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# ON APPROXIMATE SOLUTION OF CERTAIN EQUATIONS

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***Abstract.*** In this paper, we consider problems of discrete approximation of special integral operators with the Calderon–Zygmund kernel. We introduce discrete spaces and bounded discrete operators acting in these spaces; then we use these operators for the search for approximate solutions of the corresponding equations. We state theorems on the solvability of equations with discrete operators, compare integral operators with their discrete analogs, and obtain estimates of errors of approximate solutions.

***Keywords and phrases*:** Calderon–Zygmund kernel, discrete operator, symbol, approximation measure, approximate solution.

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**1. Introduction.** Calderon–Zygmund operators and the corresponding multidimensional singular equations were studied by many authors in various functional spaces (see, e.g., [6]). Such equations often appear in various problems of mathematical physics (see [2, 5, 6]). However, as a rule, only one-dimensional singular integral equations are considered in theoretical studies (see [2, 4, 5]); in certain cases, multidimensional compact operators are considered (see [7]). In recent years, the theory of $C^*$-algebras (see, e.g., [3]) is used for the theoretical justification of approximate (projection) methods, but errors were not estimated in these studies.

This paper provides an overview of the author's results [8–19] related to the justification of the discretization scheme of the simplest types of such equations and finding an approximate solution and error estimate for discrete solutions.

**2. Calderon–Zygmund operators and their discrete analogs.** A *multidimensional singular integral equation* in the space $\mathbb{R}^m$ is the equation of the form

$$a(x)u(x) + \int\limits_{\mathbb{R}^m} K(x, x-y)u(y)dy = v(x), \quad x \in \mathbb{R}^m, \tag{1}$$

where the kernel of $K(x,y)$ is the so-called Calderon–Zygmund kernel (see [6]), and the integral in (1) is meant in the sense of the principal value

$$\int\limits_{\mathbb{R}^m} K(x, x-y)u(y)dy = \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int\limits_{\varepsilon<|x-y|<N} K(x,x-y)u(y)dy.$$

**Definition 2.1.** A function $K(x,y)$ defined on $\mathbb{R}^m \times (\mathbb{R}^m \setminus \{0\})$ is called a *Calderon–Zygmund kernel* if it satisfies the following conditions:

(1) $K(x,tx) = t^{-m}K(x,y)$ for all $x \in \mathbb{R}^m$ and all $t > 0$;

(2) $\displaystyle\int\limits_{S^{m-1}} K(x,\omega) = 0$ for all $x \in \mathbb{R}^m$;

(3) $|K(x,y)| \le C$, $K(x,\omega)$ is differentiable with respect to $S^{m-1}$ for all $x \in \mathbb{R}^m$, where $S^{m-1}$ is the unit sphere in the $m$-dimensional space, and $C$ is a constant.

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We consider the simplest equation (1) in the case where the kernel $K(x,y)$ is dependent of the pole $x$, i.e., has the form

$$au(x) + \int_{\mathbb{R}^m} K(x-y)u(y)dy = v(x), \quad x \in \mathbb{R}^m. \tag{2}$$

One would think that Eq. (2) can be solved by applying the Fourier transform, but this is not so. From the computational point of view, discrete (and also finite) sets of points imitating (simulating) Eq. (2) are needed. Hence we first replace Eq. (2) by a discrete system and then consider its possible finite approximations. Some preliminary considerations related to this were described in [2, 4, 5, 7].

*2.1. Discrete analogs.* In the $m$-dimensional space $\mathbb{R}^m$, we introduce the discrete grid $h\mathbb{Z}^m$ on which the functions $u_d(\tilde{x})$ of a discrete argument $\tilde{x} \in h\mathbb{Z}^m$ are defined; we set $K(0) = 0$ and denote by $K_d$ the restriction of the kernel $K(x)$ to $h\mathbb{Z}^m$.

For the multidimensional singular integral operator

$$(Ku)(x) = \int_{\mathbb{R}^m} K(x-y)u(y)dy$$

we consider the following discrete analog:

$$(K_d u_d)(x) = \sum_{\tilde{y} \in h\mathbb{Z}^m} K_d(\tilde{x}-\tilde{y})u_d(\tilde{y})h^m, \quad \tilde{x} \in h\mathbb{Z}^m. \tag{3}$$

The sum of the series (3) means the limit of partial sums

$$\lim_{N\to\infty} \sum_{\tilde{y} \in h\mathbb{Z}^m \cap Q_N} K_d(\tilde{x}-\tilde{y})u_d(\tilde{y})h^m,$$

where

$$Q_N = \left\{ x \in \mathbb{R}^m : \max_{1\le k\le m} |x_k| \le N \right\}.$$

The symbol $\ell_h^2$ denotes the Hilbert space $L_2(h\mathbb{Z}^m)$ of functions of a discrete argument with the inner product

$$(u_d, v_d) = \sum_{\tilde{x} \in h\mathbb{Z}^m} u_d(\tilde{x})\overline{v_d(\tilde{x})}h^m$$

and the corresponding norm

$$\|u_d\|_{\ell_h^2} = \left( \sum_{\tilde{x} \in h\mathbb{Z}^m} |u_d(\tilde{x})|^2 h^m \right)^{1/2}.$$

It is well known that under the conditions stated for the kernel, the operator $K$ acts boundedly in the space $L_2(\mathbb{R}^m)$ (see [6]). Taking this into account, we can easily prove the following theorem.

**Theorem 2.1.** *The estimate*

$$\left\| K_d u_d \right\|_{\ell_h^2} \le c\|u_d\|_{\ell_h^2}$$

*is valid, where the constant $c$ is independent of $h$.*

Thus, the family of discrete operators (3) is uniformly bounded on $h$.

*2.2. Comparison of symbols.*

**Definition 2.2.** The *symbol* of an operator $K$ is the Fourier transform of the kernel $K(x)$ in the sense of the principal value:

$$\sigma(\xi) = \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int\limits_{\varepsilon<|x|<N} K(x)e^{i\xi\cdot x}dx.$$

Applying the Fourier transform to Eq. (2), we obtain the equation

$$(a+\sigma(\xi))\tilde{u}(\xi) = \tilde{v}(\xi);$$

the necessary and sufficient condition of its solvability in the space $L_2(\mathbb{R}^m)$ has the following form (see [6]):

$$\inf |a+\sigma(\xi)| > 0, \quad \xi \in \mathbb{R}^m.$$

The function $a+\sigma(\xi)$ is called the *symbol* of the operator $aI+K$; here $I$ is the identity operator.

We associate the symbol $\sigma_d(\xi)$, $\xi \in [-\pi h^{-1}, \pi h^{-1}]^m$, defined by the multidimensional Fourier series

$$\sigma_d(\xi) = \sum_{\tilde{x}\in h\mathbb{Z}^m} K(\tilde{x})e^{-i\tilde{x}\cdot\xi}h^m,$$

with the discrete operator $K_d$. Its partial sums are taken over discrete cubes $Q_N \cap h\mathbb{Z}^m$; they are periodic functions in $\mathbb{R}^m$ with the main cube of periods $[-\pi h^{-1}, \pi h^{-1}]^m$.

Similarly, the function $a+\sigma_d(\xi)$, $\xi \in [-\pi h^{-1}, \pi h^{-1}]^m$ is called the symbol of the discrete singular equation

$$(aI+K_d)u_d = v_d. \tag{4}$$

We say that a symbol is *elliptic* if it does not vanish anywhere.

It was established (see [10, 15, 16]) that the sets of the values of the symbols $\sigma(\xi)$ and $\sigma_d(\xi)$ coincide; this immediately implies that Eq. (2) and its discrete analog (4) are solvable or insolvable simultaneously. Thus, there is a solution of the infinite system of linear algebraic equations (4); it is natural to expect that for small $h>0$ it will be close to the solution of the original equation (2).

*2.3. Comparison of operators.* Let us denote by $P_h$ the operator of restriction to the grid $h\mathbb{Z}^m$, i.e., the operator assigning to each function defined on $\mathbb{R}^m$ the discrete set of its values at the nodes of the grid $h\mathbb{Z}^m$.

Following [12, 16], we state the following definition.

**Definition 2.3.** A *measure of approximation* of the operators $K$ and $K_d$ in a linear normed space $X$ of functions defined on $\mathbb{R}^m$ is the operator norm

$$\|P_hK - K_dP_h\|_{X_d},$$

where $X_d$ is the normed space of functions defined on the grid $h\mathbb{Z}^m$ with the norm induced by the norm of the space $X$.

We use the space $C_h$ along with the space $\ell_h^2$ as the space $X_d$, which is the function space $u_d$ of the discrete argument $\tilde{x}\in h\mathbb{Z}^m$ with the norm

$$\|u_d\|_{C_h} = \max_{\tilde{x}\in h\mathbb{Z}^m} |u_d(\tilde{x})|.$$

In other words, the space $C_h$ is the space of restrictions of functions $u\in C(\mathbb{R}^m)$ to the nodes of the grid $h\mathbb{Z}^m$. Here it is worth noting that the operator $K$ is unbounded in the space $C(\mathbb{R}^m)$; however, it is bounded in the space $L_2(\mathbb{R}^m)$. It is well known that if the right-hand side of Eq. (2) possesses a certain smoothness (for example, satisfies the Hölder condition), then the solution of Eq. (2) (if it exists in $L_2(\mathbb{R}^m)$) possesses the same smoothness (see [6]).

We define the discrete space $C_h(\alpha,\beta)$ as the space of functions of the discrete argument $\tilde{x} \in h\mathbb{Z}^m$ with the finite norm

$$\|u_d\|_{C_h(\alpha,\beta)} = \|u_d\|_{C_h} + \sup_{\tilde{x},\tilde{y}\in h\mathbb{Z}^m} \frac{|\tilde{x}-\tilde{y}|^\alpha}{(\max\{1+|\tilde{x}|,\, 1+|\tilde{y}|\})^\beta},$$

satisfying the conditions

$$|u_d(\tilde{x})| \le \frac{c}{(1+|\tilde{x}|)^{\beta-\alpha}}, \quad |u_d(\tilde{x}) - u_d(\tilde{y})| \le c \quad \frac{|\tilde{x}-\tilde{y}|^\alpha}{(\max\{1+|\tilde{x}|,\, 1+|\tilde{y}|\})^\beta}$$

for all $\tilde{x},\tilde{y} \in \mathbb{R}^n$, $\alpha,\beta > 0$, $0<\alpha<1$.

The continual analog of these spaces is the space $H^\alpha_\beta(\mathbb{R}^m)$ of continuous functions on $\mathbb{R}^m$ satisfying the Hölder conditions with a constant $0<\alpha<1$ and the weight $(1+|x|)^\beta$ (see [1]). In particular, results of [1] imply that the operator $K$ is a linear bounded operator $K : H^\alpha_\beta(\mathbb{R}^m) \to H^\alpha_\beta(\mathbb{R}^m)$ under the condition $m < \beta < \alpha + m$.

For the spaces $C_h(\alpha,\beta)$, the following statement holds.

**Theorem 2.2.** *The estimate*

$$\|K_d u_d\|_{C_h(\alpha,\beta)} \le c\|u_d\|_{C_h(\alpha,\beta)}$$

*holds, where $m < \beta < \alpha + m$ and the constant $c$ is independent of $h$.*

We provide an estimate of the approximation measure of the operators $K$ and $K_d$ in the space $C_h(\alpha,\beta)$. This allows us to estimate the error of the solution when replacing the continual operator $K$ with its discrete analog $K_d$.

**Theorem 2.3.** *For the approximation measure of the operators $K$ and $K_d$, the estimate*

$$\left\|P_h K - K_d P_h\right\|_{C_h(\alpha,\beta)} \le ch^{\tilde{\alpha}}$$

*holds, where the constant $c$ is independent of $h$, $\tilde{\alpha} < \alpha$, and $\tilde{\beta} > \beta$.*

**3. Analog of Eq.** (2) **in a half-space.** An analog of Eq. (2) in a half-space is the equation

$$au(x) + \int\limits_{\mathbb{R}^m_+} K(x-y)u(y)dy = v(x), \quad x \in \mathbb{R}^m_+, \tag{5}$$

in $L_2(\mathbb{R}^m_+)$, $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x = (x', x_m),\ x_m > 0\}$.

This equation is well studied. Using the Fourier transform, we can reduce it to the classical boundary-value Riemann problem for the upper and lower complex half-plane with the coefficient $\sigma(\xi',\xi_m)$, where $\xi$ is the Fourier dual variable and $\xi' = (\xi_1,\dots,\xi_{m-1})$ plays the role of a parameter.

It is well known that in the case of discrete convolutions, it is possible to use a similar scheme with the discrete Fourier transform

$$(F_d u_d)(\xi) = \frac{1}{(2\pi)^m} \sum_{\tilde{x}\in h\mathbb{Z}^m} u_d(\tilde{x}) e^{-i\tilde{x}\cdot\xi} h^m \equiv \tilde{u}_d(\xi),\ \xi \in [-h^{-1}\pi, h^{-1}\pi]^m,$$

which leads us to a periodic Riemann problem with the coefficient $\sigma_d(\xi',\xi_m)$ and the parameter $\xi'$ in a band. The solvability of both problems is determined by the topological indexes $\sigma$ and $\sigma_d$ with respect to the variable $\xi_m$.

The topological index of this boundary-value Riemann problem in the simplest case is the increment of the argument of the function $\sigma(\cdot,\xi_m)$ when $\xi_m$ changes from $-\infty$ to $+\infty$; it is independent of $\xi'$ ($m \ge 3$). The same is valid for the discrete equation (4) in the discrete half-space

$$au_d(\tilde{x}) + \sum_{\tilde{y}\in h\mathbb{Z}^m_+} K(\tilde{x}-\tilde{y})u_d(\tilde{y})h^m = v_d(\tilde{x}), \quad \tilde{x} \in h\mathbb{Z}^m_+, \tag{6}$$

and its solvability is determined by the increment of the argument $\sigma_d(\cdot,\xi_m)$ when $\xi_m$ changes in the interval $[-\pi h^{-1},\pi h^{-1}]$. There is a connection between these two symbols (see below).

*3.1. Periodic Riemann problem.* Here we explain the appearance of the periodic Riemann problem and denote its connections with Eq. (6).

Let us consider the one-dimensional case, $h=1$. If we denote projectors onto $\mathbb{Z}_\pm$ by $P_\pm$, then it is easy to calculate that

$$(F_d P_\pm u_d)(\xi) = 1/2\tilde{u}_d(\xi) \mp i/2 \lim_{s\to 0\pm} \int\limits_{-\pi}^{\pi} \cot\frac{\zeta-\tau}{2}\tilde{u}_d(\tau)d\tau, \quad \zeta=\xi+is.$$

It is worth noting that such singular integrals are obtained by summing Fourier series with the help of the Dirichlet kernel with the subsequent passage to the limit in partial sums; this is a periodic analog of the Hilbert transform

$$(Hu)(x) = \text{v.p.} \int\limits_{-\pi}^{\pi} \cot\frac{x-t}{2}u(t)dt.$$

Let us consider the function

$$\Phi(\zeta) = \frac{1}{4\pi i}\int\limits_{-\pi}^{\pi} \cot\frac{\zeta-t}{2}\phi(t)dt$$

and assume that $\phi(t)$ satisfies the Hölder condition on $[-\pi,\pi]$:

$$\left|\phi(t_1)-\phi(t_2)\right| \le c|t_1-t_2|^\alpha$$

for all $t_1,t_2\in[-\pi,\pi]$, $0<\alpha\le 1$, $\phi(-\pi)=\phi(\pi)$. The limit values of this function on the line can be expressed by a singular integral $H$.

**Theorem 3.1.** *The following formulas hold*:

$$\Phi^\pm(\xi) = \pm\frac{\phi(t)}{2} + \frac{1}{2\pi i}\text{v.p.}\int\limits_{-\pi}^{\pi} \cot\frac{\xi-t}{2}\phi(t)dt,$$

*where $\Phi^\pm(\xi)$ are the limit values of $\Phi^\pm(\zeta)$ as $s\to\pm 0$.*

These formulas (the Sokhotski—Plemelj formulas) lead to the following formulation of the periodic boundary-value Riemann problem: find a pair of functions $\Phi^\pm(z)$ that are analytic in the half-bands

$$\Pi_\pm = \Big\{z\in\mathbb{C} : z=t+is,\ t\in[-\pi,\pi],\ \pm s>0\Big\},$$

whose boundary values as $s\to 0\pm$ are related by the linear relation

$$\Phi^+(t) = G(t)\Phi^-(t)+g(t), \quad t\in[-\pi,\pi],$$

where $G(t)$ and $g(t)$ are given functions on $[-\pi,\pi]$.

If we assume that $G(t)\in C[-\pi,\pi]$ and $G(-\pi)=G(\pi)$, then the index of the function $G$ on the interval $[-\pi,\pi]$ is defined as the increment $\arg G(t)$ divided by $2\pi$ when $t$ changes from $-\pi$ to $\pi$. This integer is called the *index of problem* and is denoted by $\varkappa\equiv \operatorname{Ind} G(t)$.

**Theorem 3.2.** *If $G(t)$ satisfies the Hölder condition and $\varkappa=0$, then the periodic Riemann problem has a unique solution $\Phi^\pm(t)\in L_2[-\pi,\pi]$, which is constructed by using the function $\Phi(\zeta)$.*

*3.2. Discrete half-space.* By a similar reasoning on a multidimensional grid with a step $h$ in a half-space and using the operator (see [10, 15])

$$(H^{per}_{\xi'} u_d)(\xi_m) = \frac{1}{2\pi i} \int\limits_{-\pi h^{-1}}^{\pi h^{-1}} u(t) \cot \frac{h(t-\xi_m)}{2} dt,$$

we arrive at the periodic Riemann problem with the parameter $\xi'$. The index of this problem is

$$\varkappa_d = \int\limits_{-\infty}^{+\infty} d \arg \sigma_d(\cdot, \xi_m).$$

Since the Calderon–Zygmund kernel is very specific, we conclude that under additional condition $\sigma(0,\dots,0,-1) = \sigma(0,\dots,0,+1)$ on the character of the operator, the indices of the corresponding boundary-value problems are the same (see [10, 15]); this immediately implies the following statement.

**Theorem 3.3.** *Equations* (**??**) *and* (6) *are simultaneously solvable or insolvable in the spaces* $L_2(\mathbb{R}^m_+)$ *and* $L_2(h\mathbb{Z}^m_+)$ *for all* $h > 0$.

*3.3. Discrete approximations and comparison.* Let $x' = (x_1, \dots, x_{m-1})$, $x = (x', x_m)$; we consider a weight function of the form

$$\omega(\tilde{x}) = (1 + |\tilde{x}|)^\alpha \left( \frac{\tilde{x}_m}{1 + \tilde{x}_m} \right)^\beta .$$

Let us introduce the discrete space $H^{\alpha,\beta}_\gamma(h\mathbb{Z}^m_+)$ as the space of functions $u_d(\tilde{x})$ defined on $h\mathbb{Z}^m_+$ ($\mathbb{Z}^m_+ = \mathbb{Z}^m \cap \mathbb{R}^m_+$) with the norm

$$\|u_d\|_{\alpha,\beta,\gamma} \equiv \|\omega \cdot u_d\|_\gamma,$$

$0 < \gamma < 1$, $0 < \alpha + \gamma < m$, $\gamma < \beta < \gamma + 1$, where

$$\|u_d\|_\gamma \equiv \max_{\tilde{x} \in h\mathbb{Z}^m_+} \left| u_d(\tilde{x}) \right| + \max_{\tilde{x},\tilde{y} \in h\mathbb{Z}^m_+} \frac{\left| u_d(\tilde{x}) - u_d(\tilde{y}) \right|}{|\tilde{x} - \tilde{y}|^\gamma}.$$

Note that this space also has a continual analog $H^{\alpha,\beta}_\gamma(\mathbb{R}^m_+)$ (see [1]).

**Theorem 3.4.** *The operator* $K_d$ *is a linear bounded operator in the space* $H^{\alpha,\beta}_\gamma(h\mathbb{Z}^m_+)$, *and its norm is independent of* $h$.

We introduce the restriction operator

$$l_h : H^{\alpha,\beta}_\gamma(\mathbb{R}^m_+) \longrightarrow H^{\alpha,\beta}_\gamma(h\mathbb{Z}^m_+);$$

then for a function $u \in H^{\alpha,\beta}_\gamma(\mathbb{R}^m_+)$ we have the following result.

**Theorem 3.5.**

$$\left| \left[ (l_h K - K^h_d l_h) u \right] (\tilde{x}) \right| \le c h^\gamma \ln \frac{1}{h};$$

*moreover, the constant is independent of* $h$.

If we assume, as above, that $\sigma(0,\dots,0,-1) = \sigma(0,\dots,0,+1)$, then we obtain a result similar to Theorem 3.3.

**Theorem 3.6.** *The operators* $K$ *and* $K_d$ *are simultaneously invertible or not in the spaces* $H^{\alpha,\beta}_\gamma(\mathbb{R}^m_+)$ *and* $H^{\alpha,\beta}_\gamma(h\mathbb{Z}^m_+)$, *respectively.*

The last considerations imply that such discrete operators are useful for constructing convenient finite-dimensional approximations.

## 4. Approximate solutions and error estimates.

*4.1. Cyclic convolution.* We introduce a special discrete periodic kernel $K_{d,N}(\tilde{x})$ as follows. We take the restriction of the discrete kernel $K_d(\tilde{x})$ to the discrete cube $Q_N \cap h\mathbb{Z}^m \equiv Q_N^d$ and periodically extend it to all $h\mathbb{Z}^m$. Next, we consider discrete periodic functions $u_{d,N}$ with a discrete cube of periods $Q_N^d$ and define a cyclic convolution for a pair of such functions $u_{d,N}$ and $v_{d,N}$ by the formula

$$(u_{d,N} * v_{d,N})(\tilde{x}) = \sum_{\tilde{y} \in Q_N^d} u_{d,N}(\tilde{x} - \tilde{y}) v_{d,N}(\tilde{y}) h^m.$$

Then we define the finite Fourier transform

$$(F_{d,N} u_{d,N})(\tilde{\xi}) = \sum_{\tilde{x} \in Q_N^d} u_{d,N}(\tilde{x}) e^{i\tilde{x}\cdot\tilde{\xi}} h^m, \quad \tilde{\xi} \in R_N^d,$$

where $R_N^d = [-h^{-1}\pi, h^{-1}\pi]^m \cap Q_N^d$; here $\tilde{\xi}$ is a discrete variable. Now we introduce the operator

$$K_{d,N} u_{d,N}(\tilde{x}) = \sum_{\tilde{y} \in Q_N^d} K_{d,N}(\tilde{x} - \tilde{y}) u_{d,N}(\tilde{y}) h^m$$

on periodic discrete functions $u_{d,N}$ and finite Fourier transform for its kernel

$$\sigma_{d,N}(\tilde{\xi}) = \sum_{\tilde{x} \in Q_N^d} K_{d,N}(\tilde{x}) e^{i\tilde{x}\cdot\tilde{\xi}} h^m, \quad \tilde{\xi} \in R_N^d.$$

**Definition 4.1.** The function $\sigma_{d,N}(\tilde{\xi})$, $\tilde{\xi} \in R_N^d$, is called the *symbol* of the operator $K_{d,N}$. A symbol is said to be *elliptic* if $\sigma_{d,N}(\tilde{\xi}) \neq 0$ for all $\tilde{\xi} \in R_N^d$.

**Theorem 4.1.** *Let $\sigma_d(\xi)$ be an elliptic symbol. Then for sufficiently large $N$, the symbol $\sigma_{d,N}(\tilde{\xi})$ is also elliptic.*

As above, the elliptic symbol $\sigma_{d,N}(\tilde{\xi})$ corresponds to the invertible operator $K_{d,N}$ in the space $L_2(Q_N^d)$.

*4.2. Approximation measure.* Let $A : B \to B$ be a linear bounded operator acting in a Banach space $B$, $B_N \subset B$ be a finite-dimensional subspace, $P_N : B \to B_N$ be a projector, and $A_N : B_N \to B_N$ be a linear finite-dimensional operator (see [17, 18]).

**Definition 4.2.** An *approximation measure* of the operators $A$ and $A_N$ is the operator norm

$$\|P_N A - A_N P_N\|_{B \to B_N}.$$

Obtaining such an operator estimate is a difficult problem, so we present a weaker version of the estimate on a particular element of the space $C_h(\alpha, \beta)$. Note that under our assumptions, $C_h(\alpha, \beta) \subset L_2(h\mathbb{Z}^m)$.

**Theorem 4.2.** *For the operators $K_d$ and $K_{d,N}$ we have the estimate*

$$\|(P_N K_d - K_{d,N} P_N) u_d\|_{L_2(Q_N^d)} \leq C N^{m+2(\alpha-\beta)}$$

*for arbitrary $u_d \in C_h(\alpha, \beta)$, where $\beta > \alpha + m/2$.*

*4.3. Comparison of solutions.* Here we consider the equation

$$(aI_{d,N} + K_{d,N})u_{d,N} = P_N v_d \tag{7}$$

instead of the equation

$$(aI_d + K_d)u_d = v_d \tag{8}$$

and compare solutions of these two equations.

Assume that the operator $aI_d + K_d$ is invertible in the space $L_2(h\mathbb{Z}^m)$.

**Theorem 4.3.** *If $v_d \in C_h(\alpha, \beta)$, $\beta > \alpha + m/2$, $u_d$ is a solution of Eq.* (8), *and $u_{d,N}$ is a solution of Eq.* (7), *then the estimate*

$$\left\| u_d - u_{d,N} \right\|_{L_2(h\mathbb{Z}^m)} \le C N^{m+2(\alpha-\beta)}$$

*holds, where $C$ is a constant independent of $N$.*

*4.4. Relationship between $N$, $h$, and the location of $\tilde{x}$.* Consider the original equation

$$(aI + K)u = v \tag{9}$$

in the space $L_2(\mathbb{R}^m)$ and the equation

$$(aI_d + K_d)u_d = P_d v \equiv v_d, \tag{10}$$

where $P_d$ is the projection operator, which, given a continuous function $v$ defined on $\mathbb{R}^m$, transfers the function of a discrete argument to $\mathbb{Z}^m$. It is natural to compare the solutions of the whole triple of Eqs. (9), (10), and (7). The connection between the solutions (7) and (10) is described in Theorem 4.3.

**Theorem 4.4.** *If $v \in H^\alpha_\beta(\mathbb{R}^m)$, $0 < \alpha < 1$, $m < \beta < \alpha + m$, $u$ is a solution of Eq.* (9), *and $u_d$ is a solution of Eq.* (10), *then the following estimate holds*:

$$\left| u(\tilde{x}) - u_d(\tilde{x}) \right| \le c h^\alpha \ln \frac{1}{h} \quad \forall \tilde{x} \in h\mathbb{Z}^m,$$

*where the constant $c$ in independent of $h$.*

We denote by $r(\tilde{x})$ the distance between $\tilde{x} \in h\mathbb{Z}^m \cap Q_N$ and $\partial Q_N$.

**Theorem 4.5.** *Let $v_d \in C_h(\alpha, \beta)$, $u_d$ be a solution of Eq.* (10), *and $u_{d,N}$ be a solution of Eq.* (7). *Then for all $\tilde{x} \in \mathbb{Z}^m \cap Q_N$, the following estimate holds*:

$$\left| u_d(\tilde{x}) - u_{d,N}(\tilde{x}) \right| \le c_1 \begin{cases} N^{\alpha-\beta} \ln\left(1 + c_2 N/h\right), & \text{if } r(\tilde{x}) \sim N^{-1}, \\ N^{\alpha-\beta} & \text{otherwise}; \end{cases}$$

*here $c_1$ and $c_2$ are constants independent of $h$ and $N$.*

Of course, not all available results are presented in this paper; in particular, estimates of the error of the discrete solution in the semi-space are not presented. In addition, there are some results related to the Fredholm property of more general discrete operators, which allow us to hope for a generalization of the available results into wider classes of operators and equations.

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