NORMAL FAMILY AND THE SHARED POLYNOMIALS
OF MEROMORPHIC FUNCTIONS

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Abstract. In the paper, we study the uniqueness and the shared fixed-points of meromorphic functions and prove two main theorems which improve the results of Fang and Fang and Qiu.

Keywords: meromorphic functions, fixed-points, holomorphic coefficients, shared polynomials.

1 Introduction and main results

Schwick [8] was the first to draw a connection between values shared by functions in $\mathcal{F}$ (and their derivatives) and the normality of the family $\mathcal{F}$. Specifically, he showed that if there exist three distinct complex numbers a_1, a_2, a_3 such that f and f' share a_j $(j = 1, 2, 3)$ IM in D for each $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.

In 2006, Wang and Yi [9]proved a uniqueness theorem for entire functions that share a polynomial with their derivatives, as follows

Theorem A. *Let f be a nonconstant entire function, let $Q(z)$ be a polynomial of degree $q \geq 1$, and let $k > q$ be an integer. If f and f' share $Q(z)$ CM, and if $f^{(k)}(z) - Q(z) = 0$ whenever $f(z) - Q(z) = 0$, then $f = f'$.*

According to Bloch˘Ys principle, numerous normality criteria have been obtained by starting from Picard type theorems. On the other hand, by Nevanlinna's famous five point theorem and Montel's theorem, it is interesting to establish normality criteria by using conditions known from a sharing values theorem.

In this note, we obtain the following normal family related to Theorem A.

Theorem 1.1 *Let $\mathcal{F}$ be a family of holomorphic functions in a domain D; let $Q(z)$ be a polynomial of degree $q \geq 1$, and let $k \geq 2q + 1$ be an integer. If, for each $f \in \mathcal{F}$, we have*

$$f(z) = Q(z) \rightleftharpoons f'(z) = Q(z) \Rightarrow f^{(k)} = Q(z),$$

then $\mathcal{F}$ is normal in D.

In order to prove theorem 1.1, we need the following results, which are interesting in their own rights.

Proposition 1. *Let $\mathcal{F}$ be a family of holomorphic functions in a domain D; let $h(z)$ be a polynomial of degree $q \geq 1$; let $k > q$ be an integer. If, for each $f \in \mathcal{F}$, we have $h(z) = 0 \Rightarrow f(z) = 0$ and $f(z) = 0 \rightleftharpoons f'(z) = h(z) \Rightarrow |f^{(k)}(z)| \leq M$, where M is a positive number, then $\mathcal{F}$ is normal in D.*

Proposition 2. *Let $\mathcal{F}$ be a family of holomorphic functions in a domain D; let $Q(z)$ be a polynomial of degree $q \geq 1$; let $k \geq 2q + 1$ be an integer. If, for each $f \in \mathcal{F}$, we have $Q(z) - Q'(z) = 0 \Rightarrow f(z) \neq 0$ and $f(z) = 0 \rightleftharpoons f'(z) = Q(z) - Q'(z) \Rightarrow f^{(k)}(z) = Q(z)$, then $\mathcal{F}$ is normal in D.*

2　Some Lemmas

Lemma 2.1 *[9] Let $\mathcal{F}$ be a family of functions meromorphic in a domain D, all of whose zeros have multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$, if $\mathcal{F}$ is not normal at $z_0 \in D$, then for each $0 \leq \alpha \leq k$ there exist,*

(a) points $z_n \in D$, $z_n \to z_0$;
(b) functions $f_n \in \mathcal{F}$, and
(c) positive number $\rho_n \to 0$ such that $\rho_n^{-\alpha} f_n(z_n + \rho_n \zeta) = g_n(\zeta) \to g(\zeta)$ locally uniformly, where g is a nonconstant meromorphic function in C, all of whose zeros have multiplicity at least k, such that $g^\sharp(\zeta) \leq g^\sharp(0) = kA + 1$. In particular, if $\mathcal{F}$ is a family of holomorphic functions, then $\rho(g) \leq 1$.

Lemma 2.2 *[2] Let g be a nonconstant entire function with $\rho(g) \leq 1$; let $k \geq 2$ be a positive integer; and let a be a nonzero finite value. If $g(z) = 0 \Rightarrow g'(z) = a$, and $g'(z) = a \Rightarrow g^{(k)}(z) = 0$, then $g(z) = a(z - z_0)$, where z_0 is a constant.*

Lemma 2.3 *[2] Let $\mathcal{F}$ be a family of holomorphic functions in a domain D; let $k \geq 2$ be a positive integer; and let α be a function holomorphic in D, such that $\alpha(z) \neq 0$ for $z \in D$. If for every $f \in \mathcal{F}$, $f(z) = 0 \Rightarrow f'(z) = \alpha(z)$ and $f'(z) = \alpha(z) \Rightarrow |f^{(k)}(z)| \leq h$, where h is a positive number, then $\mathcal{F}$ is normal in D.*

In order to prove theorem 1.1, we need some definitions.

Let $\Delta = \{z : |z| < r_0\}$, let $Q(z)$ be a polynomial of degree $q \geq 1$ and $R(z) = Q(z) - Q'(z) = z^m P(z)$, $P(z) \neq 0$, when $z \in \Delta$. Define that $Q_a(z) = Q(z + a)$, where a is a constant, then

$$R_a(z) = Q_a(z) - Q'_a(z) = (z + a)^m P_a(z).$$

Define $\lambda_a = \frac{f' - R_a}{f}$ and $\lambda_a(0) \neq 0$, where f is holomorphic function in Δ. Thus we get $f' = \lambda_a f + R_a = \lambda_{a1} f + \mu_{a1}$. By mathematic induction we get $f^{(k)} = \lambda_{ak} f + \mu_{ak}$ $(k \geq q + 2)$, where

$$\mu_{ak} = R_a\{\lambda_a^{k-1} + P_{k-2}[\lambda_a]\} + R'_a\{\lambda_a^{k-2} + P_{k-3}[\lambda_a]\} + \dots + R_a^{(q)}\{\lambda_a^{k-(q+1)} + P_{k-(q+2)}[\lambda_a]\} \quad (2.1)$$

and $P_{k-2}[\lambda_a], \dots, P_{k-(q+2)}[\lambda_a]$ are differential polynomial in λ_a with degree at most $k - 2, \dots, k - (q + 2)$ respectively. Let $\mu_{ak}(0) - Q_a(0) \neq 0$. Define $\psi_a(0) \neq 0$ where

$$\psi_a = \frac{R_a f^{(k)} - Q_a f'}{f}. \quad (2.2)$$

Define $\varphi_a(0) \neq 0$ where

$$\varphi_a = -[1 + (\frac{1}{\psi_a})'Q_a + \frac{1}{\psi_a}Q'_a]R_a - \frac{1}{\psi_a}Q_a R'_a. \quad (2.3)$$

Lemma 2.4 *Let $f(z)$ be analytic in the disc $\Delta = \{z : |z| < r_0\}$; let a be a complex number such that $|a| < r_0$; let $k \geq q+2$ be a positive integer. If Q_a, R_a, λ_a, μ_{ak}, ψ_a and φ_a are defined as above; if $f(0) \neq 0$, $|f' - R_a|_{z=0} \neq 0$, $R_a = 0 \Rightarrow f(z) \neq 0$ and*

$$f(z) = 0 \rightleftharpoons f'(z) = R_a \Rightarrow f^{(k)}(z) = Q_a,$$

then

$$T(r,f) \leq LD[r,f] + M\log|\frac{f' - R_a}{\psi_a^9 \varphi_a(\mu_{ak} - Q_a)f}|_{z=0} + \log|f(0)|, \quad (2.4)$$

where

$$LD[r,f] = M_1[m(r,\frac{f'}{f}) + m(r,\frac{f^{(k)}}{f}) + m(r,\frac{f^{(k)}}{f'}) + m(r,\frac{f'' - R_a'}{f' - R_a}) + m(r,\frac{f^{(k)}}{f' - R_a})]$$
$$+ M_1 m(r,\frac{f^{(k+1)}}{f' - R_a}) + M_2[m(r,\frac{\psi_a'}{\psi_a}) + m(r,\frac{\lambda_a'}{\lambda_a}) + ... + m(r,\frac{\lambda_a^{(k-2)}}{\lambda_a})]$$
$$+ M_3[m(r,R_a) + m(r,R_a') + ... + m(r,R_a^{(q)}) + m(r,Q_a) + m(r,Q_a') + \log 2],$$

and M_1, M_2, M_3 are positive numbers.

Lemma 2.5 *[1] Let $U(r)$ be a nonnegative, increasing function on an interval $[R_1, R_2](0 < R_1 < R_2 < +\infty)$; let a,b be two positive constants satisfying $b > (a+2)^2$; and let*

$$U(r) < a\{\log^+ U(\rho) + \log \frac{\rho}{\rho - r}\} + b$$

whenever $R_1 < r < \rho < R_2$. Then, for $R_1 < r < R_2$,

$$U(r) < 2a\log \frac{R_2}{R_2 - r} + 2b.$$

Lemma 2.6 *[1] Let $g(z)$ be a transcendental entire function. Then*

$$\limsup_{|z| \to \infty} |z||g^\sharp(z)| = +\infty.$$

3 Proof of Proposition 1

Let $z_0 \in D$. If $h(z_0) \neq 0$, by Lemma 2.3, $\mathcal{F}$ is normal at z_0. Now suppose that $h(z_0) = 0$. Without loss of generality, we may assume that $z_0 = 0$, $\Delta = \{z : |z| < \delta\} \in D$ and $h(z) = z^m b(z)$, where $b(0) = 1$ and $b(z) \neq 0$ $(z \in \Delta)$. We shall prove that $\mathcal{F}$ is normal at $z = 0$.

Let $\mathcal{F}_1 = \{F = \frac{f}{z^m} : f \in \mathcal{F}\}$. We know that if $\mathcal{F}_1$ is normal at $z = 0$, then $\mathcal{F}$ is normal at $z = 0$. Thus we only need to prove $\mathcal{F}_1$ is normal at $z = 0$.

For each $f \in \mathcal{F}$, from $h(z) = 0 \Rightarrow f(z) = 0$, we get $z = 0$ is a zero of f. Thus we have

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + ... \quad (a_n \neq 0) \ (n \geq 1),$$

and

$$f'(z) - h(z) = na_n z^{n-1} + (n+1)a_{n+1} z^n + ... - (z^m + ...) \ .$$

By the assumption $f(z) = 0 \rightleftharpoons f'(z) = h(z)$, we get

$$f = \frac{1}{m+1}z^{m+1} + a_{m+2}z^{m+2} + \dots \tag{3.1}$$

Hence we get $\mathcal{F}_1$ is a family of holomorphic functions in Δ. Next we prove $\forall\, F = \frac{f}{z^m} \in \mathcal{F}_1$, $F = 0 \Rightarrow |F'| \leq M$, where $M = \max_{z \in \Delta}|b(z)| \geq 1$.

Suppose that $F(a_0) = 0$, then $f(a_0) = 0$.

If $a_0 \neq 0$, we get $F'(a_0) = \frac{f'(a_0)}{a_0^m} - \frac{m f(a_0)}{a_0^{m+1}} = b(a_0)$.

If $a_0 = 0$, we get $F'(a_0) = b(a_0) - \frac{m}{m+1} = 1 - \frac{m}{m+1} = \frac{1}{m+1}$. Thus we get $F = 0 \Rightarrow |F'| \leq M$.

Now we prove that $\mathcal{F}_1$ is normal at $z = 0$. Suppose on the contrary that $\mathcal{F}_1$ is not normal at $z = 0$, then by Lemma 2.1, we can find $z_n \to 0$, $\rho_n \to 0$ and $f_n \in \mathcal{F}$ such that

$$g_n(\zeta) = \rho_n^{-1}\frac{f_n(z_n + \rho_n\zeta)}{(z_n + \rho_n\zeta)^m} \to g(\zeta) \tag{3.2}$$

locally uniformly on C, where g is a nonconstant entire function such that $g^\sharp(\zeta) \leq g^\sharp(0) = M+1$. In particular $\rho(g) \leq 1$. Without loss of generality, we assume that $\lim\limits_{n\to\infty}\frac{z_n}{\rho_n} = c \in C$. In the following we consider two cases.

Case 1: $c = \infty$. Then $z_n \neq 0$ and $\frac{\rho_n}{z_n} \to 0$ as $n \to \infty$. Set $h_n(\zeta) = \rho_n^{-1}\frac{f_n(z_n+\rho_n\zeta)}{z_n^m}$. Then by (3.2), we get

$$h_n(\zeta) = \rho_n^{-1}\frac{f_n(z_n + \rho_n\zeta)}{(z_n + \rho_n\zeta)^m}(1 + \frac{\rho_n}{z_n}\zeta)^m \to g(\zeta). \tag{3.3}$$

We claim:

$$g(\zeta) = 0 \Rightarrow g'(\zeta) = 1 \text{ and } g'(\zeta) = 1 \Rightarrow g^{(k)}(\zeta) = 0.$$

Suppose that $g(\zeta_0) = 0$, then by Hurwitz's Theorem, there exist ζ_n, $\zeta_n \to \zeta_0$, such that (for n sufficiently large)

$$h_n(\zeta_n) = \rho_n^{-1}\frac{f_n(z_n + \rho_n\zeta_n)}{z_n^m} = 0.$$

Thus $f_n(z_n + \rho_n\zeta_n) = 0$, by the assumption we have $f'_n(z_n + \rho_n\zeta_n) = (z_n + \rho_n\zeta_n)^m b(z_n + \rho_n\zeta_n)$, then we derive that

$$g'(\zeta_0) = \lim_{n\to\infty}\frac{f'_n(z_n + \rho_n\zeta_n)}{z_n^m} = \lim_{n\to\infty}b(z_n + \rho_n\zeta_n)(1 + \frac{\rho_n}{z_n}\zeta_n)^m = b(0) = 1.$$

Thus $g(\zeta) = 0 \Rightarrow g'(\zeta) = 1$. Next we prove $g'(\zeta) = 1 \Rightarrow g^{(k)}(\zeta) = 0$. By (3.3) we know

$$\frac{f'_n(z_n + \rho_n\zeta)}{(z_n + \rho_n\zeta)^m b(z_n + \rho_n\zeta)} = \frac{f'_n(z_n + \rho_n\zeta)}{z_n^m(1 + \frac{\rho_n}{z_n}\zeta)^m b(z_n + \rho_n\zeta)} \to g'(\zeta)$$

We suppose that $g'(\zeta_0) = 1$, obviously $g' \not\equiv 1$, for otherwise $g^\sharp(0) \leq g'(0) = 1 < M + 1$, which is a contradiction. Hence by Hurwitz's Theorem, there exist ζ_n, $\zeta_n \to \zeta_0$, such that (for n sufficiently large)

$$\frac{f'_n(z_n + \rho_n\zeta_n)}{(z_n + \rho_n\zeta_n)^m b(z_n + \rho_n\zeta_n)} = 1,$$

Thus $f'_n(z_n + \rho_n\zeta_n) = h(z_n + \rho_n\zeta_n)$, by the assumption we get $|f_n^{(k)}(z_n + \rho_n\zeta_n)| \leq \overline{M}$. Then

$$|g^{(k)}(\zeta_0)| = \lim_{n\to\infty} |\frac{\rho_n^{k-1}}{z_n^m} f_n^{(k)}(z_n + \rho_n\zeta_n)| \leq \lim_{n\to\infty} |\frac{\rho_n^{k-1}}{z_n^m}|\overline{M} = 0.$$

Thus we prove the Claim. By Lemma 2.2, we get $g = \zeta - b$, where b is a constant. Thus we have $g^\sharp(0) \leq 1 < M + 1$, which is a contradiction.

Case 2: $c \neq \infty$. We set

$$G_n(\zeta) = \frac{f_n(\rho_n\zeta)}{\rho_n^{m+1}}. \tag{3.4}$$

Then

$$G_n(\zeta) = \rho_n^{-1}\frac{f_n(z_n + \rho_n(\zeta - \frac{z_n}{\rho_n}))}{(z_n + \rho_n(\zeta - \frac{z_n}{\rho_n}))^m}\zeta^m \to g(\zeta - c)\zeta^m = G(\zeta)$$

We know that $z = 0$ is zero of f_n with multiplicity $m + 1$, then we get 0 is a zero of $G(\zeta)$ with multiplicity $m + 1$ and

$$G^{(m+1)}(0) = \lim_{n\to\infty} G_n^{(m+1)}(0) = m! \tag{3.5}$$

If $G'(\zeta) \equiv \zeta^m$, we derive that $G(\zeta) = \frac{1}{m+1}\zeta^{m+1}$. Hence we obtain $g(\zeta) = \frac{1}{m+1}(\zeta + c)$. It follows that $g^\sharp(0) \leq \frac{1}{m+1} < M + 1$, a contradiction. Thus $G'(\zeta) \not\equiv \zeta^m$. Using the same argument as in the proof of Case 1, we get

$$G(\zeta) = 0 \Leftrightarrow G'(\zeta) = \zeta^m \text{ and } G'(\zeta) = \zeta^m \Rightarrow \begin{cases} G^{(k)}(\zeta) \leq \overline{M}, & k = m + 1, \\ G^{(k)}(\zeta) = 0, & k \geq m + 2. \end{cases}$$

Suppose $G(\zeta)$ is a polynomial. Let

$$G(\zeta) = b_q\zeta^q + b_{q-1}\zeta^{q-1} + ... + b_{m+1}\zeta^{m+1} \quad (b_{m+1} \neq 0). \tag{3.6}$$

From $G(\zeta) = 0 \Leftrightarrow G'(\zeta) = \zeta^m$, we get

$$G(\zeta) = \zeta(G'(\zeta) - \zeta^m)A. \tag{3.7}$$

Thus, by (3.6) and (3.7) we have $G(\zeta) = b_q\zeta^q - \frac{1}{q-(m+1)}\zeta^{m+1}$ $(q \geq m + 2)$ or $G(\zeta) = A\zeta^{m+1}$, and from (3.5), we get $G(\zeta) = \frac{1}{m+1}\zeta^{m+1}$. Then $G'(\zeta) \equiv \zeta^m$, a contradiction.

In the following we assume that $G(\zeta)$ is a transcendental entire function.

Let us consider the family $T = \{t_n : t_n(\zeta) = \frac{G((2^m)^n\zeta)}{(2^m)^{(m+1)n}}\}$, we see that t_n is a entire function satisfying

$$t_n(\zeta) = 0 \Leftrightarrow t'_n(\zeta) = \zeta^m \Rightarrow \begin{cases} t_n(\zeta) \leq \overline{M}, & k = m + 1, \\ t_n(\zeta) = 0, & k \geq m + 2. \end{cases}$$

By Lemma 2.3, we have T is normal on $D_1 = \{\zeta : (1/2)^m \leq |\zeta| \leq 2^m\}$, thus there exists a M_1 satisfying

$$t_n^\sharp(\zeta) = \frac{(2^m)^{(m+2)n}|G'((2^m)^n\zeta)|}{(2^m)^{2(m+1)n} + |(G(2^m)^n\zeta)|^2} \leq M_1.$$

Set $r(z) = \frac{G(z)}{z^{m+1}}$, then $r(z)$ is a transcendental entire function. We know that for each $z \in C$, there exists a integer n such that $z = (2^m)^n\zeta$, where $(1/2)^m \leq |\zeta| \leq 2^m$. We can get

$$|z|r^\sharp(z) \leq (2^m)^{3m+4}t_n^\sharp(\zeta) + \frac{m+1}{2} \leq (2^m)^{3m+4}M_1 + \frac{m+1}{2}. \tag{3.8}$$

From Lemma 2.6, we get

$$\limsup_{|z|\to\infty} |z| |r^{\sharp}(z)| = +\infty,$$

which contradicts with (3.8).

Thus, we prove that $\mathcal{F}_1$ is normal at $z = 0$. Hence $\mathcal{F}$ is normal at $z = 0$.

4   Proof of Proposition 2

Let $z_0 \in D$. If $[Q(z) - Q'(z)]|_{z=z_0} \neq 0$, by Lemma 2.3, $\mathcal{F}$ is normal at z_0. Now suppose that $[Q(z) - Q'(z)]|_{z=z_0} = 0$. Without loss of generality, we may assume that $z_0 = 0$, $\Delta = \{z : |z| < \delta\} \in D$ and $R(z) = Q(z) - Q'(z) = z^m P(z)$, where $P(z) \neq 0$ $(z \in \Delta)$. We shall prove that $\mathcal{F}$ is normal at $z = 0$.

Suppose on the contrary that $\mathcal{F}$ is not normal at $z = 0$, then by Lemma 2.1, we can find $z_n \to 0$, $\rho_n \to 0$ and $f_n \in \mathcal{F}$ such that

$$g_n(\zeta) = f_n(z_n + \rho_n \zeta) \to g(\zeta) \tag{4.1}$$

locally uniformly on C, where g is a nonconstant entire function. Without loss of generality, we assume that

$$\lim_{n\to\infty} \frac{z_n}{\rho_n} = c \in C.$$

First, we shall prove that $g(\zeta)$ is a transcendental entire function. In fact, we only need to prove that $g(\zeta) \neq 0$. The argument given in the proof of Proposition 1 shows that

$$g(\zeta) = 0 \Rightarrow g'(\zeta) = 0,$$

thus g only has multiple zeros. Suppose ζ_0 is a zero of $g(\zeta)$ with multiplicity $s(\geq 2)$, then $g^{(s)}(\zeta_0) \neq 0$. Thus there exists a positive number δ, such that

$$g(\zeta) \neq 0, \ g'(\zeta) \neq 0, \ g^{(s)}(\zeta) \neq 0 \tag{4.2}$$

on $D_\delta^o = \{\zeta : 0 < |\zeta - \zeta_0| < \delta\}$. By (4.1) and Rouché theorem, there exist $\zeta_{n,j}(j = 1, 2, ..., s)$ on $D_{\delta/2} = \{\zeta : |\zeta - \zeta_0| < \delta/2\}$ such that

$$g_n(\zeta_{n,j}) = f_n(z_n + \rho_n \zeta_{n,j}) = 0 \ (j = 1, 2, ..., s).$$

It follows from $R(z) = 0 \Rightarrow f(z) \neq 0$ and $f(z) = 0 \rightleftharpoons f'(z) = R(z)$ that $f'_n(z_n + \rho_n \zeta_{n,j}) = R(z_n + \rho_n \zeta_{n,j}) \neq 0$. Thus

$$g'_n(\zeta_{n,j}) = \rho_n f'_n(z_n + \rho_n \zeta_{n,j}) = \rho_n R(z_n + \rho_n \zeta_{n,j}) \neq 0 (j = 1, 2, ..., s),$$

so each $\zeta_{n,j}$ is a simple zero of $g_n(\zeta)$, that is $\zeta_{n,j} \neq \zeta_{n,i}(1 \leq i \neq j \leq s)$. On the other hand

$$\lim_{n\to\infty} g'_n(\zeta_{n,j}) = \lim_{n\to\infty} \rho_n R(z_n + \rho_n \zeta_{n,j}) = 0$$

From (4.2), we get

$$\lim_{n\to\infty} \zeta_{n,j} = \zeta_0 \ (j = 1, 2, ..., s).$$

Noting that (4.2) and $g'_n(\zeta) - \rho_n R(z_n + \rho_n \zeta)$ has s zeros $\zeta_{n,j}(j = 1, 2, ..., s)$ in $D_{\delta/2}$, then ζ_0 is a zero of $g'(\zeta)$ of multiplicity s, and thus $g^{(s)}(\zeta_0) = 0$. This is a contradiction. Hence $g(\zeta) \neq 0$ and $g(\zeta)$ is a transcendental entire function.

Now we consider five cases.

Case 1: There exist infinitely many $\{n_j\}$ such that

$$f'_{n_j}(z_{n_j} + \rho_{n_j}\zeta) \equiv R(z_{n_j} + \rho_{n_j}\zeta).$$

It follows that $g'_{n_j}(\zeta) \equiv \rho_{n_j} R(z_{n_j} + \rho_{n_j}\zeta)$. Let $j \to \infty$, we deduce that $g'(\zeta) \equiv 0$, which contradicts that g is transcendental.

Case 2: There exist infinitely many $\{n_j\}$ such that $\psi_{n_j}(z_{n_j} + \rho_{n_j}\zeta) \equiv 0$, where $\psi_n = \frac{Rf_n^{(k)} - Qf'_n}{f_n}$. Thus we have

$$(z_{n_j} + \rho_{n_j}\zeta)^m P(z_{n_j} + \rho_{n_j}\zeta)\frac{g_{n_j}^{(k)}(\zeta)}{\rho_{n_j}^k} \equiv Q(z_{n_j} + \rho_{n_j}\zeta)\frac{g'_{n_j}(\zeta)}{\rho_{n_j}}$$

and

$$\frac{g_{n_j}^{(k)}(\zeta)}{g'_{n_j}(\zeta)} = \frac{Q(z_{n_j} + \rho_{n_j}\zeta)\rho_{n_j}^{(k-(m+1))}}{P(z_{n_j} + \rho_{n_j}\zeta)(\frac{z_{n_j}}{\rho_{n_j}} + \zeta)^m}.$$

Noting that $k \geq 2q + 1 \geq 2m + 1$, let $j \to \infty$, we deduce that $g^{(k)}(\zeta) \equiv 0$, which contradicts that g is transcendental.

Case 3: There exist infinitely many $\{n_j\}$ such that $\varphi_{n_j}(z_{n_j} + \rho_{n_j}\zeta) \equiv 0$, where

$$\varphi_n = -[1 + (\frac{1}{\psi_n})'Q + \frac{1}{\psi_n}Q']R - \frac{1}{\psi_n}QR'.$$

and ψ_n is defined as above. Let

$$\Gamma(\zeta) = \rho_{n_j}^{k-(m+1)}[(\frac{z_{n_j}}{\rho_{n_j}} + \zeta)^{m-1}P_1(z_{n_j} + \rho_{n_j}\zeta)g_{n_j}^{(k)}(\zeta) + (\frac{z_{n_j}}{\rho_{n_j}} + \zeta)^m P(z_{n_j} + \rho_{n_j}\zeta)g_{n_j}^{(k+1)}(\zeta)$$
$$- \rho_{n_j}^{k-m}Q'(z_{n_j} + \rho_{n_j}\zeta)g'_{n_j}(\zeta) - \rho_{n_j}^{k-(m+1)}Q(z_{n_j} + \rho_{n_j}\zeta)g''_{n_j}(\zeta)].$$

Then

$$\frac{-\Gamma(\zeta)Q(z_{n_j} + \rho_{n_j}\zeta)}{(\frac{z_{n_j}}{\rho_{n_j}} + \zeta)^m P(z_{n_j} + \rho_{n_j}\zeta)g_{n_j}^{(k)}(\zeta) - Q(z_{n_j} + \rho_{n_j}\zeta)\rho_{n_j}^{k-(m+1)}g'_{n_j}(\zeta)} + Q'(z_{n_j} + \rho_{n_j}\zeta)\rho_{n_j}^{k-m}$$
$$+ (\frac{z_{n_j}}{\rho_{n_j}} + \zeta)^m P(z_{n_j} + \rho_{n_j}\zeta)\frac{g_{n_j}^{(k)}(\zeta)}{g_{n_j}(\zeta)} = \frac{Q(z_{n_j} + \rho_{n_j}\zeta)\rho_{n_j}^{k-(m+1)}P_1(z_{n_j} + \rho_{n_j}\zeta)}{(\frac{z_{n_j}}{\rho_{n_j}} + \zeta)P(z_{n_j} + \rho_{n_j}\zeta)},$$

where $R'(z) = z^{m-1}P_1(z)$.

Thus, let $j \to \infty$, we get $g^{(k)}(\zeta) \equiv 0$, which contradicts that g is transcendental.

Case 4: There exist infinitely many $\{n_j\}$ such that $\mu_{kn_j}(z_{n_j} + \rho_{n_j}\zeta) \equiv Q(z_{n_j} + \rho_{n_j}\zeta)$ where

$$\mu_{kn} = R\{\lambda_n^{k-1} + P_{k-2}[\lambda_n]\} + R'\{\lambda_n^{k-2} + P_{k-3}[\lambda_n]\} + ... + R^{(q)}\{\lambda_n^{k-(q+1)} + P_{k-(q+2)}[\lambda_n]\},$$

and $\lambda_n = \frac{f'_n - R}{f_n}$. Thus, let $j \to \infty$, we get

$$(\frac{g'}{g})^{k-(m+1)}[(c + \zeta)^m P(0)(\frac{g'}{g})^m + R^{(m)}(0)] \equiv 0.$$

Hence $g' \equiv 0$ or $(c + \zeta)^m P(0)(\frac{g'}{g})^m + R^{(m)}(0) \equiv 0$, which contradicts that g is a transcendental entire function.

Case 5: There exist finitely many $\{n_j\}$ such that $f'_{n_j}(z_{n_j} + \rho_{n_j}\zeta) \equiv R(z_{n_j} + \rho_{n_j}\zeta)$, $\psi_{n_j}(z_{n_j} + \rho_{n_j}\zeta) \equiv 0$, $\varphi_{n_j}(z_{n_j} + \rho_{n_j}\zeta) \equiv 0$ and $\mu_{kn_j}(z_{n_j} + \rho_{n_j}\zeta) \equiv Q(z_{n_j} + \rho_{n_j}\zeta)$.

For all n we may suppose that $f'_{n_j}(z_{n_j} + \rho_{n_j}\zeta) \not\equiv R(z_{n_j} + \rho_{n_j}\zeta)$, $\psi_{n_j}(z_{n_j} + \rho_{n_j}\zeta) \not\equiv 0$, $\varphi_{n_j}(z_{n_j} + \rho_{n_j}\zeta) \not\equiv 0$ and $\mu_{kn_j}(z_{n_j} + \rho_{n_j}\zeta) \not\equiv Q(z_{n_j} + \rho_{n_j}\zeta)$.

Take $\zeta_0 \in C$ such that $g^{(j)}(\zeta_0) \neq 0$ $(j = 0, 1, ..., k)$. In case $c \neq \infty$, choose ζ_0 to satisfy the additional conditions that $\zeta_0 \neq -c$ and

$$(c + \zeta_0)^m P(0)(\frac{g'(\zeta_0)}{g(\zeta_0)})^m + R^{(m)}(0) \neq 0.$$

Noting that $k \geq 2q + 1 \geq 2m + 1$, this facts imply that $K_n =\to 0$ as $n \to \infty$, so that $\log K_n \to -\infty$ as $n \to \infty$.

For $n = 1, 2, 3, ...$, put

$$h_n(z) = f_n(z_n + \rho_n\zeta_0 + z)$$

Since $z_n + \rho_n\zeta_0 \to 0$ as $n \to \infty$, it follows that (for sufficiently large n) h_n is defined and holomorphic on $|z| < \frac{1}{2}$. Denote

$$a_n = z_n + \rho_n\zeta_0.$$

Then, for sufficiently large n, $h_n(0) \neq 0$, $h'_n(0) - R_{a_n}(0) \neq 0$. By the assumption we get

$$h_n(z) = 0 \rightleftharpoons h'_n(z) = R_{a_n} \Rightarrow h_n^{(k)}(z) = Q_{a_n}.$$

Let $a = a_n$ and $f(z) = h_n(z)$ in Lemma 2.4, then we get

$$h_n(-a_n) = f_n(0) \neq 0, \psi_{a_n}(0) = \psi_n(a_n) \neq 0, \; \varphi_{a_n}(0) = \varphi_n(a_n) \neq 0,$$

$$[\mu_{a_n k} - Q_{a_n}]|_{z=0} = [\mu_{kn} - Q]|_{z=a_n} \neq 0,$$

thus $h_n(z)$ satisfies the assumption of Lemma 2.4.

Now applying Lemma 2.4 with $r_0 = \frac{1}{2}$, and noting that the last three terms in (2.4) are bounded for $0 < r < 1/3$, we obtain that, for sufficiently large n and $0 < r < 1/3$,

$$T(r, h_n) \leq M_1[m(r, \frac{h'_n}{h_n}) + m(r, \frac{h_n^{(k)}}{h_n}) + m(r, \frac{h_n^{(k)}}{h'_n}) + m(r, \frac{h''_n - R'_{a_n}}{h'_n - R_{a_n}}) + m(r, \frac{h_n^{(k)}}{h'_n - R_{a_n}})]$$

$$+ M_1 m(r, \frac{h_n^{(k+1)}}{h'_n - R_{a_n}}) + M_2[m(r, \frac{\psi'_{a_n}}{\psi_{a_n}}) + m(r, \frac{\lambda'_{a_n}}{\lambda_{a_n}}) + ... + m(r, \frac{\lambda_{a_n}^{(k-2)}}{\lambda_{a_n}})].$$

We can obtain, for $0 < r < \tau < 1/3$,

$$T(r, h_n) \leq C_k\{1 + \log^+\frac{1}{r} + \log^+\frac{1}{\tau - r} + \log^+ T(\tau, h_n)$$
$$+ \log^+ T(\tau, h'_n) + \log^+ T(\tau, \psi_{a_n}) + \log^+ T(\tau, \lambda_{a_n})\}.$$

$$(4.3)$$

Observe that $T(\tau, h'_n) = m(\tau, h'_n) \leq m(\tau, h_n) + m(r, \frac{h'_n}{h_n})$, hence for $1/4 < r < \rho < 1/3$ with $\tau = (r + \rho)/2$. From the above we obtain

$$T(r, h_n) \leq C_k(1 + \log^+\frac{1}{\rho - r} + \log^+ T(\rho, h_n)).$$

By Lemma 2.5 it then follows that $T(1/4, h_n) \leq A$, where A is a constant independent of n. Thus $f_n(z)$ is uniformly bounded for sufficiently large n and $|z| < 1/8$. However, from $\rho_n^2 f_n''(z_n + \rho_n \zeta_0) = g_n''(\zeta_0) \to g''(\zeta_0) \neq 0$ we see that $f(z)$ cannot bounded in $|z| < 1/8$. This is a contradiction, so the proof is complete.

5　Proof of Theorem 1.1

Let $\mathcal{G} = \{g = f - Q : f \in \mathcal{F}\}$ and $R(z) = Q(z) - Q'(z)$. Obviously, $\mathcal{G}$ is normal in D if and only if $\mathcal{F}$ is normal in D. It follows from our assumption that, for any $g \in \mathcal{G}$, we have

$$g = 0 \rightleftharpoons g' = R \Rightarrow g^{(k)} = Q. \tag{5.1}$$

Let $z_0 \in D$. Now we prove that $\mathcal{G}$ is normal at z_0. Let $\{g_n\} \subset \mathcal{G}$ be a sequence.

If $R(z_0) \neq 0$, then there exists a positive number δ such that $\Delta_\delta = \{z \in D : |z - z_0| < \delta\} \subset D$ and $R(z) \neq 0$ in Δ_δ. Then by Lemma 2.3, $\{g_n\}$ is normal at z_0.

If $R(z_0) = 0$, then there exists a positive number δ such that $\Delta_\delta = \{z \in D : |z - z_0| < \delta\} \subset D$ and $R(z) \neq 0$ in $\Delta_\delta \backslash \{z_0\}$. Suppose $\{g_n\}$ has a subsequence say, without loss of generality, itself, such that $g_n(z_0) = 0$, then $\{g_n\}$ is normal at z_0 by Proposition 1. Suppose $g_n(z_0) \neq 0$ for all but finite many of $\{g_n\}$, then $\{g_n\}$ is normal at z_0 by Proposition 2.

Thus $\mathcal{F}$ is normal in D and hence Theorem 1.1 is proved.

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НОРМАЛЬНОЕ СЕМЕЙСТВО И РАСПРЕДЕЛЕННЫЕ МНОГОЧЛЕНЫ МЕРОМОРФНЫХ ФУНКЦИЙ

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Аннотация. В работе изучается единственность и разделение неподвижной точки мероморфных функций. Доказаны две основные теоремы, улучшающие результаты Ванга и Кью.

Ключевые слова: мероморфная функция, неподвижная точка, распределенные многочлены.