УДК 517.55

SPARSE HYPERGEOMETRIC SYSTEMS

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Abstract. We study the approach to the theory of hypergeometric functions in several variables via a generalization of the Horn system of differential equations. A formula for the dimension of its solution space is given. Using this formula we construct an explicit basis in the space of holomorphic solutions to the generalized Horn system under some assumptions on its parameters.

Keywords: hypergeometric functions, Horn system of differential equations, Mellin system.

1 Introduction

There exist several approaches to the notion of a hypergeometric function depending on several complex variables. It can be defined as the sum of a power series of a certain form (such series are known as Γ-series) [10], as a solution to a system of partial differential equations [9], [11], [1], or as a Mellin-Barnes integral [15]. In the present paper we study the approach to the theory of hypergeometric functions via a generalization of the Horn system of differential equations. We consider the system of partial differential equations of hypergeometric type

$$x^{u_i} P_i(\theta) y(x) = Q_i(\theta) y(x), \; i = 1, \dots, n, \tag{1.1}$$

where the vectors $u_i = (u_{i1}, \dots, u_{in}) \in \mathbb{Z}^n$ are assumed to be linearly independent, P_i, Q_i are nonzero polynomials in n complex variables and $\theta = (\theta_1, \dots, \theta_n)$, $\theta_i = x_i \frac{\partial}{\partial x_i}$. We use the notation $x^{u_i} = x_1^{u_{i1}} \dots x_n^{u_{in}}$. If $\{u_i\}_{i=1}^n$ form the standard basis of the lattice $\mathbb{Z}^n$ then the system (1.1) coincides with a classical system of partial differential equations which goes back to Horn and Mellin (see [13] and § 1.2 of [10]). In the present paper the system (1.1) is referred to as the *sparse hypergeometric system* (or generalized Horn system) since, in general, its series solutions might have many gaps.

A sparse hypergeometric system can be easily reduced to the classical Horn system by a monomial change of variables. The main purpose of the present paper is to discuss the relation between the sparse and the classical case in detail for the benefit of a reader interested in explicit solutions of hypergeometric $\mathcal{D}$-modules. We also furnish several examples which illustrate crucial properties of the singularities of multivariate hypergeometric functions. Most of the statements in this article are parallel to or follow from the results in [16].

A typical example of a sparse hypergeometric system is the Mellin system of equations (see [7]). One of the reasons for studying sparse hypergeometric systems is the fact that knowing the structure of solutions to (1.1) allows one to investigate the so-called amoeba of the singular locus of a solution to (1.1). The notion of amoebas was introduced by Gelfand, Kapranov and Zelevinsky (see [12], Chapter 6, § 1). Given a mapping $f(x)$, its amoeba $\mathcal{A}_f$ is the image of the hypersurface $f^{-1}(0)$ under the map $(x_1, \dots, x_n) \mapsto (\log|x_1|, \dots, \log|x_n|)$. In section 5 we use the

The author was supported by the Russian Foundation for Basic Research, grant 09-01-00762-a, by grant no. 26 for scientific research groups of Siberian Federal University and by the "Dynasty"foundation.

results on the structure of solutions to (1.1) for computing the number of connected components of the complement of amoebas of some rational functions. The problem of describing the class of rational hypergeometric functions was studied in a different setting in [5], [6]. The definition of a hypergeometric function used in these papers is based on the Gelfand-Kapranov-Zelevinsky system of differential equations [9], [10], [11].

Solutions to (1.1) are closely related to the notion of a generalized Horn series which is defined as a formal (Laurent) series

$$y(x) = x^{\gamma} \sum_{s \in \mathbb{Z}^n} \varphi(s) x^s, \tag{1.2}$$

whose coefficients $\varphi(s)$ are characterized by the property that $\varphi(s+u_i) = \varphi(s)R_i(s)$. Here $R_i(s)$ are rational functions. We also use notations $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{C}^n$, $\operatorname{Re}\gamma_i \in [0,1)$, $x^s = x_1^{s_1} \ldots x_n^{s_n}$. In the case when $\{u_i\}_{i=1}^n$ form the standard basis of $\mathbb{Z}^n$ we get the definition of the classical Horn series (see [10], § 1.2).

In the case of two or more variables the generalized Horn system (1.1) is in general not solvable in the class of series (1.2) without additional assumptions on the polynomials P_i, Q_i. In section 2 we investigate solvability of hypergeometric systems of equations and describe supports of solutions to the generalized Horn system. The necessary and sufficient conditions for a formal solution to the system (1.1) in the class (1.2) to exist are given in Theorem 2.1.

In section 3 we consider the $\mathcal{D}$-module associated with the generalized Horn system. We give a formula which allows one to compute the dimension of the space of holomorphic solutions to (1.1) at a generic point under some additional assumptions on the system under study (Theorem 3.3). We give also an estimate for the dimension of the solution space of (1.1) under less restrictive assumptions on the parameters of the system (Corollary 3.4).

In section 4 we consider the case when the polynomials P_i, Q_i can be factorized up to polynomials of degree 1 and construct an explicit basis in the space of holomorphic solutions to some systems of the Horn type. We show that in the case when $R_i(s+u_j)R_j(s) = R_j(s+u_i)R_i(s)$, $Q_i(s+u_j) = Q_i(s)$ and $\deg Q_i(s) > \deg P_i(s)$, $i,j = 1,\ldots,n$, $i \neq j$, there exists a basis in the space of holomorphic solutions to (1.1) consisting of series (1.2) if the parameters of the system under study are sufficiently general (Theorem 4.1).

In section 5 we apply the results on the generalized Horn system to the problem of describing the complement of the amoeba of a rational function. We show how Theorem 2.1 can be used for studying Laurent series developments of a rational solution to (1.1). A class of rational hypergeometric functions with minimal number of connected components of the complement of the amoeba is described.

2 Supports of solutions to sparse hypergeometric systems

Suppose that the series (1.2) represents a solution to the system (1.1). Computing the action of the operator $x^{u_i}P_i(\theta) - Q_i(\theta)$ on this series we arrive at the following system of difference equations

$$\varphi(s+u_i)Q_i(s+\gamma+u_i) = \varphi(s)P_i(s+\gamma), \; i = 1,\ldots,n. \tag{2.1}$$

The system (2.1) is equivalent to (1.1) as long as we are concerned with those solutions to the generalized Horn system which admit a series expansion of the form (1.2). Let $\mathbb{Z}^n + \gamma$ denote the shift in $\mathbb{C}^n$ of the lattice $\mathbb{Z}^n$ with respect to the vector γ. Without loss of generality we assume that the polynomials $P_i(s), Q_i(s+u_i)$ are relatively prime for all $i = 1, \ldots, n$. In this section we shall describe nontrivial solutions to the system (2.1) (i.e. those ones which are not equal to zero identically). While looking for a solution to (2.1) which is different from zero on some subset S of $\mathbb{Z}^n$ we shall assume that the polynomials $P_i(s), Q_i(s)$, the set S and the vector γ satisfy the condition

$$|P_i(s+\gamma)| + |Q_i(s+\gamma+u_i)| \neq 0, \tag{2.2}$$

for any $s \in S$ and for all $i = 1, \ldots, n$. That is, for any $s \in S$ the equality $P_i(s+\gamma) = 0$ implies that $Q_i(s+\gamma+u_i) \neq 0$ and $Q_i(s+\gamma+u_i) = 0$ implies $P_i(s+\gamma) \neq 0$.

The system of difference equations (2.1) is in general not solvable without further restrictions on P_i, Q_i. Let $R_i(s)$ denote the rational function $P_i(s)/Q_i(s+u_i)$, $i = 1, \ldots, n$. Increasing the argument s in the ith equation of (2.1) by u_j and multiplying the obtained equality by the jth equation of (2.1), we arrive at the relation $\varphi(s+u_i+u_j)/\varphi(s) = R_i(s+u_j)R_j(s)$. Analogously, increasing the argument in the jth equation of (2.1) by u_i and multiplying the result by the ith equation of (2.1), we arrive at the equality $\varphi(s+u_i+u_j)/\varphi(s) = R_j(s+u_i)R_i(s)$. Thus the conditions

$$R_i(s+u_j)R_j(s) = R_j(s+u_i)R_i(s), \;\; i, j = 1, \ldots, n \tag{2.3}$$

are in general necessary for (2.1) to be solvable. The conditions (2.3) will be referred to as the compatibility conditions for the system (2.1). Throughout this paper we assume that the polynomials P_i, Q_i defining the generalized Horn system (1.1) satisfy (2.3).

Let U denote the matrix whose rows are the vectors $u_1, \ldots, u_n$. A set $S \subset \mathbb{Z}^n$ is said to be *U-connected* if any two points in S can be connected by a polygonal line with the vectors $u_1, \ldots, u_n$ as sides and vertices in S. Let $\varphi(s)$ be a solution to (2.1). We define the *support* of $\varphi(s)$ to be the subset of the lattice $\mathbb{Z}^n$ where $\varphi(s)$ is different from zero. A formal series $x^\gamma \sum_{s \in \mathbb{Z}^n} \varphi(s) x^s$ is called a *formal solution* to the system (1.1) if the function $\varphi(s)$ satisfies the equations (2.1) at each point of the lattice $\mathbb{Z}^n$. The following Theorem gives necessary and sufficient conditions for a solution to the system (2.1) supported in some set $S \subset \mathbb{Z}^n$ to exist.

Theorem 2.1 *For $S \subset \mathbb{Z}^n$ define*

$$S_i' = \{s \in S : s+u_i \notin S\}, \;\; S_i'' = \{s \notin S : s+u_i \in S\}, \;\; i = 1, \ldots, n.$$

Suppose that the conditions (2.2) are satisfied on S. Then there exists a solution to the system (2.1) supported in S if and only if the following conditions are fulfilled:

$$P_i(s+\gamma)|_{S_i'} = 0, \;\; Q_i(s+\gamma+u_i)|_{S_i''} = 0, \;\; i = 1, \ldots, n, \tag{2.4}$$

$$P_i(s+\gamma)|_{S \setminus S_i'} \neq 0, \;\; Q_i(s+\gamma+u_i)|_S \neq 0, \;\; i = 1, \ldots, n. \tag{2.5}$$

The proof of this theorem is analogous to the proof of Theorem 1.3 in [16]. Theorem 2.1 will be used in section 4 for constructing an explicit basis in the space of holomorphic solutions to the generalized Horn system in the case when $\deg Q_i > \deg P_i$ and $Q_i(s+u_j) = Q_i(s)$, $i, j = 1, \ldots, n$, $i \neq j$. In the next section we compute the dimension of the space of holomorphic solutions to (1.1) at a generic point.

3 Holomorphic solutions to sparse systems

Let G_i denote the differential operator $x^{u_i}P_i(\theta) - Q_i(\theta)$, $i = 1, \ldots, n$. Let $\mathcal{D}$ be the Weyl algebra in n variables [3], and define $\mathcal{M} = \mathcal{D}/\sum_{i=1}^{n}\mathcal{D}G_i$ to be the left $\mathcal{D}$-module associated with the system (1.1). Let $R = \mathbb{C}[z_1, \ldots, z_n]$ and $R[x] = R[x_1, \ldots, x_n] = \mathbb{C}[x_1, \ldots, x_n, z_1, \ldots, z_n]$. We make $R[x]$ into a left $\mathcal{D}$-module by defining the action of ∂_j on $R[x]$ by

$$\partial_j = \frac{\partial}{\partial x_j} + z_j. \tag{3.1}$$

Let $\Phi : \mathcal{D} \to R[x]$ be the $\mathcal{D}$-linear map defined by

$$\Phi(x_1^{a_1} \ldots x_n^{a_n}\partial_1^{b_1} \ldots \partial_n^{b_n}) = x_1^{a_1} \ldots x_n^{a_n} z_1^{b_1} \ldots z_n^{b_n}. \tag{3.2}$$

It is easily checked that Φ is an isomorphism of $\mathcal{D}$-modules. In this section we establish some properties of linear operators acting on $R[x]$. We aim to construct a commutative family of $\mathcal{D}$-linear operators $W_i : R[x] \to R[x]$, $i = 1, \ldots, n$ which satisfy the equality $\Phi(G_i) = W_i(1)$. The crucial point which requires additional assumptions on the parameters of the system (1.1) is the commutativity of the family $\{W_i\}_{i=1}^{n}$ which is needed for computing the dimension (as a $\mathbb{C}$-vector space) of the module $R[x]/\sum_{i=1}^{n} W_iR[x]$ at a fixed point $x^{(0)}$. We construct the operators W_i and show that they commute with one another under some additional assumptions on the polynomials $Q_i(s)$ (Lemma 3.1). However, no additional assumptions on the polynomials $P_i(s)$ are needed as long as the compatibility conditions (2.3) are fulfilled.

Following the spirit of Adolphson [1] we define operators $D_i : R[x] \to R[x]$ by setting

$$D_i = z_i\frac{\partial}{\partial z_i} + x_i z_i, \ i = 1, \ldots, n. \tag{3.3}$$

It was pointed out in [1] that the operators (3.3) form a commutative family of $\mathcal{D}$-linear operators. Let D denote the vector $(D_1, \ldots, D_n)$. For any $i = 1, \ldots, n$ we define operator $\nabla_i : R[x] \to R[x]$ by $\nabla_i = z_i^{-1}D_i$. This operator commutes with the operators ∂_j since both D_i and the multiplication by z_i^{-1} commute with ∂_j. Moreover, the operator ∇_i commutes with ∇_j for all $1 \le i, j \le n$ and with D_j for $i \ne j$. In the case $i = j$ we have $\nabla_i D_i = \nabla_i + D_i\nabla_i$.

Thanks to Lemma 2.2 in [16] we may define operators $W_i = P_i(D)\nabla^{u_i} - Q_i(D)$ such that for any $i = 1, \ldots, n$ W_i is a $\mathcal{D}$-linear operator satisfying the identity $\Phi(G_i) = W_i(1)$. It follows by the $\mathcal{D}$-linearity of W_i that $\sum_{i=1}^{n} W_iR[x]$ and $R[x]/\sum_{i=1}^{n} W_iR[x]$ can be considered as left $\mathcal{D}$-modules. Using Theorem 4.4 and Lemma 4.12 in [1], we conclude that the following isomorphism holds true:

$$\mathcal{M} \simeq R[x] \Big/ \left(\sum_{j=1}^{n} W_jR[x]\right). \tag{3.4}$$

In the general case the operators $W_i = P_i(D)\nabla^{u_i} - Q_i(D)$ do not commute since D_i does not commute with ∇_i. However, this family of operators may be shown to be commutative under some assumptions on the polynomials $Q_i(s)$ in the case when the polynomials $P_i(s), Q_i(s)$ satisfy the compatibility conditions (2.3). The following Lemma holds.

Lemma 3.1 *The operators $W_i = P_i(D)\nabla^{u_i} - Q_i(D)$ commute with one another if and only if the polynomials $P_i(s), Q_i(s)$ satisfy the compatibility conditions (2.3) and for any $i, j = 1, \ldots, n$, $i \ne j$, $Q_i(s + u_j) = Q_i(s)$.*

Proof Since $\nabla_i = z_i^{-1} + D_i z_i^{-1}$ it follows that $\nabla_i D_i = \nabla_i + D_i \nabla_i$ and that ∇_i commutes with D_j for $i \neq j$. Hence for any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$

$$\nabla_i D_1^{\alpha_1} \ldots D_n^{\alpha_n} = D_1^{\alpha_1} \ldots (D_i + 1)^{\alpha_i} \ldots D_n^{\alpha_n} \nabla_i. \tag{3.5}$$

Let E_i^t denote the operator which increases the ith argument by t, that is, $E_i^t f(x) = f(x + te_i)$. Here $\{e_i\}_{i=1}^n$ denotes the standard basis of $\mathbb{Z}^n$. It follows from (3.5) that

$$\nabla_i P_j(D) = (E_i^1 P_j)(D) \nabla_i. \tag{3.6}$$

For $\alpha \in \mathbb{Z}^n$ let E^α denote the composition $E_1^{\alpha_1} \circ \ldots \circ E_n^{\alpha_n}$. Using (3.6) we compute the commutator of the operators W_i, W_j :

$$W_i W_j - W_j W_i = \Big(P_i(D)(E^{u_i} P_j)(D) - P_j(D)(E^{u_j} P_i)(D) \Big) \nabla^{u_i + u_j} +$$

$$\Big((E^{u_j} Q_i)(D) - Q_i(D) \Big) P_j(D) \nabla^{u_j} + \Big(Q_j(D) - (E^{u_i} Q_j)(D) \Big) P_i(D) \nabla^{u_i}. \tag{3.7}$$

Let us define the grade $g(x^\alpha z^\beta)$ of an element $x^\alpha z^\beta$ of the ring $R[x]$ to be $\alpha - \beta$. Notice that $g(D_i(x^\alpha z^\beta)) = \alpha - \beta$ and that $g(\nabla_i(x^\alpha z^\beta)) = \alpha - \beta + e_i$, for any $\alpha, \beta \in \mathbb{N}_0^n$. The result of the action of the operator in the right-hand side of (3.7) on $x^\alpha z^\beta$ consists of three terms whose grades are $\alpha - \beta + u_i + u_j$, $\alpha - \beta + u_j$ and $\alpha - \beta + u_i$. Thus the operators W_i, W_j commute if and only if

$$Q_i(D) = (E^{u_j} Q_i)(D), \; i, j = 1, \ldots, n, \quad i \neq j, \tag{3.8}$$

and

$$P_i(D)(E^{u_i} P_j)(D) = P_j(D)(E^{u_j} P_i)(D), \; i, j = 1, \ldots, n. \tag{3.9}$$

It follows from (3.8) that the condition $Q_i(s + u_j) = Q_i(s)$, $i, j = 1, \ldots, n$, $i \neq j$ is necessary for the family $\{W_i\}_{i=1}^n$ to be commutative. Under this assumption on the polynomials $Q_i(s)$ the compatibility conditions (2.3) can be written in the form

$$P_i(s + u_j) P_j(s) = P_j(s + u_i) P_i(s), \; i, j = 1, \ldots, n$$

and they are therefore equivalent to (3.9). The proof is complete.

For $x^{(0)} \in \mathbb{C}^n$ let $\hat{\mathcal{O}}_{x^{(0)}}$ be the $\mathcal{D}$-module of formal power series centered at $x^{(0)}$. Let $\mathbb{C}_{x^{(0)}}$ denote the set of complex numbers $\mathbb{C}$ considered as a $\mathbb{C}[x_1, \ldots, x_n]$-module via the isomorphism $\mathbb{C} \simeq \mathbb{C}[x_1, \ldots, x_n]/(x_1 - x_1^{(0)}, \ldots, x_n - x_n^{(0)})$. We use the following isomorphism (see Proposition 2.5.26 in [4] or [1], § 4) between the space of formal solutions to $\mathcal{M}$ at $x^{(0)}$ and the dual space of $\mathbb{C}_{x^{(0)}} \otimes_{\mathbb{C}[x]} \mathcal{M}$

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \hat{\mathcal{O}}_{x^{(0)}}) \simeq \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}_{x^{(0)}} \otimes_{\mathbb{C}[x]} \mathcal{M}, \mathbb{C}). \tag{3.10}$$

This isomorphism holds for any finitely generated $\mathcal{D}$-module. Using (3.4) and fixing the point $x = x^{(0)}$ we arrive at the isomorphism

$$\mathbb{C}_{x^{(0)}} \otimes_{\mathbb{C}[x]} \left(R[x] \Big/ \sum_{i=1}^n W_i R[x] \right) \simeq R \Big/ \sum_{i=1}^n W_{i, x^{(0)}} R, \tag{3.11}$$

where $W_{i,x^{(0)}}$ are obtained from the operators W_i by setting $x = x^{(0)}$. Combining (3.10) with (3.11) we see that

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \hat{\mathcal{O}}_{x^{(0)}}) \simeq \mathrm{Hom}_{\mathbb{C}}\left(R \Big/ \sum_{i=1}^{n} W_{i,x^{(0)}} R,\ \mathbb{C}\right).$$

Thus the following Lemma holds true.

Lemma 3.2 *The number of linearly independent formal power series solutions to the system (1.1) at the point $x = x^{(0)}$ is equal to* $\dim_{\mathbb{C}} R / \sum_{i=1}^{n} W_{i,x^{(0)}} R$.

For any differential operator $P \in \mathcal{D}$, $P = \sum_{|\alpha| \le m} c_\alpha(x) \left(\frac{\partial}{\partial x}\right)^\alpha$ its principal symbol $\sigma(P)(x,z) \in R[x]$ is defined by $\sigma(P)(x,z) = \sum_{|\alpha|=m} c_\alpha(x) z^\alpha$. Let $H_i(x,z) = \sigma(G_i)(x,z)$ be the principal symbols of the differential operators which define the generalized Horn system (1.1). Let $J \subset \mathcal{D}$ be the left ideal generated by $G_1, \ldots, G_n$. By the definition (see [3], Chapter 5, § 2) the characteristic variety $\mathrm{char}(\mathcal{M})$ of the generalized Horn system is given by

$$\mathrm{char}(\mathcal{M}) = \{(x,z) \in \mathbb{C}^{2n} : \sigma(P)(x,z) = 0, \text{ for all } P \in J\}.$$

Let us define the set $U_{\mathcal{M}} \subset \mathbb{C}^n$ by $U_{\mathcal{M}} = \{x \in \mathbb{C}^n : \exists z \neq 0 \text{ such that } (x,z) \in \mathrm{Char}(\mathcal{M})\}$. Theorem 7.1 in [3, Chapter 5] yields that for $x^{(0)} \notin U_{\mathcal{M}}$

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \hat{\mathcal{O}}_{x^{(0)}}) \simeq \mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{O}_{x^{(0)}}).$$

It follows from [18] (pages 146,148) that the $\mathbb{C}$-dimension of the factor of the ring R with respect to the ideal generated by the regular sequence of homogeneous polynomials $H_1(x^{(0)}, z), \ldots, H_n(x^{(0)}, z)$ is equal to the product $\prod_{i=1}^{n} \deg H_i(x^{(0)}, z)$. Since a sequence of n homogeneous polynomials in n variables is regular if and only if their common zero is the origin, it follows that $U_{\mathcal{M}} = \emptyset$ in our setting. Using Lemmas 3.1,3.2, and Lemma 2.7 in [16], we arrive at the following Theorem.

Theorem 3.3 *Suppose that the polynomials $P_i(s), Q_i(s)$ satisfy the compatibility conditions (2.3) and that $Q_i(s + u_j) = Q_i(s)$ for any $i, j = 1, \ldots, n$, $i \neq j$. If the principal symbols $H_1(x^{(0)}, z), \ldots, H_n(x^{(0)}, z)$ of the differential operators $G_1, \ldots, G_n$ form a regular sequence at $x^{(0)}$ then the dimension of the space of holomorphic solutions to (1.1) at the point $x^{(0)}$ is equal to* $\prod_{i=1}^{n} \deg H_i(x^{(0)}, z)$.

Using Lemma 2.7 in [16], we obtain the following result.

Corollary 3.4 *Suppose that the principal symbols $H_1(x^{(0)}, z), \ldots, H_n(x^{(0)}, z)$ of the differential operators $G_1, \ldots, G_n$ form a regular sequence at $x^{(0)}$. Then the dimension of the space of holomorphic solutions to (1.1) at the point $x^{(0)}$ is less than or equal to* $\prod_{i=1}^{n} \deg H_i(x^{(0)}, z)$.

In the next section we, using Theorem 3.3, construct an explicit basis in the space of holomorphic solutions to the generalized Horn system under the assumption that P_i, Q_i can be represented as products of linear factors and that $\deg Q_i > \deg P_i$, $i = 1, \ldots, n$.

4 Explicit basis in the solution space of a sparse hypergeometric system

Throughout this section we assume that the polynomials $P_i(s), Q_i(s)$ defining the generalized Horn system (1.1) can be factorized up to polynomials of degree one. Suppose that $P_i(s), Q_i(s)$ satisfy the following conditions: $Q_i(s+u_j) = Q_i(s)$ and $\deg Q_i > \deg P_i$ for any $i, j = 1, \ldots, n$, $i \neq j$. In this section we will show how to construct an explicit basis in the solution space of such a system of partial differential equations under some additional assumptions which are always satisfied if the parameters of the system under study are sufficiently general.

Recall that U denotes the matrix whose rows are $u_1, \ldots, u_n$ and let U^T denote the transpose of U. Let $\Lambda = (U^T)^{-1}$, let $(\Lambda s)_i$ denote the ith component of the vector Λs and $d_i = \deg Q_i$. Under the above conditions the polynomials $Q_i(s)$ can be represented in the form

$$Q_i(s) = \prod_{j=1}^{d_i} ((\Lambda s)_i - \alpha_{ij}), \quad i = 1, \ldots, n, \quad \alpha_{ij} \in \mathbb{C}.$$

By the Ore–Sato theorem [17] (see also § 1.2 of [10]) the general solution to the system of difference equations (2.1) associated with (1.1) can be written in the form

$$\varphi(s) = t_1^{s_1} \ldots t_n^{s_n} \frac{\prod_{i=1}^{p} \Gamma(\langle A_i, s\rangle - c_i)}{\prod_{i=1}^{n} \prod_{j=1}^{d_i} \Gamma((\Lambda s)_i - \alpha_{ij} + 1)} \phi(s), \tag{4.1}$$

where $p \in \mathbb{N}_0$, $t_i, c_i \in \mathbb{C}$, $A_i \in \mathbb{Z}^n$ and $\phi(s)$ is an arbitrary function satisfying the periodicity conditions $\phi(s+u_i) \equiv \phi(s)$, $i = 1, \ldots, n$. (Given polynomials P_i, Q_i satisfying the compatibility conditions (2.3), the parameters p, t_i, c_i, A_i of the solution $\varphi(s)$ can be computed explicitly. For a concrete construction of the function $\varphi(s)$ see [16]. The following Theorem holds true.

Theorem 4.1 *Suppose that the following conditions are fulfilled.*
1. For any $i, j = 1, \ldots, n$, $i \neq j$ it holds $Q_i(s+u_j) = Q_i(s)$ and $\deg Q_i > \deg P_i$.
2. The difference $\alpha_{ij} - \alpha_{ik}$ is never equal to a real integer number, for any $i = 1, \ldots, n$ and $j \neq k$.
3. For any multi-index $I = (i_1, \ldots, i_n)$ with $i_k \in \{1, \ldots, d_k\}$ the product $\prod_{i=1}^{p}(\langle A_i, s\rangle - c_i)$ never vanishes on the shifted lattice $\mathbb{Z}^n + \gamma_I$, where $\gamma_I = (\alpha_{1i_1}, \ldots, \alpha_{ni_n})$.

Then the family consisting of $\prod_{i=1}^{n} d_i$ functions

$$y_I(x) = x^{\gamma_I} \sum_{s \in \mathbb{Z}^n \cap K_U} t^{s+\gamma_I} \frac{\prod_{i=1}^{p} \Gamma(\langle A_i, s+\gamma_I\rangle - c_i)}{\prod_{k=1}^{n} \prod_{j=1}^{d_k} \Gamma((\Lambda s)_k + \alpha_{ki_k} - \alpha_{kj} + 1)} x^s \tag{4.2}$$

is a basis in the space of holomorphic solutions to the system (1.1) at any point $x \in (\mathbb{C}^*)^n = (\mathbb{C} \setminus \{0\})^n$. Here K_U is the cone spanned by the vectors $u_1, \ldots, u_n$.

Proof It follows from Theorem 2.1 and the assumptions 2,3 of Theorem 4.1 that the series (4.2) formally satisfies the generalized Horn system (1.1). Let χ_k denote the kth row of Λ. Since $\deg Q_i(s) > \deg P_i(s)$, $i = 1, \ldots, n$ it follows by the construction of the function (4.1) (see [16]) that all the components of the vector $\triangle = \sum_{i=1}^{p} A_i - \sum_{i=1}^{n} d_i \chi_i$ are negative. Thus for any multi-index I the intersection of the half-space $\mathrm{Re}\langle \triangle, s\rangle \geq 0$ with the shifted octant $K_U + \gamma_I$ is a bounded set. Using the Stirling formula we conclude that the series (4.2) converges everywhere in $(\mathbb{C}^*)^n$ for any multi-index I.

The series (4.2) corresponding to different multi-indices I, J are linearly independent since by the second assumption of Theorem 4.1 their initial monomials $x^{\gamma_I}, x^{\gamma_J}$ are different. Finally, the conditions of Theorem 3.3 are satisfied in our setting since the first assumption of Theorem 4.1 yields that the sequence of principal symbols $H_1(x^{(0)}, z), \ldots, H_n(x^{(0)}, z) \in R$ of hypergeometric differential operators defining the generalized Horn system is regular for $x^{(0)} \in (\mathbb{C}^*)^n$. Hence by Theorem 3.3 the number of linearly independent holomorphic solutions to the system under study at a generic point equals $\prod_{i=1}^{n} d_i$. In this case $U_{\mathcal{M}} = \{x^{(0)} \in \mathbb{C}^n : x_1^{(0)} \ldots x_n^{(0)} = 0\}$. Thus the series (4.2) span the space of holomorphic solutions to the system (1.1) at any point $x^{(0)} \in (\mathbb{C}^*)^n$. The proof is complete.

In the theory developed by Gelfand, Kapranov and Zelevinsky the conditions 2 and 3 of Theorem 4.1 correspond to the so-called nonresonant case (see [9], § 8.1). Thus the result on the structure of solutions to the generalized Horn system can be formulated as follows.

Corollary 4.2 *Let $x^{(0)} \in (\mathbb{C}^*)^n$ and suppose that $Q_i(s + u_j) = Q_i(s)$ and $\deg Q_i > \deg P_i$ for any $i, j = 1, \ldots, n$, $i \neq j$. If the parameters of the system (1.1) are nonresonant then there exists a basis in the space of holomorphic solutions to (1.1) near $x^{(0)}$ whose elements are given by series of the form (1.2).*

5 Examples

In this section we use the results on the structure of solutions to the generalized Horn system for computing the number of Laurent expansions of some rational functions. This problem is closely related to the notion of the amoeba of a Laurent polynomial, which was introduced by Gelfand et al. in [12] (see Chapter 6, § 1). Given a Laurent polynomial f, its amoeba $\mathcal{A}_f$ is defined to be the image of the hypersurface $f^{-1}(0)$ under the map $(x_1, \ldots, x_n) \mapsto (\log|x_1|, \ldots, \log|x_n|)$. This name is motivated by the typical shape of $\mathcal{A}_f$ with tentacle-like asymptotes going off to infinity. The connected components of the complement of the amoeba are convex and each such component corresponds to a specific Laurent series development with the center at the origin of the rational function $1/f$ (see [12], Chapter 6, Corollary 1.6). The problem of finding all such Laurent series expansions of a given Laurent polynomial was posed in [12] (Chapter 6, Remark 1.10).

Let $f(x_1, \ldots, x_n) = \sum_{\alpha \in S} a_\alpha x^\alpha$ be a Laurent polynomial. Here S is a finite subset of the integer lattice $\mathbb{Z}^n$ and each coefficient a_α is a non-zero complex number. The Newton polytope $\mathcal{N}_f$ of the polynomial f is defined to be the convex hull in $\mathbb{R}^n$ of the index set S. The following result was obtained in [8].

Theorem 5.1 *Let f be a Laurent polynomial. The number of Laurent series expansions with the center at the origin of the rational function $1/f$ is at least equal to the number of vertices of the Newton polytope $\mathcal{N}_f$ and at most equal to the number of integer points in $\mathcal{N}_f$.*

In the view of Corollary 1.6 in Chapter 6 of [12], Theorem 5.1 states that the number of connected components of the complement of the amoeba $\mathcal{A}_f$ is bounded from below by the number of vertices of $\mathcal{N}_f$ and from above by the number of integer points in $\mathcal{N}_f$. The lower bound has already been obtained in [12]. In this section we describe a class of rational functions for which the number of Laurent expansions attains the lower bound given by Theorem 5.1. Our main tool is Theorem 2.1 which allows one to describe supports of the Laurent series expansions of a rational function which can be treated as a solution to a generalized Horn system. In the following three examples we let $u_1, \dots, u_n \in \mathbb{Z}^n$ be linearly independent vectors, $p \in \mathbb{N}$ and let $a_1, \dots, a_n \in \mathbb{C}^*$ be nonzero complex numbers. We denote by U the matrix with the rows $u_1, \dots, u_n$ and use the notation $(\lambda_{ij}) = \Lambda = (U^T)^{-1}$ and $\nu_i = \lambda_{1i} + \dots + \lambda_{ni}$. The conclusions in all of the following examples can be deduced from Theorem 7 in [14].

Example 5.2 The function $y_1(x) = (1 - a_1 x^{u_1} - \dots - a_n x^{u_n})^{-1}$ satisfies the following system of the Horn type

$$\begin{pmatrix} a_1 x^{u_1} \\ \dots \\ a_n x^{u_n} \end{pmatrix} (\nu_1 \theta_1 + \dots + \nu_n \theta_n + 1)\, y(x) = \Lambda \begin{pmatrix} \theta_1 \\ \dots \\ \theta_n \end{pmatrix} y(x). \tag{5.1}$$

Indeed, after the change of variables $x_i(\xi_1, \dots, \xi_n) = \xi_1^{\lambda_{1i}} \dots \xi_n^{\lambda_{ni}}$ (whose inverse is $\xi_i = x^{u_i}$) the system (5.1) takes the form

$$a_i \xi_i \left(\theta_{\xi_1} + \dots + \theta_{\xi_n} + 1\right) y(\xi) = \theta_{\xi_i} y(\xi), \quad i = 1, \dots, n. \tag{5.2}$$

The function $(1 - a_1 \xi_1 - \dots - a_n \xi_n)^{-1}$ satisfies (5.2) and therefore the function $y_1(x)$ is a solution of (5.1). The hypergeometric system (5.1) is a special instance of systems (5.3) and (5.5). We treat this simple case first in order to make the main idea more transparent.

By Theorem 3.3 the space of holomorphic solutions to (5.1) has dimension one at a generic point and hence $y_1(x)$ is the only solution to this system. Thus the supports of the Laurent series expansions of $y_1(x)$ can be found by means of Theorem 2.1. There exist $n+1$ subsets of the lattice $\mathbb{Z}^n$ which satisfy the conditions in Theorem 2.1 and can give rise to a Laurent expansion of $y_1(x)$ with nonempty domain of convergence. These subsets are $S_0 = \{s \in \mathbb{Z}^n : (\Lambda s)_i \geq 0, i = 1, \dots, n\}$ and $S_j = \{s \in \mathbb{Z}^n : \nu_1 s_1 + \dots + \nu_n s_n + 1 \leq 0, (\Lambda s)_i \geq 0, i \neq j\}, j = 1, \dots, n$. Besides $S_0, \dots, S_n$ there can exist other subsets of $\mathbb{Z}^n$ satisfying the conditions in Theorem 2.1. (Such subsets "penetrate" some of the hyperplanes $(\Lambda s)_i = 0$, $\nu_1 s_1 + \dots + \nu_n s_n + 1 = 0$ without intersecting them; subsets of this type can only appear if $|\det U| \geq 1$). However, none of these additional subsets gives rise to a convergent Laurent series and therefore does not define an expansion of $y_1(x)$. Indeed, in any series with the support in a "penetrating" subset at least one index of summation necessarily runs from $-\infty$ to ∞. Letting all the variables, except for that one which corresponds to this index, be equal to zero, we obtain a hypergeometric series in one variable. The classical result on convergence of one-dimensional hypergeometric series (see [10], § 1) shows that this series is necessarily divergent. Thus the number of Laurent series developments of $y_1(x)$ cannot exceed $n+1$. The Newton polytope of the polynomial $1/y_1(x)$ has $n+1$ vertices since the vectors $u_1, \dots, u_n$ are linearly independent. Using Theorem 5.1 we conclude that the number of Laurent series expansions of $y_1(x)$ equals $n+1$. Thus the lower bound for the number of connected components of the amoeba complement is attained.

Example 5.3 Recall that θ denotes the vector $\left(x_1 \frac{\partial}{\partial x_1}, \dots, x_n \frac{\partial}{\partial x_n}\right)$ and let $(\Lambda\theta)_i$ denote the ith component of the vector $\Lambda\theta$. Let $\mathcal{G}$ be the differential operator defined by

$$\mathcal{G} = (\Lambda\theta)_1 + \dots + (\Lambda\theta)_{n-1} + p(\Lambda\theta)_n + p.$$

The function $y_2(x) = ((1 - a_1 x^{u_1} - \dots - a_{n-1} x^{u_{n-1}})^p - a_n x^{u_n})^{-1}$ is a solution to the following system of differential equations of hypergeometric type

$$\begin{cases} a_i x^{u_i} \mathcal{G} y(x) = (\Lambda\theta)_i y(x), \quad i = 1, \dots, n-1, \\ a_n x^{u_n} \left(\prod_{j=0}^{p-1} (\mathcal{G} + j) \right) y(x) = \left(\prod_{j=0}^{p-1} \left(p(\Lambda\theta)_n + j\right) \right) y(x). \end{cases} \tag{5.3}$$

Indeed, the same monomial change of variables as in Example 5.2 reduces (5.3) to the system

$$\begin{cases} a_i\xi_i\,\tilde{\mathcal{G}}y(x) = \theta_{\xi_i}y(x), \;\; i = 1,\ldots,n-1, \\ a_n\xi_n\left(\prod_{j=0}^{p-1}(\tilde{\mathcal{G}}+j)\right)y(x) = \left(\prod_{j=0}^{p-1}(p\,\theta_{\xi_n}+j)\right)y(x), \end{cases} \tag{5.4}$$

where $\tilde{\mathcal{G}} = \theta_{\xi_1} + \cdots + \theta_{\xi_{n-1}} + p\theta_{\xi_n} + p$. The system (5.4) is satisfied by the function $((1 - a_1\xi_1 - \cdots - a_{n-1}\xi_{n-1})^p - a_n\xi_n)^{-1}$. This shows that $y_2(x)$ is indeed a solution to (5.3). Thus the support of a Laurent expansion of $y_2(x)$ must satisfy the conditions in Theorem 2.1. Notice that unlike (5.1), the system (5.3) can have solutions supported in subsets of the shifted lattice $\mathbb{Z}^n+\gamma$ for some $\gamma \in (0,1)^n$. Yet, such subsets are not of interest for us since we are looking for Laurent series developments of $y_2(x)$. The subsets $S_0 = \{s \in \mathbb{Z}^n : (\Lambda s)_i \geq 0,\, i = 1,\ldots,n\}$ and $S_j = \{s \in \mathbb{Z}^n : (\Lambda s)_1 + \cdots + (\Lambda s)_{n-1} + p(\Lambda s)_n + p \leq 0,\, (\Lambda s)_i \geq 0,\, i \neq j\}$, $j = 1,\ldots,n$ satisfy the conditions in Theorem 2.1. The same arguments as in Example 5.2 show that no other subsets of $\mathbb{Z}^n$ satisfying the conditions in Theorem 2.1 can give rise to a convergent Laurent series which represents $y_2(x)$. This yields that the number of expansions of $y_2(x)$ is at most equal to $n+1$. The Newton polytope of the polynomial $1/y_2(x)$ has $n+1$ vertices since the vectors $u_1,\ldots,u_n$ are assumed to be linearly independent. Using Theorem 5.1 we conclude that the number of Laurent series developments of $y_2(x)$ equals $n+1$.

Example 5.4 Let $\mathcal{H}$ be the differential operator defined by $\mathcal{H} = p(\Lambda\theta)_2 + \cdots + p(\Lambda\theta)_n + p$. Using the same change of variables as in Example 5.2, one checks that $y_3(x) = ((1 - a_1x^{u_1})^p - a_2x^{u_2} - \cdots - a_nx^{u_n})^{-1}$ solves the system

$$\begin{cases} a_1x^{u_1}\left((\Lambda\theta)_1 + \mathcal{H}\right)y(x) = (\Lambda\theta)_1\,y(x), \\ a_ix^{u_i}\frac{1}{p}\mathcal{H}\left(\prod_{j=0}^{p-1}\left((\Lambda\theta)_1 + \mathcal{H} + j\right)\right)y(x) = \\ \qquad (\Lambda\theta)_i\left(\prod_{j=0}^{p-1}(\mathcal{H} - p + j)\right)y(x), \;\; i = 2,\ldots,n. \end{cases} \tag{5.5}$$

Analogously to Example 5.2, we apply Theorem 2.1 to the system (5.5) and conclude that the number of Laurent expansions of $y_3(x)$ at most equals $n+1$. Thus it follows from Theorem 5.1 that the number of such expansions equals $n+1$.

Example 5.5 The Szegö kernel of the domain $\{z \in \mathbb{C}^2 : |z_1| + |z_2| < 1\}$ is given by the hypergeometric series

$$h(x_1,x_2) = \sum_{s_1,s_2\geq 0}\frac{\Gamma(2s_1+2s_2+2)}{\Gamma(2s_1+1)\Gamma(2s_2+1)}x_1^{s_1}x_2^{s_2} = \frac{(1-x_1-x_2)(1+2x_1x_2-x_1^2-x_2^2)+8x_1x_2}{\left((1-x_1-x_2)^2-4x_1x_2\right)^2}. \tag{5.6}$$

(See [2], Chapter 3, § 14.) This series satisfies the system of equations

$$x_i\,(2\theta_1 + 2\theta_2 + 3)\,(2\theta_1 + 2\theta_2 + 2)\,y(x) = 2\theta_i(2\theta_i - 1)y(x), \;\; i = 1,2.$$

There exist three subsets of the lattice $\mathbb{Z}^n$ which satisfy the conditions in Theorem 2.1, namely $\{s \in \mathbb{Z}^2 : s_1 \geq 0, s_2 \geq 0\}$, $\{s \in \mathbb{Z}^2 : s_1 \geq 0, s_1 + s_2 + 1 \leq 0\}$, $\{s \in \mathbb{Z}^2 : s_2 \geq 0, s_1 + s_2 + 1 \leq 0\}$. Using Theorem 2.1 we conclude that the number of Laurent expansions centered at the origin of the Szegö kernel (5.6) at most equals 3. The Newton polytope of the denominator of the rational function (5.6) is the simplex with the vertices $(0,0)$, $(4,0)$, $(0,4)$. By Theorem 5.1 the number of Laurent series developments of the Szegö kernel at least equals 3. Thus the number of Laurent expansions of (5.6) (or, equivalently, the number of connected components in the complement of the amoeba of its denominator) attains its lower bound.

Example 5.6 Let $u_1 = (1,0), u_2 = (1,1)$ and consider the system of equations

$$\left\{ \begin{array}{ll} x^{u_1} y(x) & = \left(x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} \right) y(x), \\ x^{u_2} y(x) & = \left(x_2 \frac{\partial}{\partial x_2} \right) y(x). \end{array} \right. \tag{5.7}$$

The principal symbols $H_1(x,z), H_2(x,z) \in R[x]$ of the differential operators defining the system (5.7) are given by $H_1(x,z) = -x_1 z_1 + x_2 z_2$, $H_2(x,z) = -x_2 z_2$. By Theorem 3.3 the dimension of the solution space of (5.7) at a generic point is equal to 1 since $\dim_{\mathbb{C}} R/(H_1(x,z), H_2(x,z)) = 1$ for $x_1 x_2 \neq 0$. For computing the solution to (5.7) explicitly we choose $\gamma = 0$ and consider the corresponding system of difference equations

$$\left\{ \begin{array}{ll} \varphi(s+u_1)(s_1 - s_2 + 1) & = \varphi(s), \\ \varphi(s+u_2)(s_2+1) & = \varphi(s). \end{array} \right. \tag{5.8}$$

The general solution to (5.8) is given by $\varphi(s) = (\Gamma(s_1 - s_2 + 1)\Gamma(s_2+1))^{-1}\phi(s)$, where $\phi(s)$ is an arbitrary function which is periodic with respect to the vectors u_1, u_2.

There exists only one subset of $\mathbb{Z}^2$ satisfying the conditions of Theorem 2.1, namely $S = \{(s_1, s_2) \in \mathbb{Z}^2 : s_1 - s_2 \geq 0,\ s_2 \geq 0\}$. Choosing $\phi(s) \equiv 1$ and using (4.2), we obtain the solution to (5.7):

$$y(x) = \sum_{\substack{s_1 - s_2 \geq 0, \\ s_2 \geq 0}} \frac{x_1^{s_1} x_2^{s_2}}{\Gamma(s_1 - s_2 + 1)\Gamma(s_2 + 1)} = \exp(x_1 x_2 + x_1). \tag{5.9}$$

It is straightforward to check that the solution space of (5.7) is indeed spanned by (5.9).

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РАЗРЯЖЕННЫЕ ГИПЕРГЕОМЕТРИЧЕСКИЕ СИСТЕМЫ

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Аннотация. Описывается подход к изучению теории гипергеометрических функций от нескольких переменных с помощью обобщенной системы дифференциальных уравнений типа Горна. Получена формула для вычисления размерности пространства решений этой системы, основываясь на которой строится в явном виде базис ее пространства голоморфных решений при некоторых ограничениях на параметры системы.

Ключевые слова: гипергеометрические функции, системы дифференциальных уравнений типа Горна, система Меллина.