zz} + a_3 u_t + a_4 u_z.$$

We make the following assumption:

(A) All coefficients a_i in (1.1) are polynomials and when $a_6 = 0$ there are no nonconstant polynomials u satisfying the system

$$\begin{cases} Du = 0, \\ Lu = 0. \end{cases}$$

For technical reason, here we study only meromorphic functions of finite orders. The order of a meromorphic function of several variables may be defined by using its Nevanlinna's characteristic function (cf. [12], [22]).

Theorem 1.3 *Assume that the assumption* (A) *holds. Let* $f(t, z)$ *be a nonconstant meromorphic solution of* (1.1) *such that* $\mathrm{ord}(f) < \infty$ *and let* g *be a nonconstant meromorphic function of finite order on* $\mathbb{C}^2$. *If* f *and* g *share* $0, 1, \infty$ *counting multiplicity, one of the following five cases is occurred:*

(a) $g = f$;

(b) $gf = 1$;

(c) $a_6 = 0$, $gf = f + g$;

(d) $a_6 = 0$, *and there exist a constant* $b \notin \{0, 1\}$ *and a polynomial* β *such that*

$$f = \frac{1}{b-1}\left(e^{\beta} - 1\right), \ g = \frac{b}{b-1}\left(1 - e^{-\beta}\right);$$

(e) $a_6 \neq 0$, $f^2 g^2 = 3fg - f - g$.

When $a_6 \neq 0$, the case (b) may happen. For example, we consider the differential equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial u}{\partial t} - u = 0, \tag{1.6}$$

which has an entire solution of order 1

$$f(t, z) = e^{t+z}.$$

Let's compare f with the following entire function of order 1

$$g(t, z) = e^{-t-z}.$$

Obviously, f and g share 0, 1, -1, ∞ counting multiplicity, but $g \neq f$, $gf = 1$. Now the differential equation

$$Lu + Du + a_6 = 0$$

has a nonconstant polynomial solution

$$u(t, z) = t + z.$$

The condition (A) is meaningful. For example, Theorem 1.1 shows that the differential equation (1.2) has a lot of entire solutions of finite orders. Obviously, the condition (A) associated to the differential equation (2) holds, and hence we can obtain the fact:

Corollary 1.4 *Let $f(t, z)$ be a nonconstant meromorphic solution of* (1.2) *such that* $\mathrm{ord}(f) < \infty$ *and let g be a nonconstant meromorphic function of finite order on $\mathbb{C}^2$. If f and g share* 0, 1, ∞ *counting multiplicity, then we have either $g = f$ or $gf = 1$ or $f^2g^2 = 3fg - f - g$.*

The case (b) in Theorem 1.3 may really happen for $a_6 = 0$. For example, we consider the differential equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial z} = 0, \tag{1.7}$$

which has an entire solution $f(t, z) = e^{t+z}$ of order 1 such that the assumption (A) holds obviously. The entire solution f and the function $g = e^{-t-z}$ share 0, 1, ∞ counting multiplicity, and satisfy $gf = 1$, that is, the case (b) in Theorem 1.3 happens for the case $a_6 = 0$.

For a real number x, let $[x]$ denote the maximal integer $\leq x$. We give the following result that is an analogue of Anastassiadis's theorem [1] on uniqueness of entire functions of one variable.

Theorem 1.5 *Let $f(t, z)$ and $g(t, z)$ be transcendental entire solutions of* (1.2) *such that* $\mathrm{ord}(f) < \infty$, $\mathrm{ord}(g) < \infty$, *and*

$$\frac{\partial^{2j} f}{\partial t^j \partial z^j}(0, 0) = \frac{\partial^{2j} g}{\partial t^j \partial z^j}(0, 0), \ j = 0, 1, ..., q,$$

where

$$q = \max\{[\mathrm{ord}(f)], [\mathrm{ord}(g)]\}.$$

If there exists a complex number a with $(a, f(0, 0)) \neq (0, 0)$ such that f and g share a counting multiplicity, then we have $f = g$.

Theorem 1.3 shows that when $a_6 = 0$, global solutions of the equation (1.1) can be quite complicated, however, when $a_6 \neq 0$, these solutions have normal properties. Next result also supports this view. Theorem 1.6 extends a theorem (cf. Theorem 5.8 of [10]) on meromorphic solutions of linear ordinary differential equations.

Theorem 1.6 *Assume that all a_k in (1.1) are entire functions on $\mathbb{C}^2$ which grow slower than a meromorphic solution of equations (1.1) on $\mathbb{C}^2$. If $a_6 \not\equiv 0$, then the deficiency of the solution for each non-zero complex number is zero.*

For example, the telegraph equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial z^2} + 2\alpha \frac{\partial u}{\partial t} + \alpha^2 u = 0$$

has entire solutions

$$u(t, z) = e^{-\alpha t}\{f(z + ct) + g(z - ct)\},$$

where f and g are entire functions on $\mathbb{C}$. If $\alpha \neq 0$, Theorem 1.6 shows that the deficiency of a non-constant $u(t, z)$ for each non-zero complex number a is zero, which means that the equation

$$f(z + ct) + g(z - ct) - ae^{\alpha t} = 0$$

has zeros.

Let $\mathbb{Z}_+$ denote the set of non-negative integers. For $z = (z_1, ..., z_m) \in \mathbb{C}^m$, $\mathbf{i} = (i_1, ..., i_m) \in \mathbb{Z}_+^m$, we write

$$\partial_{z_k} = \frac{\partial}{\partial z_k},\ k = 1, ..., m;\ \partial^{\mathbf{i}} = \partial_z^{\mathbf{i}} = \partial_{z_1}^{i_1} \cdots \partial_{z_m}^{i_m};\ |\mathbf{i}| = i_1 + \cdots + i_m.$$

We have interesting in the following problem:

Conjecture 1.7 *If f is a meromorphic function in $\mathbb{C}^m$ such that f and $\partial^{\mathbf{l}} f$ have no zeros for some $\mathbf{l} = (l_1, ..., l_m) \in \mathbb{Z}_+^m$ with $l_k \geq 2$ $(1 \leq k \leq m)$ and such that the set of poles of f is algebraic, then there exists a partition*

$$\{1, ..., m\} = I_0 \cup I_1 \cup \cdots \cup I_k$$

such that $I_i \cap I_j = \emptyset$ $(i \neq j)$, and

$$f(z_1, ..., z_m) = \exp\left(\sum_{i \in I_0} A_i z_i + B_0\right) \prod_{j=1}^{k} \left(\sum_{i \in I_j} A_i z_i + B_j\right)^{-n_j},$$

where A_i, B_j are constants with $A_i \neq 0$, and n_j are positive integers.

This is open if $m > 1$. For detail discussion, see [16]. When $m = 1$, the conclusion of Conjecture 1.7 was obtained by Tumura [24], and Hayman [8] gave a proof for the case $l = l_m = 2$. Later, as a correction of the gap in Tumura's proof, Clunie [6] gave a valid proof of the assertion for any $l > 1$.

Let f be a meromorphic function in $\mathbb{C}^m$ which we shall assume to be not constant. We shall be concerned largely with meromorphic functions h which are polynomials in f and the partial derivatives of f with coefficients a of the form

$$\| \quad T(r, a) = o(T(r, f)), \tag{1.8}$$

where $T(r, f)$ is the *Nevanlinna's characteristic function* of f, and where the symbol "$\|$"means that the relation holds outside a set of r of finite linear measure. Such functions h will be called *differential polynomials* in f. To study Conjecture 1.7, the following result will play a crucial role.

Theorem 1.8 *Suppose that f is meromorphic and not constant in $\mathbb{C}^m$, that*

$$g = f^n + P_{n-1}(f), \tag{1.9}$$

where $P_{n-1}(f)$ is a differential polynomial of degree at most $n-1$ in f, and that

$$\| \quad N(r, f) + N\left(r, \frac{1}{g}\right) = o(T(r, f)),$$

where $N(r, f)$ is the Nevanlinna's valence function of f for poles. Then

$$g = \left(f + \frac{a}{n}\right)^n,$$

where a is a meromorphic function of the form (1.8) in $\mathbb{C}^m$ determined by the terms of degree $n-1$ in $P_{n-1}(f)$ and by g.

When $m = 1$, Theorem 1.8 is due to Hayman ([9], Theorem 3.9, p.69). By using Theorem 1.8, we can give a proof of Conjecture 1.7, under a condition on non-vanishing of the partial derivatives of order > 1 that differs from the one posed in the conjecture, as follows:

Theorem 1.9 *If f is a meromorphic function in $\mathbb{C}^m$ such that f, $\partial_{z_1}^{l_1} f$, ..., $\partial_{z_m}^{l_m} f$ have no zeros for some $l_k \geq 2$ $(1 \leq k \leq m)$ and such that the set of poles of f is algebraic, then there exists a partition*

$$\{1, ..., m\} = I_0 \cup I_1 \cup \cdots \cup I_k$$

such that $I_i \cap I_j = \emptyset$ $(i \neq j)$, and

$$f(z_1, ..., z_m) = \exp\left(\sum_{i \in I_0} A_i z_i + B_0\right) \prod_{j=1}^{k} \left(\sum_{i \in I_j} A_i z_i + B_j\right)^{-n_j},$$

where A_i, B_j are constants with $A_i \neq 0$, and n_j are positive integers.

In particular, if f is entire, the function f in Theorem 1.9 has only an exponential form

$$f(z_1, ..., z_m) = \exp\left(A_1 z_1 + \cdots + A_m z_m + B_0\right).$$

We shall utilize the methods developed in [9], [12] and [13] and generalized Clunie lemma to prove the main results.

Bibliography

1. J. Anastassiadis. Recherches algébriques sur le théorème de Picard-Montel, Exposés sur la théorie des fonctions XVIII, Hermann, Paris, 1959.

2. C.A. Berenstein, B.Q. Li. On certain first-order partial differential equations in $\mathbb{C}^n$, Harmonic Analysis, Signal Processing, and Complexity, 29-36, Progr. Math., 238, Birkhäuser Boston, 2005.

3. S.N. Bernšteĭn. Sur la nature analytique des solutions de certaines équations aux dériveés parttielles du second order, C. R. Acad. Sci. Paris 137 (1903), 778-781.

4. G. Brosch. Eindeutigkeitssätze für meromorphe Funktionen, Thesis, Technical University of Aachen, 1989.

5. W.D. Brownawell. On the factorization of partial differential equations, Can. J. Math. Vol. XXXIX (1987), No. 4, 825-834.

6. J. Clunie. On integral and meromorphic functions, J. London Math. Soc. 37(1962), 17-27.

7. Ph. Griffiths, J. King. Nevanlinna theory and holomorphic mappings between algebraic varieties, Acta Math. 130(1973), 145–220.

8. W.K. Hayman. Picard values of meromorphic functions and their derivatives, Ann. of Math. 70(1959), 9-42.

9. W.K. Hayman. Meromorphic functions, Oxford University Press, 1964.

10. Y.Z. He, X.Z. Xiao. Algebroid functions and ordinary differential equations (Chinese), Science Press, Beijing, 1988.

11. E.W. Hobson. Spherical and Ellipsoidal Harmonics, Cambridge, 1931.

12. P.C. Hu, P. Li. C.C. Yang. Unicity of meromorphic mappings, Kluwer Academic Publishers, 2003.

13. P.C. Hu, C.C. Yang. Malmquist type theorem and factorization of meromorphic solutions of partial differential equations, Complex Variables 27 (1995), 269-285.

14. P.C. Hu, C.C. Yang. Value distribution theory related to number theory, Birkhäuser, 2006.

15. P.C. Hu, C.C. Yang. Global solutions of homogeneous linear partial differential equations of the second order, a talk at Joint Sino-Russia Symposium "Complex Analysis and its Applications Belgorod State University, Russia, 2009; to appear in Michigan Math. J. (2009).

16. P.C. Hu, C.C. Yang. Tumura-Clunie theorem in several complex variables, a talk at Joint Sino-Russia Symposium "Complex Analysis and its Applications Belgorod State University, Russia, 2009.

17. H. Lewy. Neuer Beweis des analytischen Charakters der Losungen elliptische Differentialgleichungen, Math. Ann 101(1929),609-619.

18. B.Q. Li. Entire solutions of certain partial differential equations and factorization of partial derivatives, Trans. Amer. Math. Soc. 357 (2005), no. 8, 3169-3177 (electronic).

19. B.Q. Li, E.G. Saleeby. Entire solutions of first-order partial differential equations, Complex Variables 48 (2003), No. 8, 657-661.

20. I.G. Petrovskiĭ. Dokl. Akad. Nauk SSSR 17(1937), 343-346; Mat. Sb. (N.S.) 5(47) (1939), 3-70.

21. N. Steinmetz. Über die faktorisierbaren Lösungen gewöhnlicher Differentielgleichungen, Math. Z. 170 (1980), 169-180.

22. W. Stoll. Holomorphic functions of finite orders in several complex variables, Conference Board of the Mathematical Science, Regional Conference Series in Mathematics 21, AMS, 1974.

23. W. Stoll. Value distribution on parabolic spaces, Lecture Notes in Math. 600(1977), Springer-Verlag.

24. Y. Tumura. On the extensions of Borel's theorem and Saxer-Csillag's theorem, Proc. Phys. Math. Soc. Japan (3), 19(1937), 29-35.

25. G. Valiron. Lectures on the general theory of integral functions, Toulouse: Édouard privat, 1923.

26. Z.X. Wang, D.R. Guo. Special functions, World Scientific, 1989.

ТЕОРЕМЫ ЕДИНСТВЕННОСТИ МЕРОМОРФНЫХ ФУНКЦИЙ НЕСКОЛЬКИХ КОМПЛЕКСНЫХ ПЕРЕМЕННЫХ

Пей Чу Ху[1], **Чунг Чун Янг**[2]
[1] Шаньдунский университет,
Жиньян 250100, Шаньдун, П. Р. Китай e-mail: pchu@sdu.edu.cn
[2] Гонконгский университет науки и технологии
Клеар Уотер Бэй, Ковлун, Гонконг e-mail: chungchun.yang@gmail.com

Аннотация. В работе исследуются вопросы существования, единственности и распределния значений мероморфных (или целых) решений линейных дифференциальных уравнений в частных производных второго порядка с полиномиальными коэффициентами.

Ключевые слова: мероморфные функции, однородные линейные дифференциальные уравнения в частных производных, теория Неванлинны распределения ценности.