UNIQUENESS OF MEROMORPHIC FUNCTIONS THAT SHARE THREE VALUES

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Abstract. In this paper, we study the uniqueness of meromorphic functions that share three values or three small functions with the same multiplicities and prove some results on this topic given by G. Brosch, X. H. Hua and M. L. Fang, etc.

Keywords: uniqueness, meromorphic function, value sharing.

1. Introduction and Results

It is assumed that the reader is familiar with the usual notations and the fundamental results of R. Nevanlinna theory of meromorphic function as found in [5].

Let f, g be nonconstant meromorphic functions. We say that a meromorphic function $a(z)(\not\equiv\infty)$ is a small function of f if T(r,a)=S(r,f). If N(r,1/(f-a))=S(r,f), then we say that a is an exceptional function of f. Moreover, we denote by N(r,f=a=g) the counting function of those common zeros of f-a and g-a, where z_0 is counted $\min\{p,\ q\}$ times if z_0 is a common zero of f-a and g-a with multiplicity p and q respectively; as usual, by $\overline{N}(r,f=a=g)$ the corresponding reduced counting function; and by $N_E(r,f=a=g)$ the counting function which "counts" only those common zeros of f-a and g-a with the same multiplicity in N(r,f=a=g). These notations will be used throughout the paper.

Let f, g be two nonconstant meromorphic functions, and let a be a small function of f and g or a be a constant. We say that f and g share a CM if f-a and g-a have the same zeros with the same multiplicity; if we ignore the multiplicity, then we say that f and g share a IM. For the statement of our results, we may need a slightly generalization of the definitions of CM and IM (see [6],[8]).

In 1997, Hua and Fang proved the following result.

Theorem A[6]. Let f and g be two nonconstant meromorphic functions, and let $a_j(z)$ $(j = 1, \dots, 4)$ be distinct small functions of f and g. If f and g share $a_j(z)$ (j = 1, 2, 3) CM, and share $a_4(z)$ IM. Then f and g satisfy one of the following cases.

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\begin{array}{ll} \text{(i)} \ f \equiv g, & \text{(ii)} \ F \equiv -G \ \text{with} \ a(z) \equiv -1, & \text{(iii)} \ F + G \equiv 2 \ \text{with} \ a(z) \equiv 2, \\ \text{(iv)} \ (F - 1/2)(G - 1/2) \equiv 1/4 \ \text{with} \ a(z) \equiv 1/2, & \text{(v)} F \cdot G \equiv 1 \ \text{with} \ a(z) \equiv -1, \\ \text{(vi)} \ (F - 1)(G - 1) \equiv 1 \ \text{with} \ a(z) \equiv 2, & \text{(vii)} \ F + G \equiv 1 \ \text{with} \ a(z) \equiv 1/2, \\ \text{where} \ F \equiv \frac{f - a_1}{f - a_3} \frac{a_2 - a_3}{a_2 - a_1}, \ G \equiv \frac{g - a_1}{g - a_3} \frac{a_2 - a_3}{a_2 - a_1}, \ \text{and} \quad a(z) \equiv \frac{a_4 - a_1}{a_4 - a_3} \frac{a_2 - a_3}{a_2 - a_1}. \end{array}
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Remark 1. From the proof of Lemma 6 and Lemma 7 in [6], it is easy to see that the conclusion is still true if we replace IM with "IM" in Theorem A.

Project supported by the National Natural Science Foundation(Grant No.10771121) of China, and the Natural Science Foundation(Grant No.Y6090641) of Zhejiang Province.

For the meromorphic functions that share three values, G. Brosch proved

Theorem B(see [1] or [11]). Let two meromorphic functions f and g share $0, 1, \infty$ CM. If there exists a finite value $a(\neq 0, 1)$ such that g(z) = a whenever f(z) = a. Then f is a Möbius transformation of g.

In 2008, two of the present authors proved a result on this topic.

Theorem C(see [15, Theorem 2]). Let two nonconstant meromorphic functions f and g share $0,1,\infty$ CM. If there exists a small entire function $a(z) (\not\equiv 0,1,\infty)$ of f and g such that g(z) - a(z) = 0 whenever $f(z) \stackrel{(p)}{=} a(z)$ for p = 1,2. Then f and g must satisfy one of the following ten cases.

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(i) f \equiv g, (ii) f \equiv ag, where a(z) (\not\equiv -1), 1 are exceptional functions of f, (iii) f-1 \equiv (1-a)(g-1), where a(z) (\not\equiv 2), 0 are exceptional functions of f, (iv) (f-a)(g-1+a) \equiv a(1-a), where a(z) (\not\equiv \frac{1}{2}), \infty are exceptional functions of f, (v) f \equiv -g with a(z) \equiv -1, (vi) f+g \equiv 2 with a(z) \equiv 2, (vii) (f-\frac{1}{2})(g-\frac{1}{2}) \equiv \frac{1}{4} with a(z) \equiv \frac{1}{2}, (viii) f \cdot g \equiv 1 with a(z) \equiv -1, (ix) (f-1)(g-1) \equiv 1 with a(z) \equiv 2, (x) f+g \equiv 1 with a(z) \equiv \frac{1}{2}.
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The main purpose of this paper is further to study the uniqueness of meromorphic functions that share three values or three small functions with the same multiplicities, and to prove the following three results.

Theorem 1. Let two nonconstant meromorphic functions f and g share $0,1,\infty$ CM. If there exists a small function $a(z) (\not\equiv 0,1,\infty)$ of f and g such that $N(r,f=a=g) \neq S(r,f)$. Then f and g satisfy one of the following five cases.

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(i) f \equiv g, (ii) f \cdot g \equiv 1 with a(z) \equiv -1, (iii) f + g \equiv 1 with a(z) \equiv \frac{1}{2}, (iv) (f-1)(g-1) \equiv 1 with a(z) \equiv 2, (v)f(z) = \frac{e^{\int a(z)\gamma'(z)dz}-1}{e^{\gamma(z)}-1}, g(z) = \frac{e^{-\int a(z)\gamma'(z)dz}-1}{e^{\gamma(z)}-1}, where \gamma(z) is a nonconstant entire function, and a(z) \not\equiv -1, 1/2, 2.
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Let f be a meromorphic function, let a be a small function of f or be a constant, and let p be a positive integer. We denote by $f(z_0) \stackrel{(p)}{=} a$ that z_0 is a zero of f-a with multiplicity p. By the above Theorem 1, we can prove the following result which generalize the small function a(z) in Theorem C from entire to meromorphic, and is also a great improvement of Theorem B. In order to avoid needless duplication, we shall omit the details of the proof of the following Theorem 2 in this paper.

Theorem 2. Let two nonconstant meromorphic functions f and g share $0,1,\infty$ CM. If there exists a small function $a(z) (\not\equiv 0,1,\infty)$ of f and g such that g(z)-a(z)=0 whenever $f(z) \stackrel{(p)}{=} a(z)$ for p=1,2. Then the conclusion of Theorem C still holds.

From Theorem 2, we can immediately obtain the following result which improves and generalizes Theorem A.



Theorem 3. Let F and G be nonconstant meromorphic functions, and let $a_j(z)(j=1,2,3,4)$ be distinct small functions of F and G. If F and G share $a_j(z)(j=1,2,3)$ CM, and if $G(z)=a_4(z)$ whenever $F(z)=a_4(z)$. Then f and g satisfy the conclusion of Theorem G, where $f\equiv \frac{F-a_1}{F-a_3}\frac{a_2-a_3}{a_2-a_1}$, $g\equiv \frac{G-a_1}{G-a_3}\frac{a_2-a_3}{a_2-a_1}$ and $a\equiv \frac{a_4-a_1}{a_4-a_3}\frac{a_2-a_3}{a_2-a_1}$.

2. Lemmas

Lemma 1 (see [16]). Suppose that f_1, f_2, \dots, f_n $(n \ge 3)$ are meromorphic functions which are not constants except for f_n . Furthermore, let $\sum_{j=1}^n f_j(z) \equiv 1$. If $f_n(z) \not\equiv 0$, and

$$\sum_{j=1}^{n} N(r, 1/f_j) + (n-1) \sum_{j=1}^{n} \overline{N}(r, f_j) < (\lambda + o(1))T(r, f_k),$$

where $r \in I, k = 1, 2, \dots, n-1$ and $\lambda < 1$, then $f_n(z) \equiv 1$.

Lemma 2(see [16]). Let f_1 , f_2 be nonconstant meromorphic functions and c_1 , c_2 , c_3 be nonzero constants. If $c_1f_1 + c_2f_2 \equiv c_3$, then

$$T(r, f_1) < \overline{N}(r, 1/f_1) + \overline{N}(r, 1/f_2) + \overline{N}(r, f_1) + S(r, f_1).$$

Lemma 3(see [6, Lemma 5]). Let f and g be two nonconstant meromorphic functions that share $0, 1, \infty$ CM. If $f \not\equiv g$, then for any small function $a(z) (\not\equiv 0, 1, \infty)$ of f and g, we have

$$N_{(3}\left(r,\frac{1}{f-a}\right)+N_{(3}\left(r,\frac{1}{g-a}\right)=S(r,f).$$

3. The Proof of Theorem 1

We suppose first that $f \not\equiv g$. Since f and g share $0, 1, \infty$ CM, by the second fundamental theorem due to R. Nevanlinna, we have

$$\begin{array}{lcl} (1+o(1))T(r,f) & \leq & N(r,f)+N(r,\frac{1}{f})+N(r,\frac{1}{f-1}) \\ & \leq & N(r,g)+N(r,\frac{1}{g})+N(r,\frac{1}{g-1}) \leq (3+o(1))T(r,g). \end{array} \eqno(3.1)$$

Similarly, we obtain

$$(1 + o(1))T(r,g) \le (3 + o(1))T(r,f). \tag{3.2}$$

From (3.1) and (3.2), it follows that

$$S(r,f) = S(r,g). \tag{3.3}$$

Set

$$\varphi := \frac{f'(f-a)}{f(f-1)} - \frac{g'(g-a)}{g(g-1)}. \tag{3.4}$$

If $\varphi \not\equiv 0$, then from (3.3), (3.4), the fundamental estimate of the logarithmic derivative, and the hypothesis that f and g share $0, 1, \infty$ CM, we have

$$T(r,\varphi) = S(r,f) + S(r,g) = S(r,f).$$
 (3.5)

Since f and g share $0, 1, \infty$ CM, thus by (3.4) and (3.5) we deduce that

$$N(r,f=a=g) \leq N(r,1/\varphi) + S(r,f) \leq T(r,\varphi) + S(r,f) = S(r,f),$$

which contradicts the hypothesis of Theorem 1. Hence, we have $\varphi \equiv 0$, namely

$$\frac{f'(f-a)}{f(f-1)} \equiv \frac{g'(g-a)}{g(g-1)}. (3.6)$$

Noting that f and g share $0, 1, \infty$ CM, thus there exist two entire functions α and β such that

$$\frac{f}{g} = e^{\alpha}, \qquad \frac{f-1}{g-1} = e^{\beta}. \tag{3.7}$$

Since $f \not\equiv g$, by (3.7) we can deduce that $e^{\alpha} \not\equiv 1$, $e^{\beta} \not\equiv 1$ and $e^{\beta-\alpha} \not\equiv 1$. Set $\gamma := \beta - \alpha$, then from (3.7) we have

$$f = \frac{e^{\beta} - 1}{e^{\gamma} - 1}, \quad g = \frac{e^{-\beta} - 1}{e^{-\gamma} - 1}.$$
 (3.8)

Rewriting (3.6) as

$$(1-a)\left(\frac{f'}{f-1}-\frac{g'}{g-1}\right)\equiv a\left(\frac{g'}{g}-\frac{f'}{f}\right). \tag{3.9}$$

By (3.7) and the fact that $\alpha = \beta - \gamma$, we obtain

$$\frac{f}{g} = e^{\beta - \gamma}, \quad \frac{f - 1}{g - 1} = e^{\beta}, \tag{3.10}$$

from (3.10), it follows that

$$\frac{f'}{f} - \frac{g'}{g} = \beta' - \gamma', \quad \frac{f'}{f-1} - \frac{g'}{g-1} = \beta'.$$
 (3.11)

Substitution (3.11) into (3.9) gives

$$\beta' \equiv a\gamma'. \tag{3.12}$$

From (3.8) and (3.12), we have

$$f = \frac{e^{\int a\gamma'} - 1}{e^{\gamma} - 1}, \quad g = \frac{e^{-\int a\gamma'} - 1}{e^{-\gamma} - 1}.$$
 (3.13)

We now claim that $[a(z)+1][a(z)-\frac{1}{2}][a(z)-2]\equiv 0$ if and only if f and g satisfy one of the cases (ii)-(iv) of the conclusion of Theorem 1, and thus f is a Möbius transformation of g.

In fact, if $a(z) \equiv \frac{1}{2}$, then from (3.12) we have $\gamma \equiv 2\beta + c$, where c is a constant. Thus, by (3.7) and the fact that $\alpha = \beta - \gamma$, it follows that

$$\frac{g}{f} \equiv e^{\gamma - \beta} \equiv e^{\beta + c} \equiv e^{c} \frac{f - 1}{g - 1}. \tag{3.14}$$



Noting that $N(r, f = a = g) \neq S(r, f)$, we can deduce that there exists a point z_0 such that $f(z_0) = g(z_0) = a(z_0) \neq 0, 1, \infty$, which and (3.14) imply that $e^c = 1$, and thus we obtain from (3.14) that $(g-f)(g+f-1) \equiv 0$, that is $f+g \equiv 1$. Similarly, if $a(z) \equiv -1$ or $a(z) \equiv 2$, then from (3.7), (3.12), the fact $\alpha = \beta - \gamma$, and the hypothesis of Theorem 1, we can also obtain that $f \cdot g \equiv 1$ or $(f-1)(g-1) \equiv 1$, respectively.

On the other hand, suppose that there exist four finite complex numbers c_j (j = 1, 2, 3, 4)such that $f = \frac{c_1g + c_2}{c_3g + c_4}$, where $c_1c_4 \neq c_2c_3$. By this and (3.13) we get

$$2c_3 + c_4 - 2c_2 - c_1 = c_1 e^{\gamma - \int a\gamma'} + (c_3 - c_1)e^{-\int a\gamma'} + (c_3 + c_4)e^{\int a\gamma'} - c_4 e^{-\gamma + \int a\gamma'} - (c_1 + c_2)e^{\gamma} + (c_4 - c_2)e^{-\gamma}.$$
 (3.15)

We note first that γ is not a constant. Otherwise, from (3.12) we know that β is also a constant, and thus by (3.8) we can deduce that f is a constant, a contradiction. So from this and the fact that $a(z) \not\equiv 0, 1$, we can also derive that both $\gamma - \int a\gamma'$ and $\int a\gamma'$ are not constants. In the sequel, by repeatedly applying Lemma 1 to equality (3.15) and its modified forms, and noting the fact that $c_1c_4 \neq c_2c_3$, and that $a(z) \not\equiv 0, 1$, we can prove that one of the following

- (a) $\gamma 2 \int a\gamma' \equiv \text{constant}$, that is $a(z) \equiv \frac{1}{2}$, (b) $2\gamma \int a\gamma' \equiv \text{constant}$, that is $a(z) \equiv 2$, and (c) $\gamma + \int a\gamma' \equiv \text{constant}$, that is $a(z) \equiv -1$.

For this purpose, we shall divide our argument into two cases.

Case 1. $A := 2c_3 + c_4 - 2c_2 - c_1 = 0$.

From (3.15) we have

$$c_1 e^{\gamma - \int a \gamma'} + (c_3 - c_1) e^{-\int a \gamma'} + (c_3 + c_4) e^{\int a \gamma'} - c_4 e^{-\gamma + \int a \gamma'} - (c_1 + c_2) e^{\gamma} + (c_4 - c_2) e^{-\gamma} \equiv 0. \quad (3.16)$$

We now need to consider the following seven subcases.

Subcase 1.1. $c_1c_4(c_3-c_1)(c_3+c_4)(c_1+c_2)(c_4-c_2) \neq 0$. Rewrite (3.16) as

$$\frac{c_1}{c_2-c_4}e^{2\gamma-\int a\gamma'} + \frac{c_3-c_1}{c_2-c_4}e^{\gamma-\int a\gamma'} + \frac{c_3+c_4}{c_2-c_4}e^{\gamma+\int a\gamma'} - \frac{c_4}{c_2-c_4}e^{\int a\gamma'} - \frac{c_1+c_2}{c_2-c_4}e^{2\gamma} \equiv 1. \tag{3.17}$$

Suppose that $\gamma + \int a\gamma' \not\equiv constant$. Noting the fact that $\gamma - \int a\gamma'$, $\int a\gamma'$, and γ are all not constant, so we can get by applying Lemma 1 to (3.17) that $\frac{c_1}{c_2-c_4}e^{2\gamma-\int a\gamma'}\equiv 1$, and thus from (3.17) it follows that

$$\frac{c_3 - c_1}{c_1 + c_2} e^{-\gamma - \int a\gamma'} + \frac{c_3 + c_4}{c_1 + c_2} e^{-\gamma + \int a\gamma'} - \frac{c_4}{c_1 + c_2} e^{-2\gamma + \int a\gamma'} \equiv 1. \tag{3.18}$$

By Lemma 1 and (3.18), we get $-\frac{c_4}{c_1+c_2}e^{-2\gamma+\int a\gamma'}\equiv 1$. From this and (3.18) we get $\int a\gamma'\equiv$ constant, a contradiction.

Suppose that $\gamma + \int a\gamma' \equiv constant$. Then we must have $2\gamma - \int a\gamma' \not\equiv constant$. Otherwise, we shall find that γ is a constant, which is impossible. Thus, from (3.17) and Lemma 1 we get $\frac{c_3+c_4}{c_2-c_4}e^{\gamma+\int a\gamma'}\equiv 1$, and thus again from (3.17) and Lemma 1 we have

$$\frac{c_1}{c_1+c_2}e^{-\int a\gamma'} + \frac{c_3-c_1}{c_1+c_2}e^{-\gamma-\int a\gamma'} - \frac{c_4}{c_1+c_2}e^{-2\gamma+\int a\gamma'} \equiv 1. \tag{3.19}$$

Noting the assumption $\gamma + \int a\gamma' \equiv constant$, so we must have $-2\gamma + \int a\gamma' \not\equiv constant$. By applying Lemma 1 to (3.19), we deduce that $\gamma - \int a\gamma' \equiv constant$, this is also a contradiction. Therefore, the subcase 1.1 can not occur.

Next, we can use the similar method to deal with the following six subcases: $c_1=0$; $c_4=0$ but $c_1\neq 0$; $c_3-c_1=0$, but $c_1c_4\neq 0$; $c_3+c_4=0$, but $c_1c_4(c_3-c_1)\neq 0$; $c_1+c_2=0$, but $c_1c_4(c_3-c_1)(c_3+c_4)\neq 0$; $c_2-c_4=0$. For the sake of simplicity, we omit the details.

Case 2. $A := 2c_3 + c_4 - 2c_2 - c_1 \neq 0$.

In fact, we shall verify that the case 2 can not occur by dividing it into five subcases. In case 2, from (3.15) we have

$$\frac{c_1}{A} e^{\gamma - \int a \gamma'} + \frac{c_3 - c_1}{A} e^{-\int a \gamma'} + \frac{c_3 + c_4}{A} e^{\int a \gamma'} - \frac{c_4}{A} e^{-\gamma + \int a \gamma'} - \frac{c_1 + c_2}{A} e^{\gamma} + \frac{c_4 - c_2}{A} e^{-\gamma} \equiv 1. \quad (3.20)$$

If $c_1c_4(c_3-c_1)(c_3+c_4)(c_1+c_2)(c_4-c_2) \neq 0$, then by (3.20) and Lemma 1, we can get a contradiction by noting that γ , $\int a\gamma'$ and $\gamma - \int a\gamma'$ are all not constants. So we know that at least one of the six numbers is zero.

Next, we consider the following five subcases.

Subcase 2.1. $c_1 = 0$. In this subcase, we have $c_2c_3 \neq 0$. By (3.20) we obtain

$$\frac{c_3}{A}e^{-\int a\gamma'}+\frac{c_3+c_4}{A}e^{\int a\gamma'}-\frac{c_4}{A}e^{-\gamma+\int a\gamma'}-\frac{c_2}{A}e^{\gamma}+\frac{c_4-c_2}{A}e^{-\gamma}\equiv 1. \eqno(3.21)$$

If $c_3+c_4=0$, then $c_4=-c_3\neq 0$. So, from (3.21) and Lemma 1 we get $c_4-c_2=0$, and thus a contradiction.

If $c_3+c_4\neq 0$, then we must have $c_4\neq 0$. Otherwise, by applying Lemma 1 to (3.21), we can get a contradiction. Now again by (3.21) and Lemma 1 we get $c_4-c_2=0$, and thus a contradiction. Thus we have $c_1\neq 0$.

We can easily dealt with the other four subcases $c_4 = 0$; $c_3 - c_1 = 0$; $c_3 + c_4 = 0$; $c_1 + c_2 = 0$ by the similar method.

In the above five subcases, we have shown that $c_1c_4(c_3-c_1)(c_3+c_4)(c_1+c_2) \neq 0$. Therefore, we can always obtain a contradiction by using Lemma 1 to (3.20) whether $c_4-c_2=0$ holds or not. The proof of Theorem 1 is completed.

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ЕДИНСТВЕННОСТЬ МЕРОМОРФНЫХ ФУНКЦИЙ С ТРЕМЯ РАЗДЕЛЕННЫМИ ЗНАЧЕНИЯМИ

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Аннотация. В работе изучаются единственность мероморфных функций с тремя разделяющимися значениями или малыми функциями той же кратности.

Ключевые слова: единственность, мероморфная функция, разделяющиеся значения.