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THE FIRST ORDER SYSTEM EQUATIONS OF A PRINCIPAL TYPE ON THE PLANE

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Abstract. Boundary value problems for the system equations of a principal type with constant coefficients on the plane are studied. The half-infinite domains with noncharacteristic boundary and finite domains with such property are considered. The representation solutions of this systems through solutions of canonical elliptic and hyperbolic systems is obtained. Also the index formula for associated problems in Holder weighted classes is founded.

Keywords: principal type equations, noncharacteristic boundary, index formula, function theoretical approach, canonical systems of first order.

1 Integral representation

On the (x_1, x_2) - plane $\mathbb{R}^2$ we consider a system of linear partial differential equations

$$\frac{\partial u}{\partial x_2} - a\frac{\partial u}{\partial x_1} = 0, \tag{1}$$

where $u(x)$ is an unknown l - vector-valued function and $a \in \mathbb{R}^{l\times l}$ is a constant matrix. The system (1) is said to be of a composite type (principal one [1, 2]) if it's characteristic equation

$$\det(a - \nu) = 0 \tag{2}$$

has $s_1 \geq 1$ complex roots with the positive imaginary part and $s_2 \geq 1$ real roots, $2s_1 + s_2 = l$.

Let $b_1 \in C^{s_1\times l}$, $b_2 \in \mathbb{R}^{s_2\times l}$ be constant matrices such that nonsingular matrix $b = (b_1|\overline{b_1}|b_2)$ reduces a to the Jordan normal form

$$b^{-1}ab = \operatorname{diag}(J_1, \overline{J_1}, J_2), \tag{3}$$

where the block matrixes $J_k \in \mathbb{C}^{s_k\times s_k}$, $k = 1, 2$, are composed from Jordan cells. Here J_1 has complex eigenvalues with positive imaginary part and $J_2 \in \mathbb{R}^{s_2\times s_2}$ has only real eigenvalues. Let $k_2 \leq s_2$ denote the maximum of orders of Jordan cells composing J_2.

It is valid the following representation theorem [3].

Theorem 1. *Any regular solution u of the system (1) can be represented in the form*

$$u = 2\mathrm{Re}\, b_1\Phi + b_2\Psi, \tag{4}$$

where Φ is a regular solution of the canonical elliptic system

$$\frac{\partial \Phi}{\partial x_2} = J_1\frac{\partial \Phi}{\partial x_1}, \tag{5}$$

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and Ψ *is a regular solution of the canonical hyperbolic system*

$$\frac{\partial \Psi}{\partial x_2} = J_2 \frac{\partial \Psi}{\partial x_1}. \tag{6}$$

Solutions of the system (5) are said to be a J_1 – analytical functions. It is known [4] that a general solution of this systems can be represented in the form

$$\Phi(x) = \left[\exp(x_2 J_{1,0}) \frac{\partial}{\partial x_1}\right] \phi(x_1 + \nu x_2),$$

where $J_1 = \nu + J_{10}$ is decomposition of the matrix J_1 into diagonal ν and nilpotent parts. Here the s_1- vector $\phi(x_1 + \nu x_2)$ consists from components $\phi_j(x_1 + \nu_j x_2)$, $1 \le j \le s_1$, where the functions $\phi_j(z)$ are analytic in the corresponding domain of the complex plane. The similar representation

$$\Psi(x) = \left[\exp(x_2 J_{2,0}) \frac{\partial}{\partial x_1}\right] \psi(x), \tag{7}$$

there exists for a s_2-vector-valued function $\psi = (\psi_j,\ 1 \le j \le s_2)$, where $J_2 = \nu + J_{2,0}$ and $\psi_j(x) = \tilde{\psi}_k(x_1 + \nu_j x_2)$. Note that ψ_j satisfies the hyperbolic equation

$$\frac{\partial \psi_j}{\partial x_2} = \nu_j \frac{\partial \psi_j}{\partial x_1}, \quad \nu_j \in \mathbb{R}.$$

2 Fredholm solvability in the half-infinite domain

Let $C^{\mu}(\overline{D})$ be the space of functions satisfying the Holder condition on the the closed domain $\overline{D}$ with exponent $0 < \mu \le 1$ (and bounded if $\overline{D}$ is infinite). The space $C^{\mu,n}(\overline{D})$ consists of the functions with partial derivatives in $C^{\mu,n-1}$, $n \ge 1$, $(C^{\mu,0} = C^{\mu})$. These spaces are Banach with respect to the corresponding norm. It is convenient to write $C^{\mu+0,n}$ for the class $\cup_{\varepsilon>0} C^{\mu+\varepsilon,n}$.

If the domain D is infinite we also use the space $C^{\mu,n}(\hat{D})$ for the set $\hat{D} = \overline{D} \cup \{\infty\}$ considered as the compact on the Riemann sphere $\hat{\mathbb{C}}$. These definitions also applies to the classes $C^{\mu,n}$ on curves $\Gamma \subseteq \mathbb{C}$.

Let D be a half-infinite domain on the complex plane i.e. it is a simple connected domain with smooth boundary Γ on the Riemann sphere. So the curve Γ permits a smooth parametrization $z = \gamma(t)$, $t \in \mathbb{R}$, where

$$\gamma'(t) \in C^{\mu,k_2}(\hat{\mathbb{R}}). \tag{8}$$

We consider a boundary value problem

$$Cu = f \text{ on } \Gamma, \tag{9}$$

for the system (1) where C is a $(s_1 + s_2) \times l$ matrix-valued function, and f is a $(s_1 + s_2)$ vector-valued function on $\Gamma = \partial D$. This problem is considered in the class $C^{\mu,1}(\overline{D})$ of solutions (1) such that the functions Φ and Ψ belong to this class in the representation (4). More exactly we say that the vector-valued function Ψ defined by (7) belongs to the class $C^{\mu,1}$ if the components of $\tilde{\psi}$ belong to the class $C^{s_2+1-j,\mu}$, $j = 1, \ldots, s_2$, as functions of one variable. For brevity it is assumed here that J_2 consists from one Jordan cell, in the general case this definition is regarded with respect to each Jordan's block of J_2. In what follows it is assumed that the characteristics $x_1 + \nu_j x_2 = \text{const}$ of the system (6) don't tangent of the curve Γ, i.e.

$$\mathrm{Re}\gamma'(t) + \nu_j \mathrm{Im}\gamma'(t) \neq 0, \quad t \in \hat{\mathbb{R}},\ 1 \leq j \leq s_2. \tag{10}$$

Moreover it is assumed that Γ coincide with a straight line in a neighborhood of ∞.

It is assumed also, that

$$C, f \in C^{\mu, k_2}(\hat{\Gamma}) \tag{11}$$

and

$$|(b^{-1}u)_1| \leq \text{const}(|x|)^{-1} \tag{12}$$

as $|x| \to \infty$, where by $(b^{-1}u)_1$ we denote the first s_1 elements of the vector $b^{-1}u$.

Let us put

$$C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}\begin{matrix} s_1 \\ s_2 \end{matrix}, \qquad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\begin{matrix} s_1 \\ s_2 \end{matrix}. \tag{13}$$

Without loss of generality we can assume that

$$\det C_2 b_2 \neq 0 \text{ on } \Gamma. \tag{14}$$

Let us consider

$$A = C_1(1 - b_2(C_2 b_2)^{-1} C_2) b_1. \tag{15}$$

We say that (1),(9) is a normal type problem if

$$\det A \neq 0 \quad \text{on } \Gamma. \tag{16}$$

Theorem 2. *Suppose that the conditions* (8), (10) *for the countour* Γ *and condition* (11) *for* C, f *are fullfilled. Then the problem* (1), (8) *is fredholmian in* $C^{1,\mu}(\overline{D})$ *if and only if the normality condition* (16) *is satisfied, and its index is*

$$æ = -\frac{1}{\pi} \arg\det A\Big|_{\Gamma} + s_1. \tag{17}$$

3 The case of the basic domain

Let the hyperbolic system (7) be such that the nilpotent part J_{20} of the matrix J_2 is equal to 0 and the diagonal matrix $\nu = (\nu_i \delta_{ij})$ is composed from two real numbers. Suppose that the boundary ∂D of the finite domain $D \subseteq \mathbb{C}$ consists of two noncharacteristic smooth curves Γ_1 and Γ_2 that connect two corner points τ_1 and τ_2. We consider the following boundary value problem

$$C_j u = f_j \text{ on } \Gamma_j, \quad j = 1, 2, \tag{18}$$

for the system (1), where C_j is a $(s_1 + s_2) \times l$ matrix and f_j is a $(s_1 + s_2)$ vector.

Let us introduce the weighted Holder space $C^{\mu}_{\lambda}(\overline{D}) = C^{\mu}_{\lambda}(\overline{D}; \tau_1, \tau_2)$, $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$, of all functions $\varphi(z)$ such that

$$\varphi(z) = |z - \tau_1|^{\lambda_1 - \mu} |z - \tau_2|^{\lambda_2 - \mu} \varphi_*(z),$$

where $\varphi_*(z) \in C^\mu(\overline{D})$ and $\varphi_*(\tau_1) = \varphi_*(\tau_2) = 0$. The classes $C^{1,\mu}_\lambda$ of differentiable functions are introduced by induction under the condition

$$\varphi \in C^\mu_\lambda, \quad \partial\varphi/\partial x_i \in C^\mu_{\lambda-1}.$$

We consider the problem (1),(18) in the class $C^{1,\mu}_\lambda(D)$ of solutions (1) such that the functions Φ and Ψ belong to this class in the representation (4).

Let $\gamma_j(t) \in C^{1,\mu}[0,1]$ be the smooth parametrization $[0,1] \to \Gamma_j$, $j = 1,2$ and the complex numbers $q_{2i-1} = \gamma_i'(0)$, $q_{2i} = -\gamma_i'(1)$ are the tangent vectors at the points τ_1, τ_2. By θ_j denote the angle of the sector corresponding to the corner τ_j. Evidently, $\theta_j = \arg q_k - \arg q_r$, $0 < \arg q < 2\pi$, $0 < \theta_j < 2\pi$, $P_j = \overline{k,r}$, $j = 1,2$, (more presicely, $P_1 = \overline{1,3}$, $P_2 = \overline{4,2}$), where the rotation from vector q_k to q_r about τ_j within domain is clock-wise.

Let us put the functions of matrices

$$m_j(\zeta) = (\mathrm{Re}\ q_k + (\mathrm{Im}\ q_k)J_1)^\zeta(\mathrm{Re}\ q_r + (\mathrm{Im}\ q_r)J_1)^{-\zeta}, \ \overline{k,r} = P_j,$$

and let be

$$\begin{aligned} x_1(\zeta) &= \left(e^{2\pi i\zeta} - 1\right)^{-1}\left(A_1(\tau_1)w_1(\zeta) + \overline{A_1(\tau_1)}v_1(\zeta)w_1(\zeta)\right), \\ x_2(\zeta) &= \left(e^{2\pi i\zeta} - 1\right)^{-1}\left(\overline{A_2(\tau_2)}v_2(\zeta)w_2(\zeta) + A_2(\tau_2)w_2(\zeta)\right), \end{aligned} \tag{19}$$

where

$$v_j = \begin{pmatrix} 0 & m_j(\zeta) \\ \overline{m_j^{-1}(\overline{\zeta})} & 0 \end{pmatrix}, \quad w_j(\zeta) = e^{2\pi i\zeta}\overline{v_j(\overline{\zeta})} - 1, \ j = 1,2,$$

$$A_j = c_{j,1}(1 - b_2(c_{j,2}b_2)^{-1}c_{j,2})b_1, \ j = 1,2.$$

Theorem 3. *Suppose that the conditiones (8), (10) for the curves Γ_1 and Γ_2 including the corner τ_1, τ_2 are fullfilled. Let C, f belong to C^μ_λ and the normality condition*

$$\det A_j(t) \neq 0, \quad t \in \Gamma_j, \ j = 1,2 \tag{20}$$

be satisfied.

Then the problem (1), (18) is Fredholm in $C^{1,\mu}_\lambda(\overline{D})$ if and only if

$$\det x_k(\zeta) \neq 0, \quad \mathrm{Re}\zeta = \lambda_k, \ k = 1,2, \tag{21}$$

and its index is

$$æ = -\frac{1}{\pi}\arg\det(A_1(t)A_2^{-1}(t))\Big|_0^1 - \frac{1}{2\pi}\sum_{k=1,2}\arg\det x_k(\zeta)\Big|_{\zeta=\lambda_k - i\infty}^{\lambda_k + i\infty} - s_1. \tag{22}$$

4 Some generalazations

We now consider the problem

$$C_j u = f_j \text{ on } \Gamma_j, \ j = 1,2, \tag{23}$$

in finite domains D, whose boundary ∂D consists of two curves Γ_1 and Γ_2 with the corners τ_1 and τ_2. We assume that the matrix $C_1(C_2)$ of order $(s_1 + s_2) \times l$ $(s_1 \times l)$, and the vector $f_1(f_2)$

of order $s_1 + s_2$ (s_1) are prescribed on $\Gamma_1(\Gamma_2)$ and f_j, C_j are belonged $H^k_{\mu,\lambda}$, $k = \overline{s}_2$, $j = 1, 2$. Here the curves Γ_j satisfy conditions (9), (10).

Theorem 4. *The assertion of theorem 3 remains in force also for the problem* (1), (23), *provided only that* A_j *mean matrices* $A_1 = C_{1,1}(1 - b_2(C_{1,2}b_2)^{-1}C_{1,2})b_1$, $A_2 = C_2 b_1$, *and the last term* $-s_1$ *in the formula* (22) *must be replaced by the* s_1.

We also studied the questions of asymptotics of the solutions near the corner points and the smoothness of the solutions up to the boundary. We generalized this approach for the systems of higher order and for a class of the admissible finite domain with piecewise smooth boundary. If the order of C_1 in the last problem is not equal to $s_1 + s_2$ then we investigated this problem only for the case $k_2 = 1$.

Our study is carried out in the framework of the function-theoretic method [5]. The scheme of this method is as follows. First of all we express a general regular solution in terms of regular solutions Φ and Ψ and use an anologue of a theorem of Vekua on integral representations of Φ and some notions about Ψ which arises from (7). By substituting that into the corresponding boundary conditions we reduce the problem to system of integral equations on the boundary of the domain. Another approuch see in [6].

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СИСТЕМЫ УРАВНЕНИЙ ПЕРВОГО ПОРЯДКА ГЛАВНОГО ТИПА НА ПЛОСКОСТИ

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Аннотация. В работе изучаются краевые задачи для систем уравнений первого порядка главного типа с постоянными коэффициентами на плоскости. При этом рассматриваются как полубесконечные области с нехарактеристической границей, так и конечные области типа луночки. Дано представление решений этих систем через решения более простых, так называемых, канонических систем первого порядка эллиптического и гиперболического типов. Получены также формулы для индекса соответствующих задач в весовых классах Гёльдера.

Ключевые слова: уравнения главного типа, нехарактеристическая граница, канонические системы первого порядка.